Set 5 – due 17 February

“The ideas of space and time which I wish to develop before you grew from the soil of experimental physics. Therein lies their strength. Their tendency is radical. From now on, space by itself and time by itself must sink into the shadow, while only a union of the two preserves independence.” – H. Minkowski (1908)

1) [5 points] Jackson 11.4 part (a). (b) [5 points] Instead of Jackson’s part (b), do the following: “Length contraction” occurs when an observer infers a length from an elapsed time. Suppose a rod moves parallel to its length at velocity \( v \), past an observer. The (unprimed) observer sees one end of the rod pass the origin at time \( t_1 = 0 \). The other end passes the origin at time \( t_2 \). The observer infers a length \( l = vt_2 \). A (primed) observer on the rod also has \( (x'_1, t'_1) = (0, 0) \). The other end of the rod is at \( l_0 \) and the moving first observer passes this end at time \( t'_2 = l_0/v \). Show \( l = l_0/\gamma \).

2) [10 points] Consider a stick with a mechanism which, in its proper rest frame, can simultaneously release a drop of ink from each end. The stick moves parallel to its length with a velocity \( v \) along the floor. When the mechanism is set off, how far apart (in the rest frame of the floor) are the marks on the floor? Justify your result from the point of view of an observer in the rest frame of the stick and an observer in the rest frame of the floor.

3) [15 points] Consider a stick with a mirror on the right end. At a given moment a photon and a particle moving with velocity \( u < c \) leave the left end, moving to the right along the stick. The photon reaches the mirror first, is reflected, and, moving back to the left, encounters the particle still moving to the right, a fraction \( x \) of the way along the stick. The situation at three typical moments is shown below.

Using the fact that \( x \) is an invariant quantity (think about it!) show using only the constancy of the speed of light (i.e. you don’t have to assume anything about the shrinking factor for the moving stick) that for an observer moving to the left with velocity \( v \) with respect to the stick, the velocity of the particle must be

\[
\frac{w}{1 + \frac{uv}{c^2}} = \frac{u + v}{1 + \frac{uv}{c^2}}
\]  

(1)
One tick of the clock takes a time $t'$ in the clock's frame and a time $t$ in the observer's frame. In the clock's frame, the light goes a distance $2d$ at velocity $c$ so

$$t' = \frac{2d}{c}$$

The observer sees the light going a distance $s$ in a time $t = \frac{s}{c}$ so

$$s = 2 \left[ d^2 + \left( \frac{vE}{2} \right)^2 \right]^{1/2}$$

so that

$$\frac{c^2 E^2}{4} = \frac{d^2 + v^2 E^2}{4}$$

$$\left( \frac{t'}{2} \right)^2 = \frac{d^2}{c^2 \left[ 1 - \frac{v^2}{c^2} \right]}$$

$$t = 2d \frac{1}{c \sqrt{1 - \frac{v^2}{c^2}}} = \delta t'$$.
b) \((x_1, t_1) = (x_2, t_2) = (0, 0)\) when \(\ddot{xt} = \ddot{st} = \ddot{ct} = 0\), refer to the spacetime point where the stationary observer passes the leading edge of the rod. In the rod's frame the observer is at the other end of the rod at time \(t_2^\prime\) so that spacetime point is \((0, s, t_2^\prime, c) = (x_2^\prime, t_2^\prime)\).

From the coordinates and \(t_2^\prime = \frac{lo}{c^2}\).

The unprimed coordinate is \(x_2 = 0\). The primed coordinate is \(x_2^\prime = \gamma (x_2^\prime - vt_2^\prime) = 0\).

This makes sense - the rod passes the observer but \(x_2 = x_1 = 0\). It also checks the sign of the Lorentz transformation.

\[
t_2 = \gamma \left( t_2^\prime - \frac{v}{c^2} x_2^\prime \right)
= \gamma \left[ \frac{lo}{c^2} - \frac{v}{c^2} \frac{lo}{c^2} \right]
= \gamma \left[ 1 - \frac{v^2}{c^2} \right] \frac{lo}{c^2}
= \frac{1}{\gamma} \frac{lo}{c^2}
\]

and \(\gamma = \frac{c}{V} \rightarrow \frac{V}{c} = \frac{lo}{c^2} \).
2) In the stick’s frame (I will call it the unprimed frame), the spacing between two drops is \(\Delta x' = l\), the length of the stick.

