\[ A \]

Non-Abelian Gauge Theories,
\((\text{Yang-Mills})\).

\(U(1)\) gauge invce.

\[ \psi = \bar{\psi} \left( i \partial - m \right) \psi - \frac{i}{2} F_{\mu \nu} F^{\mu \nu}. \]

\[ D_\mu \psi = \left( \partial_\mu - i e A_\mu \right) \psi. \]

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

In variant under

\[ A_\mu \rightarrow A_\mu + \epsilon_\mu \partial_\nu \chi(x), \quad \psi \rightarrow \psi' = e^{i A_\mu(x)} \psi. \]

Note covariant derivative

transforms as

\[ D_\mu \psi(x) \rightarrow (D_\mu \psi)(x') = \epsilon^{+1} i \partial_\nu \psi(x') \]

i.e. covariantly.

Abelian gauge theory \(U(1)\) gb.

Note

\[ D_\mu D_\nu - D_\nu D_\mu = -i e F_{\mu \nu}. \]
Generalize to non-Abelian group, e.g., compact matrix group $U(1)$:

$$\psi(x) \rightarrow \psi'(x) = U(\theta) \psi(x).$$

$$= e^{i \theta \cdot J} \psi(x).$$

If $g = SU(n)$ and we put $\psi$ in the defining rep $\psi' w = \begin{pmatrix} \psi_1 \\ \cdots \\ \psi_n \end{pmatrix}$

$J$ are $n \times n$ traceless matrices.

with

$$[T_i, T_j] = i \epsilon_{ijk} T_k.$$  

Structure constant.

Normalize to $(T_i, T_j) = \frac{1}{2} \delta_{ij}$. In fundamental,

Construct Lagrangian invariant under local $SU(n)$.

i.e., $\theta(x)$ functions of space-time.
Need to introduce gauge field corresponding to each parameter.

i.e. \( A^i_\mu \), \( i = 1, \ldots, \text{dim } U \) .

Thus in SU(n), \( i = 1, \ldots, n^2 - 1 \).

Introduce the matrix valued field

\[
A_\mu = \sum_{i=1}^{n^2-1} T^i A^i_\mu(x)
\]

Define covariant derivative as in Abelian case,

\[
D_\mu \psi(x) = \left( i \partial_\mu - i g A^\mu(x) \right) \psi(x)
\]

Require covariance i.e. under \( \psi \rightarrow U(\theta) \psi \)

\[
D_\mu \psi(x) \rightarrow \left( D_\mu' \psi' \right)'(x) = U(\theta) D_\mu \psi(x).
\]

i.e. \( \left( i \partial_\mu - i g A^\mu_0(x) \right) U(\theta) \psi = U(\theta) \left( \partial_\mu - i g A^\mu_0(x) \right) \psi(x) \)

or \( U^{-1} \left( \partial_\mu - i g A^\mu_0(x) \right) U(\theta) \psi = \left( \partial_\mu - i g A^\mu_0(x) \right) \psi(x) \)

i.e. \( U^{-1} \partial_\mu U \psi + \partial_\mu \psi - i g U^{-1} A^\mu_0 U \psi = \partial_\mu \psi - i g A^\mu_0 \psi \)

or \( i g U^{-1} A^\mu_0 U = i g A^\mu + U^{-1} \partial_\mu U \).

i.e. \( A^\prime_\mu(x) = U A_\mu U^{-1} - i g U^{-1} \partial_\mu U U^{-1} \).
Note this reduces to $\text{tr} U^n$ on p. 1 for Abelian case.

**Infinitesimal transformations**

$$U(\Theta) = 1 + i \Theta \sigma^I, \quad U^{-1} = 1 - i \Theta \sigma^I.$$  

$$-\frac{i}{\hbar} \partial\mu \Sigma^\mu = \frac{1}{\hbar} \partial^\mu \Theta \sigma^I \Sigma^I$$

$$U \Sigma^\mu U^{-1} = \left( 1 + i \Theta \sigma^I \right) \sigma^\mu \left( 1 - i \Theta \sigma^I \right).$$

$$= A^\mu + i \left[ \Theta \sigma^I, A^\mu \right]$$

So $A^\mu(x) \rightarrow A^\mu'(x) = A^\mu(x) + i \left[ \Theta \sigma^I, A^\mu \right] - \frac{i}{\hbar} 2 \Theta \partial \xi^I$

or $A^\mu \rightarrow A^\mu' = A^\mu - c_{\nu}^{\ i} \Theta^\nu A^\nu + \frac{1}{\hbar} \partial \Theta^\nu.$

i.e. (for const. $\Theta$) $A^\mu$ transforms in the adjoint rep.
To define the field strength $F_{\mu \nu}$ we again take the covariant derivative, commutator

$$
[D_\mu, D_\nu] \psi = \left[ \partial_\mu - ig A_\mu, \partial_\nu - ig A_\nu \right] \psi
$$

$$
= \frac{i}{2} \left[ \partial_\mu, \partial_\nu \right] - ig \left[ A_\mu, \partial_\nu \right] - ig \left[ \partial_\mu, A_\nu \right] - g^2 \left[ A_\mu, A_\nu \right] \frac{\psi}{\gamma}.
$$

$$
\equiv -ig \vec{F}_{\mu \nu} \psi
$$

or

$$
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig A_\mu A_\nu.
$$

$$
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g c^2 \sin A_\mu A_\nu.
$$

