Problem 1

[15=5+5+5 pts] Bogoliubov transformations: Generalize the scalar product we introduced in class to an invariant form

\[ (\phi_1, \phi_2) = -i \int \phi_1 \bar{\phi}_2 n^\mu d\Sigma \]  

(1)

where \( n^\mu \) is a future directed time-like unit vector orthogonal to the space-like (3d) surface \( \Sigma \) (so if \( n^\mu = (1, 0, 0, 0) \) \( d\sigma \) is \( d^3x \) the volume of the fixed time surface in the usual coordinates for Minkowski space. The above is valid for an arbitrary coordinate system. As in the lectures we introduce a complete set of (plane wave) solutions (a basis) of the Klein-Gordon equation satisfying

\[ (u_i, u_j) = \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i, u_j^*) = 0. \]  

(2)

A general solution is then \( \phi(x) = \sum_i (a_i u_i(x) + a_i^* u_i^*(x)) \) with the operator algebra \([a_i, a_j^*] = \delta_{ij}\) etc. The vacuum state relative to this mode expansion is

\[ |0>, \text{ such that } a_i|0> = 0 \forall i. \]  

(3)

Now consider a second complete orthonormal (in the above scalar product) set of modes \( \bar{u}_j(x) \) and expand the general solution in this basis as

\[ \phi(x) = \sum_i (\bar{a}_i \bar{u}_i(x) + \bar{a}_i^* \bar{u}_i^*(x)). \]

Correspondingly we can define a new vacuum

\[ |\bar{0}>, \text{ such that } \bar{a}_i |\bar{0}> = 0 \forall i. \]  

(4)

Show the following: a) 

\[ a_i = \sum_j \left( \alpha_{ij} \bar{a}_j + \beta_{ij}^* \bar{a}_j^* \right) \]

\[ \bar{a}_j = \sum_i \left( \alpha_{ji}^* a_i - \beta_{ji} a_i^* \right) \]

\( \alpha \)'s, \( \beta \)'s are called Bogoliubov coefficients - what are they in terms of the scalar products of the \( u \)'s and \( \bar{u} \)'s? b) The Bogoliubov coefficients satisfy the relations

\[ \sum_k (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) = \delta_{ij} \]

\[ \sum_k (\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}) = 0 \]
c) The vacuum of the barred modes (4) contains on average non-zero particle number of particles of the unbarred modes. i.e. if $N_i = a_i^\dagger a_i$ is the number operator for the $u$ modes show that

$$<\bar{0}|N_i|\bar{0}> = \sum_j |\beta_{ji}|^2.$$ 

These results have many uses in condensed matter physics and in QFT in curved backgrounds. In fact the famous result of Hawking on black hole radiation depends on formulae such as the last equation.

**Problem 2**

[10=3+4+3 pts] Consider the field theory of a complex scalar field with action

$$S = \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi).$$

a) Find the conjugate momenta and the Hamiltonian. b) Express the Hamiltonian in terms of creation and annihilation operators and show that the theory contains two sets of particles of mass $m$. c) This action has a $U(1)$ symmetry. Construct the corresponding Noether charge and show that the two types of particles have opposite charges.

**Problem 3**

Casimir effect in 1+1 dimensions. Consider a one dimensional box of length $L$. The ends of the box can be thought of as point like plates at which the field vanishes (take the field theory describing the physics to be a free massless scalar theory). Now insert a third point like plate inside the box at a distance $d$ from one end. a) [7 pts] Show that the zero point energy of the system is

$$E = f(d) + f(L - d)$$

where

$$f(d) = \frac{\pi}{2d} \sum_{n=1}^{n=\infty} n.$$ 

b) [7 pts] This expression is of course infinite but that is because we are summing up to arbitrarily high frequencies. In reality the plates are of finite thickness and one would expect the frequencies to be cut off for $\omega_n = k_n >> a^{-1}$ where $a$ is the thickness of the plate. Thus a regularized expression for $f$ is

$$f(d) = \frac{\pi}{2d} \sum_{n=1}^{n=\infty} ne^{-\frac{na}{d}}.$$ 

Show that this is equal to

$$\frac{\pi}{2d} \frac{e^{a \pi/d}}{(e^{a \pi/d} - 1)^2}$$

For $a/d << 1$ expand in powers of $a$ to get

$$f(d) = \frac{d}{2 \pi a^2} - \frac{\pi}{24d} + O(a^2)$$
(Hint: You will need to expand the numerator to $O(a^2)$ and the expression in parenthesis in the denominator to $O(a^3)$)

c) [7 pts] Show that in the limit $a \to 0$ the force on the middle plate is

$$F = \frac{\pi}{24} \left( \frac{1}{d^2} - \frac{1}{(L - d)^2} \right) \approx \frac{\pi}{24d^2}$$

where the last expression is valid for $d \ll L$.

d) [4 pts] How much is this force in Newtons if $d = 1mm$?

