Cataloguing Fermionic Gauge Theories

Lattice Meets Experiments: Beyond the Standard Model
Boulder, CO October 26-27 2012

Thomas Aaby Ryttov
Ladder approximation.

- Supersymmetric conformal window: Seiberg.
- Supersymmetric conformal window: Three loops in DRbar scheme.
- Non-SUSY: All-orders beta function.
- Non-SUSY: Four loops.
- Multiple representations.
- Exceptional groups and spinorial representations.

Hunt down the conformal window!
Transformation between two schemes: $S$ and $S'$

$$\alpha' = \sum_{n=1}^{\infty} h_n \alpha^n, \quad h_1 = 1$$

$$\beta'(\alpha') = \frac{\partial \alpha'}{\partial \alpha} \beta(\alpha)$$

$$\gamma'(\alpha') = \gamma(\alpha) + \frac{\partial \ln z_m(\alpha)}{\partial \alpha} \beta(\alpha)$$

The first two coefficients of the beta function and the first coefficient of the anomalous dimension are scheme independent.

The existence of a zero of the beta function is scheme independent.

The value of the anomalous dimension at a fixed point is scheme independent.
• Non-Asymptotic freedom:

Large $N_f$ - $\beta_0 < 0$

• IR fixed point:

Intermediate $N_f$ - $\beta_0 > 0$ and $\beta_1 < 0$

• QCD-like:

Small $N_f$ - $\beta_0 > 0$ and $\beta_1 > 0$
Methods and Techniques

- Ladder approximation
- All orders beta function
- Higher loop beta function and anomalous dimension
- Many more..
- Lattice

10/27/12
Thomas Aaby Ryttov
How well do they work??

Methods and Techniques

- Ladder approximation
- All orders beta function
- Higher loop beta function and anomalous dimension
- Many more..
- Lattice
Two loop fixed-point coupling

\[ \alpha_{IR} = -4\pi \frac{\beta_0}{\beta_1} \]

\( \alpha_{IR} \) becomes large as \( \beta_1 \rightarrow 0 \)

Chiral symmetry breaking could be triggered before \( \alpha_{IR} \) is reached.

The gap equation has a solution for the dynamically generated mass when the coupling reaches the value

\[ \alpha_c = \frac{\pi}{3C_2(r)} \]

Critical number of flavors

\[ \alpha_c = \alpha_{IR} \quad \Rightarrow \quad N_f = \frac{17C_2(G) + 66C_2(r) C_2(G)}{10C_2(G) + 30C_2(r) T(r)} \]
• Ladder approximation: Chiral symmetry breaking is triggered at
  $\gamma \sim 1$
Supersymmetric QCD

- Coulomb
- Free Magnetic
- Free Electric
- Confining
- Higgs
• **Exact NSVZ beta function**

\[
\beta(\alpha) = -\frac{\alpha^2}{2\pi} \left( \frac{\beta_0 - 2T(r)N_f \gamma(\alpha)}{1 - \frac{\alpha}{2\pi} C_2(G)} \right)
\]

\[
\gamma(\alpha) = C_2(r) \frac{\alpha}{\pi} + O(\alpha^2)
\]

\[
\beta_0 = 3C_2(G) - 2T(r)N_f
\]

• **Note:** relation between beta function and anomalous dimension.
• At zero of beta function
\[ \gamma = \frac{3C_2(G) - 2T(r)N_f}{2T(r)N_f} \]

• Conformality requires
\[ D(\Phi \bar{\Phi}) = 2 - \gamma \geq 1 \]

• Critical number of flavors
\[ N_{f, critical} = \frac{3}{4} \frac{C_2(G)}{T(r)} \]

• Conformal window
\[ \frac{3}{4} \frac{C_2(G)}{T(r)} < N_{f, critical} < \frac{3}{2} \frac{C_2(G)}{T(r)} \]