Since

$$
D_\mu \psi = (D_\mu^U) \psi = U(\theta) D_\mu \psi.
$$

$$
([D_\mu, D_\nu] \psi)' = U(\theta) [D_\mu, D_\nu] \psi = i g \vec{U}(\theta) \vec{F}_{\mu \nu} \psi.
$$

or

$$
[D_\mu, D_\nu]' \psi(\theta) \psi = i g \vec{F}_{\mu \nu} \psi(\theta) \psi
$$

or

$$
\psi_{\theta}(\phi) \psi = \psi_{\theta}(\phi) \psi \vec{U}(\theta) \vec{F}_{\mu \nu} \psi.
$$

or

$$
F_{\mu \nu}(x) = \vec{U}(\theta) \vec{F}_{\mu \nu}(x) \vec{U}^{-1}(\theta).
$$

Or for an infinitesimal transformation $\theta$

$$
\vec{F}_{\mu \nu} = \vec{F}_{\mu \nu} + i \left[ \mathcal{B}(x), T_{\mu \nu} \right],
$$

$$
F_{\mu \nu} = \vec{F}_{\mu \nu} + i \left[ \mathcal{B}(x), T_{\mu \nu} \right].
$$

Or

$$
\vec{F}_{\mu \nu} \vec{T}^\psi = \vec{F}_{\mu \nu} \vec{T}^\psi + i \mathcal{E}^2 \left[ \frac{T_{\mu \nu}}{\gamma}, \frac{T_{\mu \nu}}{\gamma} \right] \vec{F}_{\mu \nu}.
$$
\[ F_{\mu \nu}^2 = F_{\mu \nu}^2 + i g \epsilon^{i j k} F_{\mu \nu}^k \]

\[ \text{ie. } F_{\mu \nu} \text{ (like } A \text{ for } u/0) \text{ transforms} \]

in adjoint representation (the transformation matrices represent the algebra i.e. the } c_{ijk} \]

\[ (T^3)^i \rightarrow c_{i m n} \]

\[
\text{From the Transforming law of the only invariant kinetic term (quadratic in derivatives) we can for } u \text{ is}.
\]

\[ \text{tr. } F_{\mu \nu} F^{\mu \nu} = \frac{1}{2} F_{\mu \nu}^2 \]

\[ \Rightarrow \frac{1}{2} \text{ if rep } 3 \]

One cannot write down a free Lagrangian with which is gauge invariant for a non-abelian gauge theory.

The Yang-Mills "photon" is charged unlike the abelian photon (the neutral). It belongs to the adjoint rep of group and local gauge invariance requires the Lag be constructed out of \( F_{\mu \nu} \) only.

In particular, no invariant mass term (a new formally not) for a field.
Pure YM theory.

\[ S = \int d^2x \left( -\frac{1}{4} F_{\mu \nu}^2 + F_{\mu \nu}^4 \right) \]

\[ F_{\mu \nu} = \partial_{\mu} A^\nu - \partial_{\nu} A^\mu + g \varepsilon^{\alpha \beta \mu \nu} A_\alpha A_\beta \]

This has trilinear and quadrilinear couplings.

\[ (\partial_{\mu} A_\gamma - \partial_{\gamma} A_\mu)(\partial^\mu A^\gamma - 2 A^\mu A^\gamma) \]

\[ = (\partial_{\mu} A_\gamma - \partial_{\gamma} A_\mu)(\partial^\mu A^\gamma - 2 A^\mu A^\gamma) \]

free \ K.E \ ten \ Lagrangian

trilinear

\[ + 2g(\partial_{\mu} A_\gamma - \partial_{\gamma} A_\mu) A^\alpha A_\mu \varepsilon_{\gamma \delta \epsilon} A^\epsilon \]

\[ + g^2 \varepsilon_{\alpha \beta \gamma \delta} \varepsilon_{\epsilon \eta \mu \nu} A_\alpha A_\beta A_\epsilon A_\eta A_\mu A_\nu \]

These couplings exist because of non-linear terms in \( F \) which in turn arise from fact that \( A^\gamma \) carries \( SU(n) \) charge \( n \).

transform non-trivially (\( n \) adjoint rep) \( \varepsilon \) of \( SU(n) \).
Yang–Mills coupled to matter:

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left( i\gamma^\mu \partial_\mu - m \right) \psi \]

Note: Independent coupling for each simple (non-abelian) factor of gauge group.

\[ D_\mu = \partial_\mu - ig A_\mu \]

\[ -ig \bar{\psi} \gamma^\mu A_\mu T^2 \psi \]

\[ = + ig \bar{\psi} T^2 A_\mu \gamma^\mu \psi \]

Note that unlike the case of abelian gauge theories, the coupling constant for matter is fixed by gauge invariance. The self-interaction strength is the same and the matter coupling is governed by the same constant. Matter cannot couple to fields with strength \( \gamma g \) and preserve gauge invariance.

This is because

\[ [D_\mu, D_\nu] = -ig F_{\mu\nu} \]

occurs in Faddeev-Popov's non-linear relation for g.
For a simple group there can be only one coupling constant. For direct product groups such as $U(1) \times SU(2) \times SU(3)$, each simple factor has its own coupling constant $g_1, g_2, g_3$. No reason for equality.