This problem is a simplified version of a real effect. It has in fact been measured even though the effect is tiny. It shows that zero point energy has physical significance!
\[ \varphi(x) = \sum_j \left[ \alpha_j u_j(x) + \alpha_j^* u_j^*(x) \right] \]  
\[ (u_j, u_j^*) = \delta_{ij}, \quad (u_j^*, u_j^*) = -\delta_{ij}, \quad (u_j, u_j^*) = 0 \]  
\[ A_{ij} = \exp^{iA_{ij}} \]  
\[ \varphi(x) = \sum_j \left[ \alpha_j u_j(x) + \alpha_j^* u_j^*(x) \right] \]  
\[ (\overline{u}_j, u_j) = \delta_{ij}, \quad (\overline{u}_j^*, u_j^*) = -\delta_{ij}, \quad (\overline{u}_j, u_j^*) = 0 \]  
\[ a_i = (u_i, \varphi), \quad a_i^* = -(u_i^*, \varphi) \]  
\[ \varphi(x) = \sum_i u_i (u_i, \varphi) \rightarrow u_i^* (u_i^*, \varphi) \]  
\[ \overline{u}_j = \frac{1}{2} (\alpha_j u_j + \beta_j u_j^*) \]  
\[ \alpha_{ij} = (\overline{u}_i, u_j), \quad \beta_{ij} = - (\overline{u}_i, u_j^*) \]  
\[ \overline{u}_j^* = \frac{1}{2} (\alpha_{ij}^* u_j^* + \beta_{ij}^* u_j) \]  
\[ (\overline{u}_i, u_i) (u_i, \overline{u}_i^*) = (\overline{u}_i, u_i^*) (u_i, \overline{u}_i) = \delta_{ii} \]  
\[ = \beta_{ii} \]  
\[ \sum_i (\alpha_{ki} \alpha_{li}^* + \beta_{ki} \beta_{li}^*) = \delta_{kl} \]  
\[ \text{and similarly} \quad \sum_i (\alpha_{ki} \beta_{li} + \beta_{ki} \alpha_{li}) = 0 \]
Inner product:

\[
(\varphi_1, \varphi_2) = -2 \int \left\{ \varphi_1(\omega) \partial_\mu \varphi_2^*(\omega) - \partial_\mu (\varphi_1(\omega)) \varphi_2^*(\omega) \right\} d\omega
\]

\[
(\varphi_1^*, \varphi_2^*) = +2 \int \left\{ \varphi_1^*(\omega) \partial_\mu \varphi_2(\omega) - \partial_\mu (\varphi_1^*(\omega)) \varphi_2(\omega) \right\} d\omega
\]

\[
= - (\varphi_1, \varphi_2^*)
\]

\[
= (\varphi_2, \varphi_1)
\]
Similarly, starting from completeness in 
\[ \overline{u}_i \] 
\[ \sum_j \overline{u}_j (\overline{u}_j^*, 0) - \overline{u}_i (\overline{u}_i^*, 0) = 0 \]
we have:
\[ \frac{N_i}{\sqrt{2}} (\alpha_{ik} \overline{u}_j^* - \beta_{ik} \overline{u}_j) = S_{ik} \]
\[ \frac{N_i}{\sqrt{2}} (\alpha_{ik} \overline{u}_j^* - \beta_{ik} \overline{u}_j) = 0 \]  

Hence we also have inner relations:
\[ N_i = \sum_j (\alpha_{ij}^* \overline{u}_j^* - \beta_{ij} \overline{u}_j) \]  

\[ c) \quad \langle \overline{a}_i | \overline{a}_i \rangle = 0 \quad \forall \ i \quad \text{(10)} \]
\[ N_i = a_i^* a_i = \sum_j (\alpha_{ij}^* \overline{a}_j^* + \beta_{ij} \overline{a}_j) \sum_j \alpha_{ij} \overline{a}_j + \beta_{ij} \overline{a}_j^* \overline{a}_j \]

Using (10) and its c.c. give the result:
\[ \langle \overline{c}_i | N_i | \overline{a}_i \rangle = \sum_{j,k} \beta_{ij} \beta_{ik}^* \langle \overline{c}_i | \overline{a}_j^* \overline{a}_k \rangle \]
\[ = \sum_j \beta_{ij} \beta_{ij}^* = \frac{1}{\sqrt{2}} \beta_i \beta_i^* \]