Asymptotic freedom
• Can we match other approximate techniques to the exact results of Seiberg?
• The ladder approximation does not do a good job.
SUSY three loop beta function and anomalous dimension

- Three loop beta function and anomalous dimension are known in DRbar scheme

\[ \beta_{3\text{-loop}}(\alpha) = -\beta_0 \frac{\alpha^2}{(2\pi)^1} - \beta_1 \frac{\alpha^3}{(2\pi)^2} - \beta_2 \frac{\alpha^4}{(2\pi)^3} + O(\alpha^5) \]

\[ \gamma_{3\text{-loop}}(\alpha) = \gamma_0 \left( \frac{\alpha}{\pi} \right) + \gamma_1 \left( \frac{\alpha}{\pi} \right)^2 + \gamma_2 \left( \frac{\alpha}{\pi} \right)^3 + O(\alpha^4) \]

- Search for an infrared fixed point of the beta function.
- Evaluate the anomalous dimension at the infrared zero of the beta function.
Three loop beta function

- Fundamental representation

<table>
<thead>
<tr>
<th>N</th>
<th>Nf</th>
<th>(\alpha_{2l} )</th>
<th>(\alpha_{3l} )</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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\(3 < N_f^{\text{Seiberg}} < 6\)

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<td>0.308</td>
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\(4.5 < N_f^{\text{Seiberg}} < 9\)
Three loop anomalous dimension

- Fundamental representation

### $3 < N_f^{Seiberg} < 6$

<table>
<thead>
<tr>
<th>N</th>
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<td>0.0802</td>
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### $4.5 < N_f^{Seiberg} < 9$

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<td>8</td>
<td>0.139</td>
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SUSY anomalous dimension vs. number of flavors

![Graphs showing anomalous dimension vs. number of flavors for different values of Nf, with curves for Fundamental, Nf = 2, Nf = 3, and Nf = 4.](image)
QCD

Coulomb

Free Magnetic

Free Electric

Confining

Higgs
All-orders Beta Function

\[
\beta(\alpha) = -\frac{\alpha^2}{2\pi} \frac{a + bN_f\gamma(\alpha)}{1 - \frac{\alpha}{2\pi} c}
\]

\[
\gamma(\alpha) = 3C_2(r) \frac{\alpha}{2\pi} + O(\alpha^2)
\]

- Matching to two loop beta function

\[
a = \beta_0
\]

\[
\beta_0 c + 3C_2(r)N_f b = \beta_1
\]

- Has same form as NSVZ.
• Assume $b$ and $c$ do not depend on the number of flavors

$$b = \frac{\beta_1(\bar{N}_f)}{3C_2(r)\bar{N}_f}$$

$$c = \frac{\beta_{YM}^1}{\beta_{YM}^0}$$

• With $\bar{N}_f$ being the critical number of flavors for which asymptotic freedom is lost

$$\beta(\alpha) = \frac{\alpha^2}{2\pi} \left[ \frac{\beta_0 + \frac{\beta_1(\bar{N}_f)}{3C_2(r)\bar{N}_f} N_f \gamma(\alpha)}{1 - \frac{\alpha}{2\pi} \frac{\beta_{YM}^1}{\beta_{YM}^0}} \right]$$

• With these coefficients the anomalous dimension appearing in $\beta(\alpha)$ at a Banks-Zaks fixed point coincides with the perturbative value.
Analysis similar to SUSY case:

- At zero of beta function
  \[ \gamma = - \frac{\beta_0 \left( N_f \right) \bar{N}_f}{\beta_1 \left( \bar{N}_f \right) N_f} 3C_2(r) \]

- Conformality requires
  \[ D(\tilde{\psi}\psi) = 3 - \gamma \geq 1 \quad \Rightarrow \quad \gamma \leq 2 \]

- Ladder approximation
  \[ \gamma = 1 \]