Grand unification says

$$U(1) \times SU(2) \times SU(3) \subset SU(5)$$

The goal is a simple group with one coupling, halved by one generational power.
Note one can scale $A$ about $\theta$

$$A \rightarrow \theta^{-1} A$$

$$D_\mu = \partial_\mu - i A_\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{2} g [A_\mu, A_\nu]$$

$$\rightarrow \frac{1}{g^2} \left[ \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] \right]$$

$$= \frac{1}{g^2} F$$

Scaling $\gamma \rightarrow \frac{1}{\sqrt{\gamma}}$.

We may write

$$\lambda \rightarrow \frac{1}{\sqrt{\gamma}} \lambda \quad (g=1)$$

In both contexts

$$e^{\frac{1}{\sqrt{\gamma}}} S [A, \varphi, \bar{\varphi}]$$

$$\rightarrow e^{\frac{1}{\sqrt{\gamma} g^2}} S [A, \varphi, \bar{\varphi}]$$

in $g^2$ occurs same way that $g$ does - recall loop exp.
Equations of Motion

\[ S = -\frac{1}{4} F_{\mu \nu}^i F^{i \mu \nu} + \bar{\psi} i \gamma^\mu \gamma^\nu \psi - m \bar{\psi} \psi. \]

\[ D_\mu \equiv \partial_\mu - i \theta A_\mu. \]

\[ \delta S = \int d^4x \left[ -\frac{1}{2} F_{\mu \nu}^i \left( 2 \delta A_\alpha^i - 2 \delta A_{\alpha \nu}^i + \theta c^i_{\beta \kappa} (A_\alpha^\gamma A_\beta^\kappa + \delta A_\alpha^\gamma A_{\beta \nu}^i) \right) \right. \]

\[ + \left. g \bar{\psi} \gamma_\mu \gamma^i \psi \delta A_\mu^i \right]. \]

\[ = \int d^4x \left[ \partial_\mu \left( F_{\mu \nu}^i - \theta A_\alpha^i A_\beta^\mu F_{\beta \nu}^k + \theta \bar{\psi} \gamma_\nu \gamma^i \psi \right) \right] \delta A_\nu^i. \]

\[ \text{Faraday's law} \]

\[ \frac{\delta S}{\delta A_\nu^i} = D^\mu F_{\mu \nu}^i + g \bar{\psi} \gamma_\nu \gamma^i \psi = 0. \]

\[ \text{current} \]

\[ D^\mu F_{\mu \nu}^i \equiv (\partial_\mu A_\nu^i - \theta \gamma_\nu A_\alpha^i A^\mu_\kappa) F_{\mu \nu}^k. \]

\[ \left[ D^\nu D^\mu F_{\mu \nu} \right]^i = -g \left( D^\nu j_\nu^i \right)^i. \]

\[ \text{LHS zero since} \]

\[ \left( D^\nu D^\mu F_{\mu \nu} \right)^i = \frac{1}{2} \left( \left[ D^\nu, D^\mu \right] F_{\mu \nu}^i \right) \]

\[ = \frac{1}{2} \theta c^i_{\beta \kappa} F_{\beta \nu}^i F_{\mu \nu}^k. \]

\[ \text{From anti-symmetry of} \]

\[ \theta c^i_{\beta \kappa} F_{\beta \nu}^i F_{\mu \nu}^k = 0. \]
Hence we have the **covariant** conservation law

$$D^\mu j^{\mu} = 0.$$ 

rather than $$D^\mu j^{\mu} = 0$$ as in QED.

Reason is that $A^\mu$ also carries charge.

Fermion eqn:

$$\frac{\delta S}{\delta \bar{\psi}} = 0.$$ 

$$(i\partial - m)\psi = \{i\gamma^\mu (\partial_\mu - ig A_\mu) - m\}\psi = 0.$$ 

Gauge covariant Dirac eqn

as in QED.
Higgs Mechanism in $SU(2)$

Gauge Theory:

Fields Gauge: $W^r_m \rightarrow W^r_m = \sum_{r=1}^{3} W^r_m t^r$, $t^r = \frac{e^{ir\theta}}{\sqrt{2}} t^r$

$tr^r t^s = \frac{1}{2} \delta^{rs}$, \hspace{1cm} \begin{bmatrix} t^r \ t^s \end{bmatrix} = i \epsilon^{rsu} t^u$

Higgs field: $H = (H^+, H^0)$ - $SU(2)$ doublet,

$H^+, H^0$ - complex!

Covariant derivative: $D^r \mu H = (\partial^\mu - ig W^r_m) H$.

Field strength:

$F^r_{\mu\nu} = \partial^\mu W^r_{\nu} - \partial^\nu W^r_{\mu} + g \epsilon^{rsu} W^s_{\mu} W^u_{\nu}$

Potential: $V(H) = -\mu^2 H^+ H + \lambda (H^+ H - \frac{m^2}{2})^2$

Note: potential minimum at $\langle H^+ H \rangle = \frac{v^2}{2}$

$\nu \equiv \sqrt{\mu^2 / \lambda}$.

$SU(2)$ Gauge Invariant

Lagrangian:

$L = (\partial^\mu H)^+ (\partial^\mu H) - V(H) - \frac{1}{4} F^{r\nu}_{\mu} F^{\nu\mu}_{\mu}$

Gauge transformations: $H \rightarrow H' = e^{i \epsilon^r_t \theta^r_{\mu}} H$

$U = e^{i \epsilon^r_t \theta^r_{\mu}} A W^r_m \rightarrow W^r_m' = U W^r_m U^{-1} - \frac{i}{g} \partial^\mu U U^{-1}$.
By an $SU(2)$ transformation take
\[ \langle H \rangle = \frac{1}{2} \langle 0 \rangle \]

To display the spectrum of the theory (to zeroth order in $g$).

