Ward-Takahashi identities and restoration of supersymmetries in lattice $\mathcal{N} = 4$ super Yang-Mills

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Supersymmetry (SUSY) - interesting subject in its own right.

New particles, hierarchy problem, unification ... Expected to be significant in future experiments

Dark matter ...

- SUSY plays a significant role in many theories of BSM physics
  - MSSM - minimal extension of SM with $\mathcal{N} = 1$ SUSY
  - SUSY technicolor
  - Extra dimensional models
  - AdS/CFT duality, strings, black holes

- Many interesting features are non-perturbative. dynamical SUSY breaking, gaugino condensation, ...

- Need a non-perturbative definition of the theory $\rightarrow$ lattice construction.
Supersymmetric Yang–Mills (SYM) theories: many interesting features/results.

- confinement
- spontaneous chiral symmetry breaking
- strong coupling/weak coupling duality
- gauge theory/string theory duality

Needs lattice to study the strong coupling dynamics.

We focus on the 4D $\mathcal{N} = 4$ SYM

- It is a fascinating theory in itself
- Plays crucial role in AdS/CFT correspondence

Lattice version of this theory would allow:

- Strong coupling calculations, Monte Carlo simulations
- New ideas/approaches eg. (quantum) string corrections from finite $\mathcal{N}$, $\lambda \cdots$
The SUSY algebra, which is an extension of the Poincaré algebra, is explicitly broken on the lattice.

$\{Q, \bar{Q}\} = \gamma \cdot P$

One cannot realize infinitesimal translation on the lattice. So this relation is broken by discretization [Dondi and Nicolai 1977].

Folklore: Impossible to put SUSY on the lattice exactly.

Leads to (very) difficult fine tuning - lots of relevant SUSY breaking counter-terms in effective action.

$\mathcal{N} = 4$ SYM particularly difficult - contains scalar fields.
What if we could retain a subalgebra of the SUSY algebra on the lattice that does not generate translations?

**Exact lattice SUSY** allows this.

We could relabel the fields and supercharges of a class of SYM theories in a convenient way - a process called **topological twisting**.


Twisting leads to **exact lattice SUSY**.

- Positive energy states
- E=0 ground state
- fermion-boson spectrum degenerate under $Q$

In D=4 topological twisting results in a unique theory: the $\mathcal{N} = 4$ SYM.
**Exact lattice SUSY : Twisting $\mathcal{N} = 4$ SYM**

- $\mathcal{N} = 4$ SYM in 4D has additional flavor (R) symmetries: $SO_R(6)$.
- Twist: decompose fields under the diagonal subgroup of $SO(4)' = SO_{\text{Lorentz}}(4) \times SO_R(4)$


- Fermions: spinors under both factors - become integer spin after twisting
- Scalars transform as vectors under R-symmetry - vectors after twisting
- Gauge fields remain vectors - combine with scalars to make complex gauge fields. Gauge symmetry is still just $U(N)$.

*Twisting is just a change of variables in flat space.*
Field content: 4D gauge field, 6 scalars, 16 fermions

Twisting the theory leads to a theory compactly expressed as dimensional reduction of a 5D theory:

- 16 fermions: \( \Psi = (\eta, \psi_a, \chi_{ab}) \), \( a, b = 1, \cdots, 5 \)
- 10 bosons as 5 complex gauge fields:
  \[ A_a = A_a + iB_a, \ a = 1, \cdots, 5 \]
- 16 supercharges: \( (Q, Q_a, Q_{ab}), \ a = 1, \cdots, 5 \)

Action:

\[
S_{\mathcal{N}=4, D=4}^{\text{SYM}} = Q \int Tr \left( \chi_{ab}F_{ab} + 2\eta[\bar{D}_a, D_a] - \eta d \right) + S_{\text{closed}},
\]

\[
S_{\text{closed}} = \frac{1}{2} \int Tr \epsilon_{abcde} \chi_{de} \bar{D}_c \chi_{ab}.
\]

We have \( QS_{\mathcal{N}=4, D=4}^{\text{SYM}} = 0 \) \( \rightarrow \) action is \( Q \)-invariant.
The supercharge $Q$ is nilpotent: $Q^2 = 0$. A property that can be easily transported on to the lattice.

\[
Q A_a = \frac{1}{2} \psi_a, \quad Q \chi_{ab} = -\overline{F}_{ab}, \\
Q \overline{A}_a = 0, \quad Q \eta = \frac{1}{2} d, \\
Q \psi_a = 0, \quad Q d = 0
\]

Perform $Q$-variation and integrate $d$ to get more explicit form for the action:

\[
S_{\text{SYM}}^{N=4, D=4} = \int \text{Tr} \left[ -\overline{F}_{ab} F_{ab} + \frac{1}{2} [D_a, D_a]^2 - \frac{1}{2} \chi_{ab} D_{[a} \psi_{b]} - \eta \overline{D}_a \psi_a \right] + S_{\text{closed}}.
\]