- Critical number of flavors
  \[ N_f = \frac{121C_2(G)C_2(r)}{2(7C_2(G) + 33C_2(r))T(r)} \]
Four loop beta function and anomalous dimension are known in MSbar scheme

\[ \beta_{4\text{-loop}}(\alpha) = -\beta_0 \frac{\alpha^2}{(2\pi)^1} - \beta_1 \frac{\alpha^3}{(2\pi)^2} - \beta_2 \frac{\alpha^4}{(2\pi)^3} - \beta_3 \frac{\alpha^5}{(2\pi)^4} + O(\alpha^6) \]

\[ \gamma_{4\text{-loop}}(\alpha) = \gamma_0 \left( \frac{\alpha}{\pi} \right) + \gamma_1 \left( \frac{\alpha}{\pi} \right)^2 + \gamma_2 \left( \frac{\alpha}{\pi} \right)^3 + \gamma_3 \left( \frac{\alpha}{\pi} \right)^4 + O(\alpha^5) \]

• New group invariants enter at the four loop level.
• Search for an infrared fixed point of the beta function.
• Evaluate the anomalous dimension at the infrared zero of the beta function
Four loop beta function

- Fundamental representation

<table>
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<tr>
<th>N</th>
<th>Nf</th>
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<th>( \alpha_{3l} )</th>
<th>( \alpha_{4l} )</th>
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<td>2</td>
<td>10</td>
<td>0.231</td>
<td>0.196</td>
<td>0.200</td>
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<table>
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<tr>
<th>N</th>
<th>Nf</th>
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<th>( \alpha_{3l} )</th>
<th>( \alpha_{4l} )</th>
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<td>2.21</td>
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<td>3</td>
<td>13</td>
<td>0.468</td>
<td>0.317</td>
<td>0.337</td>
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Four loop beta function

- Adjoint representation

<table>
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<tr>
<td>2</td>
<td>2</td>
<td>0.628</td>
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<td>2</td>
<td>0.314</td>
<td>0.2295</td>
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- 2-indexed symmetric representation

<table>
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<tr>
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<td>0.842</td>
<td>0.500</td>
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<td>3</td>
<td>3</td>
<td>0.085</td>
<td>0.079</td>
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Four loop anomalous dimension

- Fundamental representation

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<th>$\gamma_{2l}(\alpha_{2l})$</th>
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<th>$\gamma_{4l}(\alpha_{4l})$</th>
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<td>0.752</td>
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<td>0.204</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>0.275</td>
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<td>0.157</td>
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<td>2</td>
<td>10</td>
<td>0.0910</td>
<td>0.0738</td>
<td>0.0748</td>
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<table>
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<tr>
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<th>$\gamma_{4l}(\alpha_{4l})$</th>
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<tbody>
<tr>
<td>3</td>
<td>11</td>
<td>1.61</td>
<td>0.439</td>
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<tr>
<td>3</td>
<td>12</td>
<td>0.773</td>
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<td>0.253</td>
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<tr>
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<td>13</td>
<td>0.404</td>
<td>0.220</td>
<td>0.210</td>
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Four loop anomalous dimension

- Adjoint representation

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<th>Nf</th>
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<th>$\gamma_{3l}(\alpha_{3l})$</th>
<th>$\gamma_{4l}(\alpha_{4l})$</th>
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<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>0.820</td>
<td>0.543</td>
<td>0.500</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.820</td>
<td>0.543</td>
<td>0.523</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.820</td>
<td>0.543</td>
<td>0.532</td>
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</table>

- 2-indexed symmetric representation

<table>
<thead>
<tr>
<th>N</th>
<th>Nf</th>
<th>$\gamma_{2l}(\alpha_{2l})$</th>
<th>$\gamma_{3l}(\alpha_{3l})$</th>
<th>$\gamma_{4l}(\alpha_{4l})$</th>
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<tr>
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<td>2.44</td>
<td>1.28</td>
<td>1.12</td>
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<tr>
<td>3</td>
<td>3</td>
<td>0.144</td>
<td>0.133</td>
<td>0.133</td>
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Anomalous dimension vs. number of flavors
• N=3 and the fundamental representation

