1) [15 points] Consider a two-state system with Hamiltonian

\[ H = \begin{pmatrix} \epsilon_1 & \lambda \Delta \\ \lambda \Delta & \epsilon_2 \end{pmatrix} \]  

As you did in set 1, first find the energy eigenvalues and eigenfunctions exactly. Then, assume that the system is almost degenerate, that \( \epsilon_2 - \epsilon_1 \equiv \epsilon \ll \lambda \Delta \). Show that the exact result you just found is close to the perturbative answer you would have, but using degenerate state perturbation theory, namely treating

\[ H_0 = \begin{pmatrix} \epsilon_1 & \lambda \Delta \\ \lambda \Delta & \epsilon_2 \end{pmatrix} \]  

as the zeroth order term and

\[ H_1 = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_2 - \epsilon_1 \end{pmatrix} \],

as the perturbation. Solve for both energies and states. This is another example about the use of degenerate-state perturbation theory.

2) [10 points] Find the eigen-energies of

\[ H = \begin{pmatrix} E_1 & b & 0 \\ b & E_1 & c \\ 0 & c & E_2 \end{pmatrix} \]  

through second order in the small parameters \( b, c \), using degenerate state perturbation theory.

3) [35 points] Consider the Hamiltonian for a quantum mechanical pendulum,

\[ H = -\frac{\hbar^2}{2ml^2} \frac{\partial^2}{\partial \theta^2} + mgl(1 - \cos \theta), \]

in perturbation theory. (a) [15 points] For \( g \) large, find the energies of all the states of the pendulum assuming that the problem is mostly a harmonic oscillator, and the \( \theta^4 \) term in the expansion of the cosine is a perturbation. (It’s an exponentially small error to extend the limits of \( \theta \) to \( \pm \infty \).) (b) [15 points] For \( g \) small, treat the whole potential as a perturbation and calculate the energies of all the states of the system to lowest nontrivial order in the potential. (c) [5 points] From your answers to (a) and (b), describe more carefully what “\( g \) large” and “\( g \) small” mean (i.e., what is the dimensionless small parameter describing the perturbative expansions)?