Put
\[ H(x) = e^{i \frac{r(x)}{\sqrt{\lambda}}} \left( \begin{array}{c} 0 \\ \frac{v + \eta}{\sqrt{\lambda}} \end{array} \right) \]

with \( \langle \bar{s} \rangle = \langle \eta \rangle = 0 \), 2 complex → 4 real

and
\[ \Gamma_{\mu} \equiv U(3) A_{\mu} U^{-1}(3) - \frac{i}{\lambda} \hat{\sigma}_{\mu} U(3) U^{-1}(3) \]

This is completely equivalent to doing a gauge transformation with gauge parameter \( \theta(x) = -r(x) \sqrt{\lambda} \).

i.e.
\[ H \rightarrow H' = U(3) H = \left( \begin{array}{c} v + \eta \\ \sqrt{\lambda} \end{array} \right) \equiv h \]

\[ W_{\mu} \rightarrow W'_{\mu} = U W_{\mu} U^{-1} - \frac{i}{\sqrt{\lambda}} \hat{\sigma}_{\mu} U U^{-1} = \Gamma_{\mu} \].
So the Lagrangian can be written as

\[ \mathcal{L} = (D^\mu h)^* (D^\mu h) - \lambda (h^\dagger h - \frac{\mu^2}{2})^2 - \frac{1}{4} F_{\mu\nu}^r F^{\mu\nu}_r \]

Redefine

\[ F_{\mu\nu}^r = \partial_\mu Y_\nu^r - \partial_\nu Y_\mu^r + \delta^{rs} \gamma^s Y_\mu^r \]

As in U(1) Higgs mechanism the "Goldstone fields" \( \frac{\phi^r}{\sqrt{2}} \) have disappeared from \( \mathcal{L} \). Note also,

\[ D_\mu h = \left( \partial_\mu - i g Y_\mu^r \right) \left( \frac{0}{\sqrt{2}} \right) \]

And \( (D_\mu h)^\dagger = \left( 0, \frac{\phi^r}{\sqrt{2}} \right) \left( \delta_\mu^s + i g Y_\mu^r \phi^s \right) \).

So \( (D_\mu h)^\dagger (D_\mu h) \) has a mass term for \( Y_\mu^r \), i.e.

\[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{\lambda}{2} \left( \phi^r \phi^s \right) \frac{\delta^r_\mu \delta^s_\nu}{\sqrt{2}} \left( \frac{\phi^r}{\sqrt{2}} \right) Y_\mu^r Y_\nu^s \]

\[ = \frac{1}{2} \left( \frac{\partial_\mu \phi}{\sqrt{2}} \right)^2 \gamma^r Y_\mu^r Y_\phi^s \]

Corresponding to a mass \( M_w = g \nu/\sqrt{2} \).

Also \( m_H = + \sqrt{2} \lambda \nu = \sqrt{2} \lambda \mu^2 \).

Typical Higgs mass

\[ \text{Notation:} \quad M_{\phi} \rightarrow M_w \quad M_\phi \rightarrow M_H \]

\[ \nu^2 \rightarrow \mu^2 \]
Constructing the Standard Model

Useful to write the fermion fields as left chiral Dirac fields

\[ \psi_0 = ( \psi_0^L ) = \begin{pmatrix} \psi_L \\ \bar{c}_L \end{pmatrix} \]

and their charge conjugates

\[ (\psi_0^c)^L = (\psi_0^L)^c = \begin{pmatrix} \bar{c}_L \\ c_L \end{pmatrix} \]

Note that these two are independent fields.

The local gauge symmetry of the standard model is

\[ G = SU(3)_c \times SU(2)_L \times U(1)_Y \]

\[ SU(2)_L \rightarrow \text{acts on left chiral fields} \]

\[ U(1)_Y \rightarrow \text{Abelian symmetry - } Y \text{-hypercharge} \]

\[ \rightarrow \text{Simplest possible group consistent with the available experimental data and theoretical constraints circa 1970}. \]
$\mathcal{L} = \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{Higgs}}$

\begin{align*}
A = 1, 2, 3 \\
SU(2)_L \text{ doublets!} \quad & Q_L^A = (u^L_L, d^L_L, (c^L_L, s^L_L), (t^L_L, b^L_L)) \\
SU(3)_{\text{coulomb}} \quad & (u^c)^A_L = (u^c)^L_L, \quad (c^c)^L_L, \quad (b^c)^L_L \\
SU(3)_{\text{triplet}} \quad & (d^c)^A_L = (d^c)^L_L, \quad (c^c)^L_L, \quad (b^c)^L_L \\
SU(2)_L \text{ singlet} \quad & (e^-)^A_L = (e^c)^L_L, \quad (\mu^-)^L_L, \quad (\tau^-)^L_L \\
SU(2)_L \text{ triplet} \quad & (e^c)^A_L = (e^c)^L_L, \quad (\mu^c)^L_L, \quad (\tau^c)^L_L
\end{align*}