An unprimed observer in the floor’s frame sees the drops placed

\[\Delta x = \gamma \left[ \Delta x' + \gamma \Delta t' \right] = \gamma l \text{ apart}\]

The drops are not placed simultaneously,

\[\Delta t = \gamma \left( \Delta t' + \gamma \Delta x' \right) = \gamma \frac{l}{c^2}\]

If you want to think about length contraction,

the dots are spaced \(\Delta x = \frac{l}{\gamma} + \gamma \Delta t\) apart,

\[\Delta x = \frac{l}{\gamma} + \gamma \gamma \frac{v^2}{c^2} l = \gamma l \left[ \frac{1}{\gamma} + \frac{v^2}{c^2} \right] = \gamma l\]

i.e., lines on floor are spaced \(\gamma l\) apart, dots are \(\gamma l\) apart.

To an observer on the red, the dots are spaced a distance \(l\), but the spacing between lines is length-contracted to \(\frac{l}{\gamma}\). In either frame,

\[
\frac{\text{spacing of dots}}{\text{spacing of lines}} = \gamma = \frac{\gamma l}{\frac{l}{\gamma}} = \frac{l}{c} \left[ \frac{c}{l} \right]
\]
In the rest frame of the rod, the particle has velocity \( u \) and goes a distance \( xL = uT \).

The photon goes \( l + (1-x)l = (2-x)l \).

So,
\[
T = \frac{xL}{u} = \frac{(2-x)l}{c}
\]

or
\[
2-x = \frac{x}{c/u} \Rightarrow 2z = \frac{c+x}{uc} \Rightarrow x = \frac{2u}{c+u}
\]

Now go to a frame where the rod has velocity \( v \).

In this frame the particle has velocity \( w \). The motion has 3 parts:

1) In the first part, the photon reaches mirror (which has moved to the right a distance \( v t_1 \)).

2) In the second part, the photon goes distance \( l_2 = c t_2 \).

Switch is still moving - end of rod now \( vt_2 \) further along.

\[
x + l_1 - vt_2 = \frac{x' + l}{c-v} \quad \text{and} \quad \frac{c}{c-v} - vt_2 = \frac{l_2}{c-v}
\]

\[
l_2 = \frac{e'(1-x)-v}{c+u}
\]

\[
t_2 = \frac{e'(1-x)}{c+u}
\]
A better way - waist to chest. Photon is at $x = t'$

For trajectory in $l' = ct_2$, no time left.

In the $t_2$ back of which moves $v t_2$

Meeting occurs at length $x l'$ along after a

in space, $v t_2 + x l' = l' - ct_2$

\[
\begin{align*}
\text{(c+v)} t_2 &= l' (1-x) \\
\frac{t_2}{(c+V)} &= \frac{l' (1-x)}{(c+V)}
\end{align*}
\]
2) Ellipsoid time is \( t_1 + t_2 \). Particle moves to location \( x' \) measured from left end of rod, but this is a distance \( x' + V(t_1 + t_2) \) from the starting point. Particle has velocity \( W \) so

\[
W \left[ t_1 + t_2 \right] = x' \; \frac{e'}{W - V} + V \left[ t_1 + t_2 \right]
\]

so \( t_1 + t_2 = \frac{x' e'}{W - V} = \frac{e'}{C - V} + \frac{e' (1 - x)}{C + V} \)

\[
\frac{x}{W - V} = \frac{1}{C - V} + \frac{1 - x}{C + V} = \frac{c(2 - x) - v x}{c^2 - v^2}
\]

\[
W - V = \frac{(c^2 - v^2)x}{(2 - x) c + x v} \Rightarrow W = \frac{(2 - x)c v + x v^2 + (c^2 - v^2)x}{(2 - x)c + x v}
\]

\[
W = \frac{(2 - x)c v + x c^2}{(2 - x)c + x v}
\]

Now we insert \( x = \frac{2u}{c + u} \) so \( 2 - x = \frac{2 - 2u}{c + u} = \frac{2c}{c + u} \)

\[
W = \frac{2c^2 v + 2uc^2}{2c^2 + 2ucv} = \frac{v + u}{1 + \frac{uv}{c^2}} \]

Again the only assumption we made were that \( x \) and \( c \) were invariants.