Reading assignment: Jackson chap 2
J # refers to a problem number from Jackson.

Problems:

1. [10 pts] A conductor at potential $\Phi = 0$ has the shape of an infinite plane except for a hemispherical bulge of radius $a$. A charge $q$ is placed above the center of the bulge, a distance $p$ from the plane (or $p - a$ from the top of the bulge). What is the force on the charge?

2. [10 pts] J 2.5

3. [10 pts] J 2.7

4. [20 pts] a) A rectangular box has sides of length $a, b, c$ along the $x, y, z$ axes respectively. The Green's function for the Dirichlet (D) problem may be expanded as

$$G(x, x') = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{mn}(z, z') \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b}.$$ 

Find the differential equation that $g_{mn}(z, z')$ must satisfy. b) Solve this equation subject to D boundary conditions and write down the result for $G(x, x')$. c) Find the potential $\Phi$ inside the box when $\Phi(x, y, c) = V(x, y)$ and $\Phi = 0$ on the other five sides using the Green's function above.

1
Image change method: Replace conductor with configuration of image charges.

charge $-q'$ with $q' = \frac{2a}{p}$ at $p' = \frac{a}{p}$.

Leaves hemisphere at $\theta = 0$.

Introduce 2 more image charges $q'$ at $z = p'$ and $-p$ at $z = -p$.

These do not change potential on hemisphere but now have $\vec{E} = 0$ on the infinite plane as well.

So total force due to conductivity Sarlak = force due to image charges.

$$F = \frac{-q q'}{(p-p')^2} + \frac{q q'}{(p+p')^2} - \frac{q^2}{(2p)^2}$$

$$= -q^2 \left( \frac{ab}{(p^2-a^2)^2} - \frac{ab}{(p^2+a^2)^2} + \frac{1}{4p^2} \right).$$
\[ W = \int 0^a F \, dy = \frac{q^2 \alpha}{4 \pi \varepsilon_0 a^2} \left[ 1 - \frac{\alpha}{a} \right] \ln \left( \frac{a}{\alpha} \right) \]

\[ = \frac{q^2 \alpha}{4 \pi \varepsilon_0 a^2} \left[ \left(1 - \frac{\alpha}{a} \right)^{1/2} \right] \ln \left( \frac{a}{\alpha} \right) \]

\[ = \frac{q^2 \alpha}{8 \pi \varepsilon_0 a} \left( \frac{1}{a^2} \right) \ln \left( \frac{a}{\alpha} \right) \]

\[ \quad \text{Eqn 2.2: } \mathbf{E}(x) = \frac{1}{4 \pi \varepsilon_0} \left[ \frac{q}{|x - z|} + \frac{q'}{|x - z'|} \right] \]

\[ \quad \text{with } q' = \frac{q}{r^2} \]

\[ \quad \text{and } z' = \frac{z}{r^2} \]

\[ \text{sect 1.11: } W' = \frac{1}{2} \int \mathbf{E}(x) \cdot \mathbf{E}(x) \, dx \]

\[ = \frac{1}{2} q \frac{q'}{4 \pi \varepsilon_0} \left[ \frac{1}{|x - z'|^2} \right] \]

\[ = - \frac{1}{8 \pi \varepsilon_0 \frac{q^2 \alpha}{a^2}} = - W \quad \text{(a)} \]

2.5b) Calculate work done against far charge sphere and image charge - i.e., \( q \) at \( y > 0 \) and \( y' \) at \( y' > 0 \) and

\[ F = \frac{q^2 \alpha}{4 \pi \varepsilon_0 \left( y^2 + (y - y')^2 \right)} \]

\[ \text{set p61} \]
\[- \frac{1}{4\pi\varepsilon_0} \left[ \int_{r_1}^{r_0} \frac{2Q}{y} dy + \frac{g^2 a}{2y^2} \right] \]

\[-\frac{1}{4\pi\varepsilon_0} \left[ \frac{2Q}{y} \bigg|_1^0 - \frac{g^2 a}{2y^2} \bigg|_1^0 \right] \]

\[-\frac{1}{4\pi\varepsilon_0} \left( \frac{2Q}{1} - \frac{g^2 a}{2r^2} + W \right) \]

\[-\frac{1}{4\pi\varepsilon_0} \left( 2Q - \frac{g^2 a}{2r^2} + W \right) \]
a) \[ \Phi(x,\beta) = \frac{1}{1-x \cdot \beta} - \frac{1}{1-x \cdot \beta} \]

\[ K = \frac{1}{1-x \cdot \beta} \]

\[ x' = (x', y' = z) \]

\[ (x-x')^2 = R^2 + (z-z')^2 \]

\[ |x-x'| = R \sqrt{z^2 + 2z' \cos \varphi + (\cos^2 \varphi)} \]

\[ g = \frac{1}{\sqrt{R^2 + (z-z')^2}} \]