Note:
\begin{align*}
Q = \frac{1}{2} P - Y \\
K_{\text{matter}} = i \overline{Q}_L^A D Q_L^A + i \overline{(u^c)^A_L D (u^c)^A_L} + i \overline{(d^c)^A_L D (d^c)^A_L} \\
& - i \overline{L}_L^A D L_L^A + i \overline{(e^c)^L_L D (e^c)^L_L} + \text{etc.}
\end{align*}

In general:
\begin{align*}
D^\mu \psi = \left( \partial^\mu + \frac{1}{2} \gamma^\mu \right) \psi + i g \frac{1}{2} \epsilon_{\mu \nu \lambda} \gamma^\nu \gamma^\lambda (\psi L) \gamma_5.
\end{align*}
Generator of

\[ T^a = \text{su}(3) \text{ rep of } \psi \]

\[ t^r = \text{su}(2) \text{ rep of } \psi \]

\[ Y = \text{Hypercharge of } \psi \]

i.e. the field \( \psi \) transforms under gauge transformations as:

\[ \text{su}(3): \quad \psi \rightarrow e^{i \theta_{\mu} T^a} \psi, \quad \text{su}(2): \quad \psi \rightarrow e^{i \theta_{\mu} t^r} \psi \]

\[ U(1)_Y: \quad \psi \rightarrow e^{i \theta} \psi \]

\[ u = e^{i \theta_{\mu} T^a} \quad \text{su}(3): \quad g_{\mu} = G_{\mu}^a \rightarrow g_{\mu}' = u g_{\mu} u^{-1} - \frac{i}{2} \partial_{\mu} u u^{-1} \]

\[ u = e^{i \theta_{\mu} t^r} \quad \text{su}(2): \quad w_{\mu} = W_{\mu}^a t^a \rightarrow w_{\mu}' = u w_{\mu} u^{-1} - \frac{i}{2} \partial_{\mu} u u^{-1} \]

\[ B_{\mu} \rightarrow B_{\mu}' = B_{\mu} + \frac{i}{8} \partial_{\mu} (u u^{-1}) \]

Using \( (\psi^c)_L = C \gamma^0 \psi^c_R \)

we may rewrite the matter Lagrangian as:

\[ \mathcal{L}_{\text{matter}} = i \bar{Q}_L \gamma^\mu \partial_{\mu} Q_L + i \bar{U}_R \gamma^\mu \partial_{\mu} U_R + i \bar{Q}_L \gamma^\mu \partial_{\mu} A_{\mu} + i \bar{U}_R \gamma^\mu \partial_{\mu} \tilde{A}_{\mu} + i \bar{e}_L \gamma^\mu \partial_{\mu} e_L + i \bar{e}_R \gamma^\mu \partial_{\mu} \tilde{e}_R \]

Note that there cannot be any
mass terms for the fermions

- No Majorana mass - All $\psi$'s carry $\gamma_5$

(Also in complex reps of $\text{SU}(3)$ and/or pseudo real $\text{SU}(2)$)

No Dirac term since that would require pairs of (say, left chiral) fermions in complex conjugate reps - and we don't have any.

(Recall $\mathcal{L} = -m \bar{\psi} \gamma = m \left( \sum L \psi_L - \psi_L^T \psi_R^* \right)$

but $\gamma = \left( \begin{array}{c} \gamma_L \\ \gamma_R^* \end{array} \right)$
Weak / E - M interaction

\[ W_{\text{int}} = \bar{\psi} \gamma^\mu \left( g \frac{1}{\sqrt{2}} W^\mu_{-} + g' \gamma B_{\mu} \right) \psi \]

\[ \psi = (u \ d) \]

\[ \gamma (\gamma^-) \]

\[ \sigma = (0 \ 1) \]

\[ \sigma_3 = 0.0 \]

\[ \sigma_1 = (2 \ 0) \]

\[ \sigma_2 = 0 \]

Identify photon as field coupling to \( Q = \frac{\sigma^3}{2} P^- + Y^- \)

Put \( W^3 = \sin \theta W A^3 + \cos \theta W Z \)

\( B_\mu = \cos \theta W A^3 - \sin \theta W Z \)

\[ g \left( \frac{\sigma^3}{2} P^- + g' \gamma \right) \left( \cos \theta W A^3 - \sin \theta W Z \right) \]

\[ = A_\mu \left( g \sin \theta W \frac{\sigma^3}{2} P^- + g' \gamma \cos \theta W \right) + Z_\mu \left( g \cos \theta W \frac{\sigma^3}{2} P^- - g' \sin \theta W \right) \]

Should be proof to \( Q = \frac{\sigma^3}{2} P^- + \frac{\sigma_1}{2} \)

\[ \frac{g'}{g} = \tan \theta \]

\[ \sin \theta W = \frac{g'}{\sqrt{g^2 + g'^2}} \]

\[ \cos \theta W = \frac{g}{\sqrt{g^2 + g'^2}} \]
So the interaction (matrix) is

\[ \frac{g g'}{\sqrt{g^2 + g'^2}} A_\mu Q' + \frac{1}{\sqrt{g^2 + g'^2}} Z_\mu \left( g^2 g'^2 P - g^2 Y \right). \]

\[ g' = \frac{g}{\sin \theta_W}, \quad g = \frac{e}{\cos \theta_W}. \]