Since \[ \langle \overline{c}_i | \overline{a}_j^* \overline{a}_j^* \rangle = \langle \overline{c}_i | [\overline{a}_j^* \overline{a}_j^+] \rangle = \delta_{ik} \]
HW III

(Solution to I incorporated below)

\[ S = \int d^4x \left( \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi \phi^* \right). \]

\[ L = \frac{1}{2} \phi^* \partial^\mu \phi - m^2 \phi \phi^*. \]

\[ \Pi = \frac{\partial S}{\partial \phi}, \quad \Pi^* = \phi. \quad \text{Canonical equal time commutation relations.} \]

\[ [\phi(x), \Pi(y)]_{x=x', y^0} = [\phi(x), \phi^*(y)]_{x=x', y^0} = i \delta^3(x^\mu - y^\mu). \]

\[ [\phi(x), \Pi^*(y)]_{x=x', y^0} = 0. \]

\[ [\phi(x), \phi(y)]_{y^0} = [\Pi(x), \Pi(y)]_{y^0} = 0 \]

Similar \textit{hermitian} conjugates.

\textit{NB.} In quantum they \( \phi^* \rightarrow \phi^+ \) etc.

\textit{Hamiltonian density.}

\[ H = \Pi \phi \phi^* - L. \]

\[ = \Pi \pi^* + \pi^* \pi - \left( \pi^* \pi - (\nabla \phi)^2 - m^2 \phi^2 \right) \]

\[ = \left( \Pi^2 + \nabla \phi^2 + m^2 \phi^2 \right). \]

\[ H = \int d^3x \, H. \]
Heisenberg equations

\[
\frac{\partial \varphi}{\partial t} = \left[ \varphi, H \right] = \int dy \left\{ \varphi(x, y), \frac{1}{\sqrt{2\pi}} \int \frac{d^2 \varphi}{(2\pi)^2} \left( \frac{1}{\sqrt{2\pi}} \int \frac{d^2 \varphi}{(2\pi)^2} \left[ \varphi(x, y), \varphi^*(x', y') \right] \right) \right\}
\]

\[
= \int_{x_0}^{x_0} dy \left\{ \varphi(x, y), \varphi^*(x, y) \right\}
\]

\[
= \left( \frac{\partial^3}{\partial x^3} \varphi(x_0 - \frac{1}{2}) \varphi^*(x) \right) = \xi \varphi^*(x)
\]

\[
\frac{d \varphi^*}{dt} = \int dy \left\{ \varphi^*(x), \frac{\partial}{\partial y} \varphi + \left[ \varphi^*, \varphi \right] \right\}
\]

\[
= \int dy \left\{ \frac{\partial}{\partial y} \left( \frac{\partial^3}{\partial x^3} \varphi(x - y) \right), \frac{\partial}{\partial y} \varphi + \left( -i \frac{\partial^3}{\partial x^3} \varphi(x - y) \right) \right\}
\]

\[
= i \left( \frac{\partial^2}{\partial x^2} - m^2 \right) \varphi(x)
\]

\[
\partial_t^2 \varphi = \partial_t \varphi^* = \left( \frac{\partial^2}{\partial x^2} - m^2 \right) \varphi
\]

\[
\rightarrow \left( \partial_t^2 - \frac{\partial^2}{\partial x^2} + m^2 \right) = (\Box + m^2) \varphi = 0
\]

\[- k = \omega \varphi \]
\[ P^2 = \int d^3x : \phi^* \partial^2 \phi + \partial \phi \partial^* \phi^* : \]
\[ = \int d^3k \kappa^2 \left[ b^*_k b_k + c^*_k c_k \right] . \]

\[ P^0 = \int d^3x : \left( \frac{i}{2} \phi \phi^* + \frac{i}{2} \partial \phi \partial \phi^* + \frac{1}{2} m^2 \phi \phi^* \right) ; \]

\[ \phi = \int d^3k \left[ b_k (-i\omega_k) u^*_k + c^*_k (i\omega_k) u_k \right] \]
\[ \phi^* = \int d^3k' \left[ b^*_k (-i\omega_k) u^*_k + c^*_k (i\omega_k) u_k \right] \]
\[ \partial \phi = \int d^3k \left[ b_k i \kappa u^*_k + c^*_k (-i\kappa) u_k \right] \]
\[ \partial \phi^* = \int d^3k' \left[ b^*_k (i\kappa) u^*_k + c^*_k (-i\kappa) u_k \right] \]

\[ \int \frac{d^3k d^3k'}{2\omega_k \omega_k'} \left[ b_k c^*_k \left( \omega_k \omega_k' - \kappa \cdot \kappa' + m^2 \right) \right] \times \delta^3(k-k') \]
\[ + b^*_k b_k \left[ + \omega_k \omega_k' + \kappa \cdot \kappa' + m^2 \right] \delta^3(k-k') \]
\[ + c^*_k c_k \left[ - \omega_k \omega_k' + \kappa \cdot \kappa' + m^2 \right] \delta^3(k-k') \]
\[ + c^*_k b^*_k \left[ - \omega_k \omega_k' + \kappa \cdot \kappa' + m^2 \right] \delta^3(k-k') \]