\[ R^2 = S^2 + S'^2 - 2 S S' \cos \varphi' \]

\[ (x, \varphi = 0) \]

\[ b) \quad \Delta(x) = -\frac{1}{4\pi} \int \frac{\Phi(x') \frac{\partial \delta}{\partial \alpha}}{\frac{\partial \delta}{\partial \alpha}} \, dx' \]

\[ \frac{\partial \delta}{\partial \alpha} = \frac{\partial \delta}{\partial z'} \]

\[ \frac{\partial \delta}{\partial \alpha} = \frac{V}{4\pi} \int_0^s \int_0^a f dz' \frac{2z}{(S^2 + z^2)^{3/2}} \]

\[ \Phi(0, \beta) = \frac{V}{4\pi^2} \int_0^s \frac{s'}{2z} \frac{dS'}{(S^2 + z^2)^{3/2}} \]

\[ = V \left[ 1 - \frac{\pi}{2\sqrt{a^2}} \right] \]

\[ d) \quad \frac{\partial G}{\partial \alpha} = -\frac{\frac{\partial \delta}{\partial \alpha}}{\frac{\partial \delta}{\partial \alpha}} = \frac{1}{2z} \frac{1}{(S^2 + 2s^2 \cos \varphi + z^2)^{3/2}} \]

\[ \frac{1}{2z} \frac{1}{(S^2 + 2s^2 \cos \varphi + z^2)^{3/2}} \]

\[ = \frac{2z}{(S^2 + z^2)^{3/2}} \left[ 1 - \frac{2}{S^2 + z^2} \left( 1 - \frac{2S^2 \cos \varphi}{S^2 + z^2} \right) \right]^{-3/2} \]

\[ = \frac{2z}{(S^2 + z^2)^{3/2}} \left[ 1 - \frac{2}{S^2 + z^2} \left( 1 - \frac{2S^2 \cos \varphi}{S^2 + z^2} \right) \right]^{-3/2} \]

\[ + \frac{15}{8} \left( \frac{S^2}{S^2 + z^2} \right)^2 \left( 1 - \frac{2S^2 \cos \varphi}{S^2 + z^2} \right) \]

So

\[ \text{Use} \quad \int \cos \varphi = 0 \quad \text{or} \quad \langle \cos \varphi \rangle = \frac{1}{2} \]
\[ \Phi(z, 0) = V \left( \frac{a}{2} \right)^2 \left[ 1 - \frac{3}{2} \frac{a^2}{2^2} + \frac{5}{8} \left( \frac{a}{2} \right)^4 + \cdots \right] \]

\[ V(1 - \frac{1}{\sqrt{1 + b^2}}) = V \left[ \frac{1}{2} \left( \frac{a}{2} \right)^2 - \frac{3}{8} \left( \frac{a}{2} \right)^4 + \frac{5}{16} \left( \frac{a}{2} \right)^6 + \cdots \right] \]
A rectangular box has sides of lengths $a$, $b$ and $c$.

a) For the Dirichlet problem in the interior of the box, the Green's function may be expanded as

$$G(x, y; z, z') = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{mn}(z, z') \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b}$$

Write down the appropriate differential equation that $g_{mn}(z, z')$ must satisfy.

Note that $\sin kx$ satisfies the completeness relation

$$\sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} = \frac{a}{2} \delta(x - x')$$

Hence the Green's function equation

$$\nabla^2_x G(x, x') = -4\pi \delta(x - x')$$

has an expansion

$$\sum_{m,n} \nabla^2_x \left( g_{mn}(z, z') \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b} \right)$$

$$= -4\pi \delta(z - z') \frac{4}{ab} \sum_{m,n} \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b}$$

Working out the $x'$ and $y'$ derivatives on the left-hand side yields

$$\sum_{m,n} \left[ \frac{d^2}{dz'^2} \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2 \right] g_{mn}(z, z') \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b}$$

$$= -\frac{16\pi}{ab} \delta(z - z') \sum_{m,n} \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b}$$
However, since $\sin kx$ forms an orthogonal basis, each term in this sum must vanish by itself. This results in the differential equation

$$
\left( \frac{d^2}{dz'^2} - \gamma_{mn}^2 \right) g_{mn}(z, z') = -\frac{16\pi}{ab} \delta(z - z')
$$

where $\gamma_{mn} = \pi \sqrt{(m/a)^2 + (n/b)^2}$ is given in part c). Note that the Fourier sine expansion automatically satisfies Dirichlet boundary conditions for $x$ and $y$. The remaining boundary condition is that $g_{mn}(z, z')$ vanishes whenever $z$ or $z'$ is equal to 0 or $c$.

b) Solve the Green's function equation for $g_{mn}(z, z')$ subject to Dirichlet boundary conditions and write down the result for $G(x, y; x', y', z')$.

