Problem 1

[40=2+3+5+5+5+20 pts] The composition law of a Lie group is given by $g(\theta)g(\phi) = g(\xi(\theta, \phi)$ where $\theta = \{\theta^i\}, \phi = \{\phi^i\}$ are n-dimensional parameter vectors and $\xi = \{\xi^i\}$. Show that a) $\xi(\theta, 0) = \xi(0, \theta) = \theta$. b) $\xi(\theta, \xi(\phi, \psi)) = \xi(\xi(\theta, \phi), \psi)$. c) Write $g(\phi)g(\theta)g^{-1}(\phi)g^{-1}(\theta) = g(\xi(\theta, \phi))$. Show that near the identity element $\xi^i = c^i_{jk} \theta^j \phi^k$. d) By evaluating the commutator $g(\phi)g(\theta)g^{-1}(\phi)g^{-1}(\theta)$ show that the generators satisfy the commutation relations $[X_j, X_k] = ic^l_{jk} X_l$. e) Deduce that $c^l_{jk} = -c^l_{kj}$, and that $c^m_{jk} c^n_{lm} + c^m_{kl} c^n_{jm} + c^m_{lj} c^n_{km} = 0$. f) For a matrix group we define the Cartan-Killing metric on the Lie algebra by $g_{ij} = tr(X_i X_j)$. i) Show that $c_{ijk} \equiv g_{il} c^l_{jk}$ is totally anti-symmetric in $i, j, k$. ii) If $U = e^{iH}$ is a unitary matrix with $det U = 1$, show that $tr H = 0$. g) Let $\psi, \phi, \ldots$ be vectors in the space of n-dimensional column vectors ($\psi = \{\psi_a\}$ etc.) which carry an n-dimensional unitary representation of some Lie group. Suppose the group elements $\{g\}$ of a group of unitary transformations on this vector space are given in some unitary representation by the matrices $D(g)$. i) Show that the totally anti-symmetric tensor $\epsilon_{i_1 \ldots i_N} = \pm 1$ (with upper(lower) sign for even(odd) permutations of 1, 2, \ldots, $N$) is an invariant of the group $SU(N)$. ii) Suppose the vector $\psi = \{\psi_i, i = 1, \ldots, N\}$ is in the fundamental (defining) representation of $SU(N)$. Then the tensor $\psi_{ij}$ transforms as the direct product of $\psi \times \psi \equiv \{\psi_i \psi_j\}$. Define the permutation operator $P$ so that $P \psi_{ij} = \psi_{ji}$. Show that $P$ commutes with the group transformation law. Show that $\psi_{ij}$ is a reducible tensor representation by demonstrating that the symmetric and anti-symmetric combinations $\psi^\pm_{ij} \equiv \frac{1}{2}(\psi_{ij} \pm \psi_{ji})$ do not mix under the group.
transformations.

**Problem 2**

[30=5+10+10+5 pts] For the Lie algebra of $SU(N)$ show that a) $C_{ijk}T_jT_k = \frac{i}{2}C_2(G)T_i$ b) Prove the completeness relation (for the generators in the fundamental representation)

$$(T_i)_{\gamma\beta}(T_i)_{\alpha\lambda} = \frac{1}{2}\left(\delta_{\beta\alpha}\delta_{\gamma\lambda} - \frac{1}{N}\delta_{\beta\gamma}\delta_{\alpha\lambda}\right)$$

c) Show that in any IR $r$ the generators satisfy the relation

$$T_iT_jT_i = \left[C_2(r) - \frac{1}{2}C_2(G)\right]T_j.$$  

d) Using this (or otherwise?) show that if the generators in the fundamental are normalized with $C(N) = \frac{1}{2}$ then $C(G) = C_2(G) = N$.

**Problem 3**

[30=5+5+5+5+5+5 pts] a) Starting from the Lorentz algebra and defining $J_i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk}$, $K_i \equiv -M_{0i}$ and $\mathcal{J}_i^\pm \equiv \frac{1}{2}(J_i \pm iK_i)$, show that

$$[\mathcal{J}_i^\pm, \mathcal{J}_j^\pm] = i\epsilon_{ijk}\mathcal{J}_k^\pm,$$

$$[\mathcal{J}_i^\pm, \mathcal{J}_j^\mp] = 0.$$  

b) Show that if $\chi_L$ is in the $(\frac{1}{2}, 0)$ representation, $\epsilon\chi_L^*$ (here $\epsilon = i\sigma_2$) is in the $(0, \frac{1}{2})$ representation (i.e. it transforms like $\chi_R$). c) Show that $\mathcal{L} = -\frac{1}{2}m\bar{\psi}_M\psi_M = m(\psi_L^T\epsilon\psi_L - \psi_L^T\bar{\epsilon}\psi_L^*)$ where $\psi_M$ is the four component Majorana spinor and $\psi_L$ is a left chiral Weyl spinor. d) Show that $(\psi_D^c)^c = \psi_D$. e) Show that $\mathcal{L} = -m[(\psi_D^c)_L^TC(\psi_D)_L + h.c.]$ is a Dirac mass term and that $\mathcal{L} = -m[\psi_D^T_DL\psi_D^DL + h.c.]$ is a Majorana mass term. Here $\psi_D$ is a four component Dirac spinor and $C$ is the charge conjugation matrix. e) Show that if $\psi_D^c = \psi_D^c$ then $\psi_D = \psi_M$. f) Show that $\mathcal{L} = -\frac{1}{2}m(\psi_D^c)_R^T\psi_D^DL + h.c.)$ is a Majorana mass.