Integrating over \( k' \) and using \( \omega^2_k = \kappa^2 + m^2 \)

\[ P^0 = \int d^3k \omega_k \left[ b^*_k b_k + c^*_k c_k \right] \]

\[ T_{ab} = \int d^3k \left[ N_b(k) + N_c(k) \right] \]
\[ N_b = b^*_k b_k , \quad N_c = c^*_k c_k . \]

\[ k^0 = \omega_k \]
\( P^n = \int d^3x \left[ (\phi^* x^4) - n^0 x^1 \right] \)

\[ P^0 = \int d^3x \left[ \phi^* x^4 + \phi x^4 - \eta^0 x^1 \right] \]

\[ \phi = \nabla \int d^3x \left[ bk u^k(x) + c^*_m u^m(x) \right] \]

\[ \phi^* = \int d^3x \left[ c^*_m u^m(x) + b^*_m u^m(x) \right] \]

\[ n^+(x) = \frac{1}{(2\pi)^2} \sqrt{2\omega_n} e^{+i\omega_n x} \]

\[ \left( \phi^* x^4 \phi \right) = \int d^3x \left[ c^*_m b^*_n u^m u^k \delta^{(4)}(x) + c^*_m c^*_n u^m u^k \delta^{(4)}(x) \right. \]

\[ \left. + b^*_m b^*_n u^m u^k \delta^{(4)}(x) + b^*_m c^*_n u^m u^k \delta^{(4)}(x) \right] \]

\[ \int d^3x \phi^* \phi = \int d^3x \frac{1}{(2\pi)^2} \sqrt{2\omega_n} \left[ c^*_m b^*_n (\omega_n) s^3(h+n) \right] \]

\[ = \int d^3x \left[ c^*_m b^*_n (\omega_n) s^3(h+k') \delta^{(4)}(x) \right] \]

\[ \left( \phi^* x^4 \phi \right) = \int d^3x \frac{1}{2} \left[ c^*_m b^*_n \delta^{(4)}(x) + c^*_m c^*_n k^2 + b^*_m b^*_n \right. \]

\[ \left. + b^*_m c^*_n k^2 \right] \]

\[ \int d^3x \phi^* \phi = \int d^3x \frac{1}{2} \left[ b^*_m c^*_n (-h^2) + c^*_m c^*_n k^2 + b^*_m b^*_n \right. \]

\[ \left. + b^*_m c^*_n k^2 \right] \]
Casimir effect in 1+1 Dimensions

Massless scalar satisfies $\Box \phi = 0$.

Boundary condition in I: $\phi(0) = c \phi(d) = 0$.

Solution: $\phi = A \sin \frac{\pi x}{d}$.

in I $\phi(d) = \phi(0) = 0$ $\phi = A \sin \frac{\pi x}{d}$

in II $\phi(d) = \phi(0) = 0$ $\phi = A \sin \frac{\pi x}{d}$

Total zero-point energy $E = \frac{4}{d} \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{2 n^2 \pi^2}$

This is infinite. Take thickness of plate into account and cutoff frequencies.

$c_{n} \gg \frac{1}{d}$

Regularized version:

$\frac{4}{d} \sum_{n=1}^{\infty} n e^{-\frac{\pi n^2}{d}} = -\frac{1}{2} \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\pi n^2}}{n^2}$

$= -\frac{1}{2} \frac{2}{\pi} \frac{1}{1 - e^{-\pi^2/d}} = \frac{\pi}{2d} \left( \frac{e^{-\pi^2/d}}{1 - e^{-\pi^2/d}} \right)$

$= \frac{\pi}{2d} \left( \frac{e^{\pi^2/d}}{(1 - e^{-\pi^2/d})^2} \right)$
\[
\begin{align*}
\left[ N_b^+ b_{n}^+ \right] &= \left[ b_{n}^+, b_{n}', b_{n}'' \right] = b_{n}^+ \left[ b_{n}', b_{n}'' \right] \\
&= b_{n}^+ s^3 (\lambda - \lambda').
\end{align*}
\]

