**Problem 1) MS beta-function with multiple dimensionful couplings**

Consider a field theory with a set of couplings $g_1, g_2, \ldots, g_\ell, \ldots$ which we’ll call $\vec{g}$. Your goal is to derive an expression for the beta-functions in the MS-scheme that is valid at any order in perturbation theory. (It will also be valid for both renormalizable theories and “non-renormalizable” theories with irrelevant operators.) Let $\Delta_l(d) = \Delta_\ell + \epsilon \rho_\ell$ be the dimension of the bare coupling $g_\ell^{\text{bare}}$ in $d = 4 - 2\epsilon$ dimensions. We can define a dimensionless running coupling in the MS-scheme by

$$g_\ell^{\text{bare}} \mu^{-\Delta_l(d)} = g_\ell(\mu, d) Z_{g_\ell}(\vec{g}),$$

where $g_\ell(\mu, d)$ is analytic in $d$ and $Z_{g_\ell}$ is a series in $1/\epsilon$. Prove that the beta-function for $g_\ell \equiv g_\ell(\mu, d)$ is

$$\beta(g_\ell, d) = \mu \frac{d}{d\mu} g_\ell = -\epsilon \rho_\ell g_\ell - \Delta_\ell g_\ell + g_\ell \sum_m \frac{da_1^\ell(\vec{g})}{dg_m} \rho_m g_m,$$

where $a_1^\ell$ is the coefficient of the $1/\epsilon$ pole term in $Z_{g_\ell}$. Find a recursion relation for the coefficients of the higher poles, $a_k^\ell(\vec{g})$. How are these results modified in the $\overline{\text{MS}}$-scheme? [Remark: in this notation the standard beta-function is $\beta(g_\ell) = \beta(g_\ell, 4)$.

**Problem 2) Scheme and gauge dependence of beta-functions**

Consider a renormalizable non-abelian gauge theory. Let $g(\mu_R)$ and $\beta(g)$ be the renormalized coupling and beta-function in a mass-independent renormalization scheme. This could be the MS-scheme, or an offshell momentum subtraction scheme with $m \to 0$, etc.

a) Prove that in any scheme the first term in the series expansion of $\beta(g)$ is gauge independent.

b) Prove that the first two terms in the series expansion of $\beta(g)$ are scheme independent.

c) Finally, prove that in the MS-scheme the renormalized coupling is gauge independent. Thus in this scheme all terms in $\beta(g)$ are gauge independent. Use this result to strengthen the statement in a).

**Problem 3) QCD beta-function in background field gauge**

Using background field gauge derive the lowest order beta-function for QCD in a massless scheme by carrying out the computation of the 3 diagrams discussed in lecture. Express your result in terms of the quadratic adjoint Casimir $C_A$ and the number of quark flavors $n_f$. What is the beta-function for an SU(2) gauge theory with 5 flavors?
Problem 4) The QCD running coupling and thresholds

We originally motivated the discussion of mass-independent renormalization schemes by considering $\mu \gg m$. However, these schemes are well defined regardless of the relation between $\mu$ and $m$, and so we can consider using them for $\mu \simeq m$ and for that matter $\mu \leq m$. In this problem we’ll explore how this works in QCD at one-loop order.

In lecture you saw that in a mass-dependent scheme for QED the electron “decouples” from the beta-function as we go below its mass scale, falling off quite rapidly, $\beta(\mu \ll m) \sim \mu^2/m^2$. However, if you consider the mass-independent QED beta-function (say in MS) with an $e^-$, $\mu^-$, and $\tau^-$ then a priori you have the same beta-function for $m_e \ll \mu \ll m_\mu$, $\mu \gg m_\tau$, or any other value of $\mu$. The issue with a mass-independent scheme is that $\alpha(\mu)$ is not smart enough to know that heavy particles in the field theory should decouple. This is an important piece of physics, and we’re going to build this into the mass-independent schemes by hand.

To do this consider evolving the coupling down from a $\mu \gg m_\tau$ with the QED beta-function with 3-leptons, $n_\ell = 3$. When we reach $\mu = m_\tau$ we’ll decree that the tau is removed from our theory, so that below this scale we switch to using a beta function with $n_\ell = 2$. Let’s call the coupling in the theory with $n_\ell$ leptons $\alpha^{(n_\ell)}(\mu)$. To ensure that this process does not disturb our field theory too much, we’ll demand that scattering amplitudes computed in the theory with $n_\ell = 3$ and $n_\ell = 2$ are the same at $\mu = m_\tau$. At lowest order in perturbation theory this just implies continuity of the coupling at the boundary, $\alpha^{(3)}(m_\tau) = \alpha^{(2)}(m_\tau)$. At each mass-threshold we’ll repeat the above procedure to build in the decoupling by hand.1

Let’s apply the same logic to QCD to give couplings $\alpha^{(n_f)}_{s}(\mu)$ which satisfy the mass-independent beta-function equation for $n_f$-flavors. From $\alpha^{(n_f)}_{s}(\mu)$ we can define the integration constant $\Lambda^{(n_f)}_{QCD}$.

a) Let $\alpha^{(5)}_{s}(m_Z) = 0.118$ with the physical $Z$-boson mass be your initial condition. Let’s take the mass of the bottom and charm quarks to be $m_b = 4.5$ GeV and $m_c = 1.6$ GeV. Using the decoupling procedure described above, compute $\alpha^{(3)}_{s}(\mu = 1.3$ GeV). Compare it numerically to $\alpha^{(5)}_{s}(\mu = 1.3$ GeV).

b) Consider QCD with $m_u = m_d = m_s = 0$ and note that in this theory the proton-mass is dominated by non-perturbative dynamics of these three quarks. Hence we expect the proton mass $m_p \propto \Lambda^{(3)}_{QCD}$. Derive a relation between $\Lambda^{(6)}_{QCD}$ and $\Lambda^{(3)}_{QCD}$ that only involves heavy-quark masses. Now imagine that the strong coupling is fixed at some very high scale (eg. at a unification scale $\sim 10^{16}$ GeV), and predict how much the proton-mass changes if you double the c-quark mass.

1In the context of effective field theory this procedure of removing particles is known as “integrating out” a massive degree of freedom, and ensuring the continuity of the S-matrix elements is known as “matching”.