Problem 1

a) [10 pts] Use the forced harmonic oscillator discussion in QM to derive the expression,

\[ Z_0[j] = e^{-\frac{1}{2} \int d^4xd'x'j(x)\Delta_F(x-x')j(x')} Z_0[0] \]

for a free scalar field theory of mass \( m \). Here

\[ \Delta_F(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - m^2 + i\epsilon}. \]

Hint: One way of doing this is to work in a box and expand the field in a (spatial) Fourier expansion

\[ \phi(x) = \frac{1}{\sqrt{V^3}2} \sum_k e^{ik\cdot x} q_k(t) \]

to convert the free scalar field Lagrangian to a sum over harmonic oscillator Lagrangians. Then use theQM result and take the continuum limit.

b) [5 pts] Show that the this Feynman propagator corresponds to having +(-) frequency solutions going forward (backward) in time.

c) [5 pts] Calculate \( G_2(x,x') \) in the operator (canonical) formalism of the free scalar field theory and show that you get \( i\Delta_F(x-x') \).

Problem 2

[10 pts] Scattering from an external potential: A real scalar field \( \phi \) has an interaction \( \mathcal{L}_I = U(x)\phi^2(x) \), with a time independent potential \( U \).

Show that to lowest order in \( U \), the S-matrix element for a incoming boson with momentum \( k = (\omega,k) \) to be scattered to a state with momentum \( k' = (\omega',k') \) is given by,

\[ \langle k' | S | k \rangle = \frac{2\pi^\delta(\omega' - \omega)}{(2\sqrt{\omega})^{1/2}(2\sqrt{\omega'})^{1/2}} 2\tilde{U}(k' - k) \]

where \( \tilde{U}(q) = \int d^3xe^xU(x)e^{-iq\cdot x}. \)

Problem 3

[30=10+5+5+10] Peskin and Schroeder Problem 4.3 (see attached)
4.3 Linear sigma model. The interactions of pions at low energy can be described by a phenomenological model called the linear sigma model. Essentially, this model consists of $N$ real scalar fields coupled by a $\phi^4$ interaction that is symmetric under rotations of the $N$ fields. More specifically, let $\Phi^i(x), i = 1, \ldots, N$ be a set of $N$ fields, governed by the Hamiltonian

$$H = \int d^3x \left( \frac{1}{4}(\Pi^i)^2 + \frac{1}{2}(\nabla \Phi^i)^2 + V(\Phi^2) \right),$$

where $(\Phi^i)^2 = \Phi \cdot \Phi$, and

$$V(\Phi^2) = \frac{1}{2}m^2(\Phi^i)^2 + \frac{1}{4}((\Phi^i)^2)^2$$

is a function symmetric under rotations of $\Phi$. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to $H$; hence, $V$ is the field potential energy.

(What does this Hamiltonian have to do with the strong interactions? There are two types of light quarks, $u$ and $d$. These quarks have identical strong interactions, but different masses. If these quarks are massless, the Hamiltonian of the strong interactions is invariant to unitary transformations of the 2-component object $(u,d)$:

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \exp(i\alpha \cdot \sigma/2) \begin{pmatrix} u \\ d \end{pmatrix}.$$}

This transformation is called an isospin rotation. If, in addition, the strong interactions are described by a vector "gluon" field (as is true in QCD), the strong interaction Hamiltonian is invariant to the isospin rotations done separately on the left-handed and right-handed components of the quark fields. Thus, the complete symmetry of QCD with two massless quarks is $SU(2) \times SU(2)$. It happens that $SO(4)$, the group of rotations in 4 dimensions, is isomorphic to $SU(2) \times SU(2)$, so for $N = 4$, the linear sigma model has the same symmetry group as the strong interactions.)
(a) Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly $N$ copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter $\lambda$. Show that the propagator is

$$\Phi^i(x) \Phi^j(y) = \delta^{ij} D_F(x - y),$$

where $D_F$ is the standard Klein-Gordon propagator for mass $m$, and that there is one type of vertex given by

$$\begin{array}{c}
  k \\
  i \\
  j \\
  l
\end{array} = -2i\lambda (\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}).$$