Similarly,
\[
\left[ N_b^+ c_{n}^+ \right] = c_{n}^+ s^3 (\lambda - \lambda').
\]

Also,
\[
N_b^+ 10 >= N_c^+ 10 >= 0 \Rightarrow P^m 10 >= 0.
\]

\[
\begin{align*}
\left[ P^m, b_{n}^+ \right] &= \frac{4\pi}{2k} \int d^3x \sum_{\ell} \left( 2 \mu (N_m^0, b_{n}^+) + N_b^+ \right) b_{n}^+ \\
&= k_m b_{n}^+ \left[ P^m, c_{n}^+ \right] = k_m c_{n}^+.
\end{align*}
\]

\[
P^m (b_{n}^+, c_{n}^+) 10 >= \left[ P^m, b_{n}^+ \right] 10 >= k_m b_{n}^+ 10 >= 0.
\]

There are two types of particles, each of mass \( m \).

**c)**
\[
\Omega = \int d^3x \left[ \left( \phi \phi^+ - \phi^+ \phi \right) \right] (\pi)
\]

\[
= \int d^3x \left[ \sum b_n c_n \left( -i\omega_n + i\omega_n \right) \right] + b_{n}^+ b_{n} \left( i\omega_n + 2i\omega_n \right) s^3 (\lambda - \lambda')
\]

\[
+ c_{n}^+ c_{n} \left( i\omega_n + i\omega_n \right) s^3 (\lambda - \lambda')
\]

\[
+ b_{n}^+ c_{n}^+ \left( i\omega_n - i\omega_n \right) s^3 (\lambda - \lambda')
\]

\[
= \int \frac{d^3k}{(2\pi)^3} \left[ c_{n}^+ c_{n} - b_{n}^+ b_{n} \right]
\]

\[
\left[ \Omega, c_{n}^+ \right] = 4\pi \leftrightarrow c_{n}^+ , \left[ \Omega, b_{n}^+ \right] = -2 b_{n}^+ , \Omega 10 >= 2
\]

Two types have equal and opposite charge.
Expand in power of $a/d$

\[-\frac{1}{d^2} \cdot (1 + \frac{a \pi}{d^2} + \frac{1}{8} \cdot \frac{a^2 \pi^2}{d^4} + \cdots) \left( \frac{a \pi}{d^2} + \frac{a^2 \pi^2}{2d^4} + \frac{a^3 \pi^3}{6d^6} + \cdots \right) \]

\[= \frac{d}{2\pi a^2} \left( 1 + \frac{a \pi}{d^2} + \frac{a^2 \pi^2}{2d^4} + \cdots \right) \left( 1 - \frac{a \pi}{d^2} + \frac{a^2 \pi^2}{2d^4} + \cdots \right) \]

\[= \frac{d}{2\pi a^2} \left( 1 - \frac{a \pi}{d^2} + \frac{2a^2 \pi^2}{2d^4} - \frac{a^3 \pi^3}{6d^6} + \frac{2a^3 \pi^3}{4d^6} + \cdots \right) \]

\[= \frac{d}{2\pi a^2} \left( 1 + \left( \frac{1}{2} + \frac{3}{4} - 1 - \frac{1}{3} \right) \frac{a^2 \pi^2}{d^2} \right) \]

\[= \frac{d}{2\pi a^2} \left( 1 - \frac{1}{12} \frac{a^2 \pi^2}{d^2} + \cdots \right) \]

\[= \frac{d}{2\pi a^2} - \frac{\pi}{24d} + O(a^2) \]

\[\frac{\partial E}{\partial a} = \delta'(d) - \delta'(L-d) \]

\[= \frac{\sqrt{a}}{2\pi a^2} + \frac{\pi}{24d^2} - \left( \frac{1}{2\pi a^2} + \frac{\pi}{24(L-d)^2} \right) \]

\[= \frac{\pi}{24} \left( \frac{1}{d^2} - \frac{1}{(L-d)^2} \right) \approx \frac{\pi}{24d^2} \quad \text{as} \quad L \gg d \]

\[F = -\frac{\partial E}{\partial a} \approx -\frac{\pi}{24a^2} \approx -\frac{\pi}{24d^2} \quad \text{for} \quad a = \frac{L-2b}{12} \]

\[F = -\frac{3\pi}{24} \times \frac{2.10^{-10} \text{N}}{11^3} \quad \text{N} \times 10^{21} \text{N} \quad \text{Attractive force.} \quad \text{N}^\circ \text{B F=20 on B=30} \]