Action on the lattice takes the form:

\[
S_{\text{SYM}}^{N=4, D=4} = \int \text{Tr} \left[ - (D_a^{(+)} U_b)^\dagger (x) (D_a^{(+)} U_b)(x) + \frac{1}{2} (\overline{D}_{a}^{(-)} U_a(x))^2 - \frac{1}{2} \chi_{ab}(x) D_{[a}^{(+)} \psi_{b]}(x) \\
- \eta(x) \overline{D}_{a}^{(-)} \psi_{a}(x) + \frac{1}{2} \epsilon_{abcde} \chi_{de}(x + \hat{e}_a + \hat{e}_b + \hat{e}_c) \overline{D}_{c}^{(-)} \chi_{ab}(x + \hat{e}_c) \right].
\]
Outstanding question: How to restore full SUSY?

- We have a lattice formulation for $\mathcal{N} = 4$ SYM. It is:
  - Local
  - Gauge invariant
  - Doubler free
  - Invariant under ONE $Q$

- There are 15 other SUSYs - $Q_a$ and $Q_{ab}$ - that are broken on the lattice.

- What about restoration of the full set of SUSY as we take the continuum limit?

- Do we have to deal with operator renormalization/mixing? tuning of lattice parameters?

- Apparent solution: Construct SUSY Ward-Takahashi (WT) identities and examine the restoration of SUSYs as we approach the continuum limit.
Outstanding question: How to restore full SUSY? [contd.]

To construct WT identities we need to know the other SUSY transformations.

We can make use of the discrete subgroups of the R-symmetries and the $Q$ symmetry.

We can find $Q_a$ and $Q_{ab}$ SUSY transformations of the fields.

Let us look at the following field interchanges (with index $a$ fixed):

\[
2\eta \rightarrow \psi_a, \quad \psi_a \rightarrow 2\eta, \quad \psi_b \rightarrow -\chi_{ab}
\]

\[
\chi_{ab} \rightarrow -\psi_b, \quad \chi_{bc} \rightarrow \frac{1}{2}\varepsilon_{bcagh}\chi_{gh}
\]

\[
\mathcal{D}_a \rightarrow \mathcal{D}_a, \quad \overline{\mathcal{D}}_a \rightarrow \overline{\mathcal{D}}_a, \quad \mathcal{D}_b \rightarrow \overline{\mathcal{D}}_b, \quad \overline{\mathcal{D}}_b \rightarrow \mathcal{D}_b
\]

The action is invariant under these field interchanges.
This leads to the SUSY transformations associated with $Q_a$:

\[
\begin{align*}
Q_a A_b &= \delta_{ab} \eta, \\
Q_a \overline{A}_b &= -\frac{1}{2} \chi_{ab}, \\
Q_a \psi_b &= \delta_{ab} d + (1 - \delta_{ab}) [\overline{D}_a, D_b], \\
Q_a \chi_{bc} &= -\frac{1}{2} \epsilon_{abcdg} F_{gh}, \\
Q_a \eta &= 0, \\
Q_a d &= 0.
\end{align*}
\]

Note: $Q_a^2 = 0$ in the continuum but not true on the lattice.
The $Q_{ab}$ SUSY transformations

The action is invariant under another set of field interchanges (with indices $a, b$ fixed).

\[ 2\eta \rightarrow \chi_{ab}, \quad \psi_a \rightarrow \psi_b, \quad \psi_b \rightarrow -\psi_a, \quad \psi_c = \frac{1}{2} \epsilon_{c a b g h} \chi_{g h} \]

\[ \chi_{ab} \rightarrow -2\eta, \quad \chi_{ac} \rightarrow \chi_{bc}, \quad \chi_{bc} \rightarrow -\chi_{ac}, \quad \chi_{c d} \rightarrow -\epsilon_{c d a b e} \psi_e \]

\[ D_{a, b} \rightarrow \overline{D}_{a, b}, \quad \overline{D}_{a, b} \rightarrow D_{a, b}, \quad D_c \rightarrow D_c, \quad \overline{D}_c \rightarrow \overline{D}_c \]

This leads to the SUSY transformations associated with $Q_{ab}$:

\[ Q_{ab} A_c = \frac{1}{4} \epsilon_{a b c g h} \chi_{g h}, \]

\[ Q_{ab} \overline{A}_c = \frac{1}{2} (\delta_{a c} \psi_b - \delta_{b c} \psi_a), \]

\[ Q_{ab} \psi_c = \epsilon_{a b c g h} \overline{F}_{g h}, \]

\[ Q_{ab} \chi_{c d} = \delta_{a c} \delta_{b d} d - \delta_{b c} [D_a, \overline{D}_d] + \delta_{a c} [D_b, \overline{D}_d], \]

\[ Q_{ab} \eta = \frac{1}{2} F_{a b}, \]

\[ Q_{ab} d = 0. \]

$Q_{a b}^2 = 0$ in the continuum but broken on the lattice.
WT identities and supercurrents

WT identities involving supercurrent and a local (or multi-local) operator.
Consider a composite operator $O(y)$. Its expectation value is

$$\langle O(y) \rangle = \frac{1}{Z} \int[d\Phi] \exp(-S[\Phi]) O(y).$$

Consider infinitesimal transformations of the fields

$$\Phi(x) \rightarrow \Phi'(x) = \Phi(x) + \delta_\kappa \Phi(x), \quad \delta_\kappa \Phi(x) = \delta_\kappa \Delta \Phi(x),$$

$\delta_\kappa$: infinitesimal Grassmann odd parameter, $\Delta$: deformation of the field.