(That is, the vertex between two $\Phi^1$s and two $\Phi^2$s has the value $(-2i\lambda)$; that between four $\Phi^1$s has the value $(-6i\lambda)$.) Compute, to leading order in $\lambda$, the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

$$\Phi^1 \Phi^2 \rightarrow \Phi^1 \Phi^2, \quad \Phi^1 \Phi^1 \rightarrow \Phi^2 \Phi^2, \quad \text{and} \quad \Phi^1 \Phi^1 \rightarrow \Phi^1 \Phi^1$$

as functions of the center-of-mass energy.

(b) Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, $V$ has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since $V$ is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting $\Phi^i$ toward the minimum of $V$. By rotational invariance, we can consider this shift to be in the $N$th direction. Write, then,

$$\Phi^i(x) = \pi^i(x), \quad i = 1, \ldots, N - 1,$$

$$\Phi^N(x) = v + \sigma(x),$$

where $v$ is a constant chosen to minimize $V$. (The notation $\pi^i$ suggests a pion field and should not be confused with a canonical momentum.) Show that, in these new coordinates (and substituting for $v$ its expression in terms of $\lambda$ and $\mu$), we have a theory of a massive $v$ field and $N - 1$ massless pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \rightarrow 0$. Construct the Feynman rules by assigning values to the propagators and vertices:

$$\begin{array}{c}
  \sigma^+ \sigma^- = \\
  \pi^i \pi^j = i \rightarrow j
\end{array}$$

(c) Compute the scattering amplitude for the process

$$\pi^i(p_1) \pi^j(p_2) \rightarrow \pi^k(p_3) \pi^l(p_4)$$
to leading order in $\lambda$. There are now four Feynman diagrams that contribute:

\[ + \quad + \quad + \quad + \]

Show that, at threshold ($p_i = 0$), these diagrams sum to zero. (Hint: It may be easiest to first consider the specific process $\pi^+ \pi^- \rightarrow \pi^+ \pi^-$, for which only the first and fourth diagrams are nonzero, before tackling the general case.) Show that, in the special case $N = 2$ (1 species of pion), the term of $O(p^2)$ also cancels.

(d) Add to $V$ a symmetry-breaking term,

\[ \Delta V = -a\Phi^N, \]

where $a$ is a (small) constant. (In QCD, a term of this form is produced if the $u$ and $d$ quarks have the same nonvanishing mass.) Find the new value of $v$ that minimizes $V$, and work out the content of the theory about that point. Show that the pion acquires a mass such that $m_p^2 \sim a$, and show that the pion scattering amplitude at threshold is now nonvanishing and also proportional to $a$.

4.4 Rutherford scattering. The cross section for scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, treat the field as a given, classical potential $A_\mu(x)$. The interaction Hamiltonian is

\[ H_I = \int d^3 x \bar{\psi} \gamma^\mu \psi A_\mu, \]

where $\psi(x)$ is the usual quantized Dirac field.

(a) Show that the $T$-matrix element for electron scattering off a localized classical potential is, to lowest order,

\[ (p'|iT|p) = -ie\bar{u}(p')\gamma^\mu u(p) \cdot \vec{A}_\mu(p' - p), \]

where $\vec{A}_\mu(q)$ is the four-dimensional Fourier transform of $A_\mu(x)$.

(b) If $A_\mu(x)$ is time independent, its Fourier transform contains a delta function of energy. It is then natural to define

\[ (p'|iT|p) = i\mathcal{M} \cdot (2\pi)^3 \delta(E_f - E_i), \]

where $E_i$ and $E_f$ are the initial and final energies of the particle, and to adopt a new Feynman rule for computing $\mathcal{M}$:

\[ \includegraphics[width=0.2\textwidth]{fourier_transform.png} = -ie\gamma^\mu \vec{A}_\mu(q), \]

where $\vec{A}_\mu(q)$ is the three-dimensional Fourier transform of $A_\mu(x)$. Given this definition of $\mathcal{M}$, show that the cross section for scattering off a time-independent,