We may solve the Green's function equation (1) by first noting that the homogeneous equation is of the form

$$
g''(z') - \gamma_{mn}^2 g(z') = 0
$$

This is a second-order linear equation with constant coefficients admitting the familiar solution

$$
g(z') = A e^{\gamma_{mn} z'} + B e^{-\gamma_{mn} z'}
$$

However, we want $g(z') = 0$ when $z' = 0$ or $z' = c$. This motivates us to write out the solutions

$$
u(z') = \sinh \gamma_{mn} z' \quad \text{for} \quad 0 < z' < z
$$

$$
v(z') = \sinh[\gamma_{mn}(c - z)] \quad \text{for} \quad z < z' < c
$$

The Green's function solution is then given by

$$
g_{mn}(z, z') = \begin{cases} 
Au(z') & z' < z \\
Bv(z') & z' > z
\end{cases}
$$

The matching conditions

$$
g_\leq = g_\geq, \quad \frac{d}{dz'} g_\leq = \frac{d}{dz'} g_\geq + \frac{16\pi}{ab}
$$

then give the system

$$
A \sinh \gamma_{mn} z = B \sinh[\gamma_{mn}(c - z)],
$$

$$
A \cosh \gamma_{mn} z = -B \cosh[\gamma_{mn}(c - z)] + \frac{16\pi}{ab \gamma_{mn}}
$$

which may be solved to yield

$$
A = \frac{16\pi \sinh[\gamma_{mn}(c - z)]}{ab \gamma_{mn} \sinh \gamma_{mn} c}
$$

$$
B = \frac{16\pi \sinh[\gamma_{mn}(c - z)] + \sinh \gamma_{mn} z \cosh[\gamma_{mn}(c - z)]}{ab \gamma_{mn} \sinh \gamma_{mn} c} = \frac{16\pi}{ab \gamma_{mn} \sinh \gamma_{mn} c} \nu(z)
$$
and
\[ B = \frac{16\pi \sinh \gamma_{mn} z}{ab\gamma_{mn} \left( \cosh \gamma_{mn} z \sinh [\gamma_{mn} (c - z)] + \sinh \gamma_{mn} z \cosh [\gamma_{mn} (c - z)] \right)} = \frac{16\pi \sinh \gamma_{mn} z}{ab\gamma_{mn} \sinh \gamma_{mn} c} = \frac{16\pi}{ab\gamma_{mn} \sinh \gamma_{mn} c} u(z) \]

As a result, the full Green's function solution is then given by
\[ g_{mn}(z, z') = \frac{16\pi}{ab\gamma_{mn} \sinh \gamma_{mn} c} \frac{u(z)}{\sinh \gamma_{mn} z} \sinh \gamma_{mn} [(c - z)] \]

Hence
\[ G(\vec{x}, \vec{x}') = \frac{16\pi}{ab} \sum_{m,n} \frac{1}{\gamma_{mn} \sinh \gamma_{mn} c} \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b} \times \sinh \gamma_{mn} z \sinh \gamma_{mn} [(c - z)] \]

(2)

\[ c) \text{ Consider the boundary value problem where the potential on top of the box is } \Phi(x, y, c) = V(x, y) \text{ while the potential on the other five sides vanish. Using the Greens' function obtained above, show that the potential may be written as} \]

\[ \Phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh \gamma_{mn} z \]

where \( \gamma_{mn} = \pi \sqrt{(m/a)^2 + (n/b)^2} \) and

\[ A_{mn} = \frac{4}{ab \sinh \gamma_{mn} c} \int_{0}^{a} dx \int_{0}^{b} dy V(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]

Since we only have to worry about the potential on the top of the box (and since we assume there is no charge inside the box), the Green's function solution may be written
\[ \Phi(\vec{x}) = \frac{1}{4\pi} \int_{z' = c} \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} da' \]

\[ = \frac{1}{4\pi} \int_{z' = c} V(x', y') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} d\vec{x}' \]
\[ (3) \]
Noting that the outward-pointing normal $\hat{n}'$ on the top of the box is in the $+z'$ direction, we compute the normal derivative of (2)

$$\frac{\partial G(\vec{x}, \vec{x}')}{\partial n'}|_{z'=c} = \frac{\partial G(\vec{x}, \vec{x}')}{\partial z'}|_{z'=c}$$

$$= \frac{16\pi}{ab} \sum_{m,n} \frac{1}{\gamma_{mn} \sinh \gamma_{mn} c} \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b}$$

$$\times \sinh \gamma_{mn} z \left( -\gamma_{mn} \cosh \gamma_{mn} (c - z') \right) |_{z'=c}$$

$$= -4\pi \sum_{m,n} \frac{4}{ab \sinh \gamma_{mn} c} \sinh \gamma_{mn} z$$

$$\times \sin \frac{m\pi x'}{a} \sin \frac{n\pi y'}{b}$$

Inserting this into (3) then straightforwardly gives the desired result. Note that the primes may be dropped from the double integral $A_{mn}$ once it has been isolated from the rest of the expression.