The functional integral is independent of relabeling of integration variables:

$$\langle O'(y) \rangle - \langle O(y) \rangle = 0.$$
This gives the relation:

$$\langle \delta_\kappa SO(y) \rangle = \langle \delta_\kappa O(y) \rangle.$$  

Making the transformation position dependent:

$$\Phi(x) \rightarrow \Phi'(x) = \Phi(x) + \delta_\kappa(x) \Phi(x), \quad \delta_\kappa(x) \Phi(x) = \delta_\kappa(x) \Delta \Phi(x),$$

In the twisted theory we have

$$\delta_\kappa(x) = \delta_\kappa A(x) Q_A,$$

$$\delta_\kappa A(x) = (\delta_\kappa_0(x), \delta_\kappa_a(x), \delta_\kappa_{ab}(x)),$$

$$Q_A = (Q, Q_a, Q_{ab}).$$

The infinitesimal variation of the action gives:

$$\delta_\kappa(x) S = \int d^4x - (\partial_m \delta_\kappa A(x)) S^m_A(x), \quad S^m_A(x) = (S_0^m(x), S_a^m(x), S_{ab}^m(x)).$$

Here $S^m_A(x)$ are the supercurrents (Noether currents).
The associated WT relations are:

\[ \langle \partial_m S^m_A(x) O(y) \rangle = \delta^{(4)}(x - y) \langle Q_A O(y) \rangle. \]

We can check how strongly these relations hold on the lattice for a given operator \( O \).

A way of measuring the amount of SUSY breaking by the lattice.

The scalar supercurrent: \( \delta^{\text{scalar}}_\kappa S \rightarrow S^m_0 \).

In the continuum it is:

\[ S^m_0 = \text{Tr} \sum_n \bar{F}_{mn} \psi_n - \frac{1}{2} d\psi_m - \frac{1}{2} \sum_{n,c,g,h} \epsilon_{cnmgh} \bar{F}_{cn} \chi_{gh}. \]
WT identities on the lattice

Additional terms appear in the lattice WT equations: \( I_0^m(x), I_a^m(x), I_{ab}^m(x) \).

WT identities on the lattice:

\[
\partial_m \langle S_A^m(x) O(0) \rangle + \langle I_A^m(x) O(0) \rangle = \delta^{(4)}(x) \langle Q_A O(0) \rangle.
\]

We can write down expressions for the supercurrents on the lattice after performing a variation of the lattice action. Symmetry breaking terms \( I_A^m(x) \) also follow from such variation.

The scalar supercurrent on the lattice:

\[
S_0^m(x) = \text{Tr} \; \mathcal{F}_{mn}^\dagger(x) \mathcal{U}_m(x) \psi_n(x + \hat{e}_m) - \frac{1}{2} d(x + \hat{e}_m) \mathcal{U}_m^\dagger(x) \psi_m(x) \\
- \frac{1}{2} \epsilon_{cnmgh} \chi_{gh}(x + \hat{e}_c + \hat{e}_n + \hat{e}_m) \mathcal{U}_m^\dagger(x + \hat{e}_m) \mathcal{F}_{cn}^\dagger(x + \hat{e}_m).
\]

\[
l_0^m(x) = 0, \text{ for scalar SUSY}.
\]
Breaking terms are $O(a)$ artifacts.

$\langle l_a^m(x) \rangle \to 0$, $\langle l_{ab}^m(x) \rangle \to 0$ as we take the continuum limit.

Things to remember while choosing the operator $O$:

- It is a composite operator.
- May or may not be gauge invariant.
- Operator mixing can happen.
- Ultraviolet effects in correlators.

We could take an appropriate lattice supercurrent as the operator and study the WT relation.

\[
O(x) = S_A^m(x)
\]
\[
\partial_m \langle S_A^m(x) S_B^m(0) \rangle + \langle l_A^m(x) S_B^m(0) \rangle = \delta^{(4)}(x) \langle Q_A S_B^m(0) \rangle.
\]
Conclusions/Further developments

- We have a lattice $\mathcal{N} = 4$ SYM that preserves exactly one supersymmetry.

- Studying WT identities on the lattice is needed to examine the restoration of other SUSYs in the continuum limit.

- Need expressions for 1-form and 2-form lattice supercurrents and appropriately chosen operators.

- What can we say about the renormalization of the supercurrent operators?

- There are subtleties like operator mixing, ultraviolet effects etc. that one has to address.