Alternative form of interaction matrix,

\[ e A_\mu Q' + e Z_\mu \left( \cot \theta_W \frac{\sigma^3}{2} P - \tan \theta_W Y \right). \]

\[ Q' = Q - \frac{\sigma^3}{2} P. \]

\[ \sin \theta_W \]

\[ e = g' \sin \theta_W \]

\[ e A_\mu Q + e Z_\mu \left( \cot \theta_W + \tan \theta_W \frac{\sigma^3}{2} P - \tan \theta_W Q \right). \]

\[ = e A_\mu Q + e Z_\mu \left( \frac{\sigma^3}{2} P - \sin^2 \theta_W Q \right). \]

\[ \rightarrow \chi_{\text{int}} = e A_\mu J_{\text{EM}}^\mu + \frac{g}{\cos \theta_W} Z_\mu J_0^0. \]

Neutral weak current

\[ J_0^0 = \bar{\nu}_\mu \gamma_\mu \left( \frac{\sigma^3}{2} P - m^2 \theta_W Q \right) \nu. \]

\[ J_{\text{EM}}^\mu = \bar{\nu} \gamma_\mu Q \cdot \nu, \quad \gamma = (\frac{\gamma}{c}). \]

\[ \rightarrow \text{etc.} \]
Yukawa coupling becomes:

\[
\chi_{\text{Yukawa}} = - s_A B - \frac{A}{u} Q_L i \sigma_2 H^*_R u_R - s_d B \frac{A}{L_i} H_i ^* d_R - s_e B \frac{A}{L_e} H_e ^* e_R + \text{h.c.}
\]

\[
\begin{align*}
\epsilon' \sigma e \\
\text{etc}
\end{align*}
\]

\[
H \rightarrow \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}
\]

After diagonalizing the $s^{AB}$ matrices:

- See next section.

Eg:

\[
\begin{align*}
5^u \bar{Q}_L^u & \in H^* u_R = 5^u (\bar{u}_L, d_L)(1, 0) (1) \frac{v + \eta}{\sqrt{2}} u_R \\
&= \frac{1}{\sqrt{2}} 5^u (v + \eta) \bar{u}_L u_e.
\end{align*}
\]

(Note: $(\bar{u}_L u_R)^+ = \bar{u} \left(1 - \frac{\nu^2}{2}\right) u$)

\[
\bar{u}_L u_R = \bar{u} \left(1 + \frac{\nu^2}{2}\right) u.
\]

So, if $\eta$ real then

\[
\begin{align*}
\frac{v}{\sqrt{2}} + \frac{1}{\sqrt{2}} 5^u (v + \eta) \bar{u}_L u_R + \text{h.c.} \\
\Rightarrow m_u = 5^u \frac{v}{\sqrt{2}}
\end{align*}
\]

\[
\text{et c.}
\]
Last piece is the Higgs Sector

\[
V(E_{\text{Higgs}}) = \lambda \left( H^+H - \frac{\mu^2}{2} \right)^2 \\
H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}
\]

As in previous discussion (for SU(2) case),

\[
H_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}, \quad v = \mu^2/\lambda
\]

at the minimum of the potential.

Put

\[
H(x) = U(3) \begin{pmatrix} 0 \\ v_+ \\ v_- \end{pmatrix}
\]

\[
U(3) = e^{i\left( \frac{\varphi}{2} \Sigma^1 + \frac{\varphi^2}{2} \Sigma^2 + \frac{i}{2} \varphi \Sigma^3 \right)}
\]

\[
\bar{\Sigma}_1 = 0, \bar{\Sigma}_2 = 0, \text{ and } \varphi = \cos \theta_W \bar{\Sigma}_3 - \tan \theta_W \gamma
\]

are the 'broken' generators in SU(2) × U(1).

As argued before, U(3) dependence disappears due to gauge invariance. So

\[
\mathcal{L}_{\text{Higgs}} = (D_{\mu} H)^+ (D^\mu H) - V(H)
\]

\[
V(H) = \lambda v^2 \eta^2 + \lambda v \eta^3 + \frac{\lambda \eta^4}{4} \quad \Rightarrow m^2_\eta = 2 \lambda v^2
\]
The vector boson masses,

\[ \chi = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right), \]

can be read off from the term of Higgs,

\[ \frac{\sigma^+}{2} x = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \]

\[ \left(D^m H \right)^+ D^m H \]

after putting again \[ H \rightarrow \frac{v}{\sqrt{2}} \chi \]

\[ \frac{\sigma^+}{2} x = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \]

\[ \frac{\sigma^z}{2} x = 0 \]

\[ \frac{\sigma^x}{2} x = 0 \]

\[ W^\pm = \Lambda \left( \begin{array}{c} W^- \\ W^+ \end{array} \right) \]

\[ \frac{V^2}{2} \chi^+ \left( g \frac{\sigma^+}{2} W^+_m + g \frac{\sigma^-}{2} W^-_m + \frac{g_4}{2\cos \theta_w} Z^+ \left( \frac{\sigma^3}{2} - \sin^2 \theta_w Q \right) + eA_Q \right) \chi \]

\[ \chi = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \]

\[ \frac{\sigma^z}{2} x = 0 \]

\[ \frac{\sigma^3}{2} x = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \]

\[ \frac{\sigma^x}{2} x = 0 \]

\[ \frac{\sigma^y}{2} x = 0 \]

\[ \frac{\sigma^x}{2} x = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \]

\[ \frac{\sigma^z}{2} x = 0 \]

\[ \frac{\sigma^3}{2} x = \frac{1}{2} x \]

So this reduces to,

\[ \frac{g^2 V^2}{4} W^+_m W^-_m + \frac{g^2 V^2}{4 \cos^2 \theta_w} \frac{1}{2} Z^+_m Z^m \]
From $\frac{1}{2} H^2$ term identify. 

$$M_W^2 = \frac{g^2 v^2}{4} \quad M_Z^2 = \frac{g^2 v^2}{4 \cos^2 \theta_W}$$

$$8 = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} = 1$$

($8$ parameter) is a consequence of having just one Higgs doublet.