- \( N_f = 8 \) breaks chiral symmetry
  \( \gamma_{\beta\text{-function}} = 0.87 \quad \gamma_{4\text{-loop}} = NA \)
- \( N_f = 12 \quad ?? \)
  \( \gamma_{\beta\text{-function}} = 0.31 \quad \gamma_{4\text{-loop}} = 0.253 \)

• N=2 and the adjoint representation

- \( N_f = 2 \) is conformal
  \( \gamma_{\beta\text{-function}} = 0.46 \quad \gamma_{4\text{-loop}} = 0.500 \)

• N=3 and the 2-indexed symmetric representation

- \( N_f = 2 \quad ?? \)
  \( \gamma_{\beta\text{-function}} = 0.83 \quad \gamma_{4\text{-loop}} = 1.12 \)
• Kuti: Theory space is enormous.

• Construct walking technicolor models with multiple representations: Smallest possible naive $S$ parameter + smallest number of additional fermions.

• Other famous examples:

  1) SU(5) grand unified theory: $\bar{5} + 10$

  2) MSSM: $\text{Adj} + F$
Beta Function – Multiple Representations

\[
\beta(\alpha) = -\frac{\alpha^2}{2\pi} + \sum_{i=1}^{k} b(r_i) N_f(r_i) \gamma_i(\alpha) \left( 1 - \frac{\alpha}{2\pi} c \right)
\]

\[
\gamma_i(\alpha) = 3C_2(r_i) \frac{\alpha}{2\pi} + O(\alpha^2)
\]

- Consider the zero of the beta function. Bound the conformal house by bounding the values of the anomalous dimensions.
• There is a critical coupling associated to the triggering of each of the condensates

\[ \alpha_c(r_1) = \frac{\pi}{3C_2(r_1)} \quad \alpha_c(r_2) = \frac{\pi}{3C_2(r_2)} \]

• Assume: \( C_2(r_1) > C_2(r_2) \)  
  Conformal House: \( \alpha_c(r_1) = \alpha_{IR}(r_1, r_2) \)
Multiple Representations

- **Gauge charges:**

<table>
<thead>
<tr>
<th></th>
<th>$\text{SU}(2)_{\text{TC}}$</th>
<th>$\text{SU}(2)_L$</th>
<th>$\text{U}(1)_Y$</th>
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<tr>
<td>$(U_L, D_L)^T$</td>
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<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$U_R$</td>
<td>$\bar{2}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
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<tr>
<td>$D_R$</td>
<td>$\bar{2}$</td>
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<td>$-\frac{1}{2}$</td>
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<tr>
<td>$\lambda^f$</td>
<td>Adj</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- **Global symmetry:**

  $$SU(4) \times SU(2) \times U(1) \quad \rightarrow \quad Sp(4) \times SO(2) \times Z_2$$
Exceptional Groups (Fundamental)

\[ n_f \]

- \( G_2 \)
- \( F_4 \)
- \( E_6 \)
- \( E_7 \)
- \( E_8 / \text{Ad} \)

**Ladder**

- Four Loops \( \gamma^* = 1 \)
- All Orders \( \gamma^* = 1 \)
## Exceptional Groups (Adjoint)

<table>
<thead>
<tr>
<th>Adj</th>
<th>$G_2$</th>
<th>$F_4$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
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<td>2.75</td>
<td>2.75</td>
<td>2.75</td>
<td>2.75</td>
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<tr>
<td>4 loops</td>
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<td>1.72</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
</tr>
<tr>
<td>All orders</td>
<td>1.51</td>
<td>1.51</td>
<td>1.51</td>
<td>1.51</td>
<td>1.51</td>
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<td>Ladder</td>
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<td>2.08</td>
<td>2.08</td>
<td>2.08</td>
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• Similar for $N$ odd.
The End