If $2 -$ doublets this would change.

Value of $v$ from Fermi coupling.

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{\sqrt{2}}$$

From charged current interaction terms,

$$\mathcal{L}_{\text{I.C.C.}} = \frac{g}{\sqrt{2}} \left( J^+_\mu W^+ \mu + J^-_\mu W^- \mu \right) \quad J^\pm_\mu = \bar{\psi} \gamma^\mu \frac{1}{2} P_{\pm} \psi$$

If only one quark.

$$\psi = (\nu, u) \quad J^\pm_\mu = \frac{1}{2} \bar{\nu} \gamma_\mu (1-\gamma_5) e + \frac{1}{2} \bar{u} \gamma_\mu (1-\gamma_5) d$$

$$G_F = \frac{g^2}{8 M_W^2} = \frac{g^2}{8 \frac{v^2}{4}} = \frac{1}{4 v^2} \quad v_F = 10^5 \text{GeV} \Rightarrow v = 250 \text{GeV}$$
\[ M_W^2 = \frac{e^2 v^2}{4 \sin^2 \theta_W} \approx \left( \frac{374 \text{GeV}}{\sin \theta_W} \right)^2 \]

From \[ g = \frac{e}{\sin \theta_W} \quad \frac{e^2}{4 \pi} \approx \frac{1}{137} \]

and \[ M_Z^2 = \frac{e^2 v^2}{4 \sin^2 \theta_W \cos^2 \theta_W} = \left( \frac{374 \text{GeV}}{\sin \theta_W \cos \theta_W} \right)^2 \]

If we knew \( \sin \theta_W \), then \( M_Z, M_W \)

are predictions.

Alternatively, if \( M_W \) (say) is known, \( M_Z, \sin \theta_W \) are predictions. Neutral current Expts (these are (CERN 1972, 1979) also predictions!) determined \( \sin^2 \theta_W \approx 0.22 \).

\[ M_W = 80 \text{GeV} \quad M_Z = 90 \text{GeV}. \]

Current values:

\[ M_W = 80.393(25) \text{GeV} / c^2 \quad \sin^2 \theta_W = 0.2319(14). \]

\[ M_Z = 91.1876(21) \text{GeV} / c^2 \quad \text{at } \sqrt{s} = \text{pole}. \]

Note: if we use above value for \( \theta_W \), \( \sin \theta_W \approx 0.23 \).
Global Symmetries

Accidental Symmetries of matter - follows from gauge invariance and ref. center, i.e. not imposed as an additional symmetry.

\[ Q^A_L \rightarrow U^A_{AB} Q^B_L \quad U^A_R \rightarrow U^A_{uR} u^B_R \]

\[ d^A_R \rightarrow U^A_{dR} d^B_R \quad L^A_L \rightarrow U^A_{eL} L^B_L \quad e^A_R \rightarrow U^A_{eR} e^B_R \]

Globally symm. of L matter \( U(3)^5 \)

Violated (mostly) by Yukawa

\[ \mathcal{L}_{\text{Yukawa}} = -\sum_u f^{AB} e^{\phi} \bar{Q}^A_L e_i \frac{g}{\sqrt{2}} H^B \left( u^R R \text{ part} \right) - \sum_d f^{AB} e^{\phi} \bar{Q}^A_L H^B \left( d^R \text{ part} \right) + \text{h.c.} \]

Only Baryon number and lepton # survive

\[ q^A_L \rightarrow e^{i\theta_{13}} q^A_L \quad L^A_L \rightarrow e^{i\theta_{23}} L^A_L \quad e^A_R \rightarrow e^{i\theta_{23}} e^A_R \]

Under \( SU(3)_C \times SU(2)_W \times U(1)_Y \).
As in general discussion of Higgs phenomenon - spontaneous symmetry breakdown (SSSB)

\[ \langle H \rangle \equiv \Phi H_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \]

Fermions (except for \( \nu \)) acquire mass terms

\[ L_M = -m_{uL}^A u_L^A u_R^B - m_d^A d_L^A d_R^B - m_e^A e_L^A e_R^B \]

\[ + h.c. \]

\[ m_{uL}^A = -f_A^u \frac{v}{\sqrt{2}} \]

Yukawa matrices \( f_A^u \) are 3-complex

\[ 3 \times 3 \quad \Rightarrow \quad 3 \times 3 \times 3 \times 2 = 5 \times 4 \]

new real parameters!

But
May-Redundant. Removed Lag field redefinitions - unitary transformation preserves kinetic terms!

\[ (\bar{\nu} \gamma \nu \rightarrow \bar{\nu} \gamma \nu, \nu \gamma \nu = \bar{\nu} \gamma \nu) \]

for \( \nu = u_L, d_L \), \( \gamma = \nu, \bar{\nu}, e, \bar{e} \).

Thm A non-singular matrix can be diagonalized by a bi-unitary transform.

\[ U_R^\dagger U_L = U_L^\dagger U_R = I \]

\[ \text{such that } U_L^\dagger M U_R = \text{diag}(m_1, m_2, m_3) \]

\[ M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \]

\[ U_L^\dagger M U_L = \begin{bmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{bmatrix} \]

\[ m_{\text{diag}} = \begin{bmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{bmatrix} \]

Proof \( m m^\dagger \) is Hermitian so \( \exists U_L \text{ s.t. } (U_L^\dagger U_L) \)

By a phase change \( U_L \Rightarrow U_L^* \)

we can take \( m_1 \) to be real with \( m_i > 0 \)

Put \( H = U_L M U_L^\dagger \Rightarrow H^\dagger = H \), and \( V = H^\dagger M \)

\[ V = M^\dagger H^\dagger \]

\[ V V^\dagger = H^\dagger M M^\dagger H^\dagger = H^\dagger U_L M^\dagger H^\dagger U_L^\dagger H^\dagger = H^\dagger H^\dagger H^\dagger = I \]

\[ H^\dagger, \nu^\dagger \text{ are unitary. So } M = H V \text{ - Polar decomposition. } \]

\[ U_L^\dagger H U = U_L^\dagger \nu^\dagger \nu U_L = M \text{ diag } \]
i.e. \( U_L \, m \, U_R = \text{diag} \, \, \, U_R = V^+ U_L \).

So each \( m \) matrix can be diagonalized by \( U \).

\[
\begin{align*}
U_L &\rightarrow U_{wL} \, u_L, \, \, U_R \rightarrow U_{wR} \, u_B \\
\bar{d}_L &\rightarrow U_{dL} \, \bar{d}_L, \, \, \bar{d}_R \rightarrow U_{dR} \, \bar{d}_B \\
e_L &\rightarrow U_{eL} \, e_L, \, \, e_R \rightarrow U_{eR} \, e_B \\
\chi^A_L &\rightarrow U_{\chi_L} \, \chi^B_L
\end{align*}
\]

\[\bar{\Psi}_L \, m \, \Psi_R \rightarrow \bar{\Psi}_L \, U_L^+ \, m \, U_R \, \Psi_R = \bar{\Psi}_L \, m_{wL} \, \Psi_R \text{ also.}\]

Note since \( m^{AB} = g^{AB} \, v_1 \, v_2 \), diagonalizes Yukawa Interactions. However \( W^\pm \) gauge interactions are not diagonal \( (\chi \, \chi^0 \, (\sim W^\pm )) \) in the new basis (charged current interaction).

\[\text{L Matter: } \bar{\phi}_L \, A^\mu \left( -ig \, \frac{\sigma^\pm}{2} \right) \chi^\mu = \frac{g}{\sqrt{2}} \, J^\pm \, W^\mu.\]

\[
W^\pm_m = \frac{1}{\sqrt{2}} \left( W^1_m + i \, W^2_m \right)
\]
\( J^+_m = \frac{1}{2} \bar{Q}^A_L \gamma_m \frac{\sigma^+}{2} Q_L = \bar{U}^A_L \gamma_m d^A_L \)

\[ \rightarrow \bar{Q}^A_L \bar{U}^A_L \gamma_m (U^\dagger_L U_{dL})^{AB} d^B_L \]

This may effectively be regarded as a rotation of the down quark basis

\[
\begin{pmatrix}
  d \\
  s \\
  b
\end{pmatrix}
\rightarrow
U_{\text{CKM}}
\begin{pmatrix}
  d \\
  s \\
  b
\end{pmatrix}
= 
\begin{pmatrix}
  d' \\
  s' \\
  b'
\end{pmatrix}
\]

\( U_{\text{CKM}} \equiv U^+_L U_{dL} \rightarrow (\text{CKM matrix}) \)

Note \( U_{\text{CKM}} U^\dagger_{\text{CKM}} = 1 \)

Point is the basis in which the mass terms are diagonal is different from the basis in which the charge current is diagonal.

* Note neutral currents remain diagonal though. *
\[ \text{Neutrino masses exist so } \Rightarrow \text{no CKM matrix for leptons} \]

\[ \text{modify this later} \]

\[ \text{need to separate lepton families are decoupled} \]

Any way this means lepton conservation forbidden.

\[ U_{12} = U_{13} \Rightarrow U_{23} \leq 1 \]
\# of parameters in CKM matrix.

3x3 Unitary matrix (9 - real #)
- 3 real angles - 6 phases.

However can remove 5 out of 6 phases - \( u_A \rightarrow e^{i\theta_A} u_A \) \( d_A \rightarrow e^{i\phi_A} d_A \), \( A = 1, 2, 3 \).

Except that a common rotation in all six will leave \( U_{CKM} \) unchanged.

So, \( U_{CKM} \) has 3 angles + 1 phase.

No phase (no \( \theta \)).

Generally if \( n \)-doublets: Complex matrix \( n^2 \) real #
for 2-generations.

1 - phase
by 3-generations

\# angles = \# of parameters in orthogonal matrix.

\# phases which can be removed = \( 2n - 1 \)

\# physical phases = \( n^2 - (2n - 1) - \frac{n(n - 1)}{2} = \frac{(n - 1)(n - 2)}{2} \)

\( n = 3 \) 1-phase.