Properties of Wavefunctions

Announcements:

• 2\textsuperscript{nd} exam is next Thursday (Nov. 7) in G2B90 from 7:30 – 9:00 pm.
• New Homework posted and due next week – it is from material in chapter 7. So I think it is part of studying.
• A past exam from 2011 has posted on D2L – both exam and solution.
What is it?
How will the detectors see gravitational waves?

When gravitational waves pass through LIGO's L-shaped detector they will decrease the distance between the test masses in one arm of the L, while increasing it in the other. These changes are minute: just $10^{-16}$ centimeters, or one-hundred-millionth the diameter of a hydrogen atom over the 4 kilometer length of the arm.

Measurement is performed by bouncing high-power laser light beams back and forth between the test masses in each arm, and then interfering the two arms' beams with each other. The slight changes in test-mass distances throw the two arms' laser beams out of phase with each other, thereby disturbing their interference and revealing the form of the passing gravitational wave.
Superposition

If $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are both solutions to the wave equation then the sum $\Psi_1(x,t) + \Psi_2(x,t)$ is also a solution.

Actually, any linear combination $A\Psi_1(x,t) + B\Psi_2(x,t)$ is a solution.

This is the superposition principle.

This seems pretty straightforward but is actually an important result.

Let’s look at the infinite square well solution.
Infinite Square Well

\[ \Psi(x,t) = \psi(x)\phi(t) = \sqrt{\frac{2}{a}} \sin \left(\frac{n\pi x}{a}\right)e^{-iE t/\hbar} \]

\( n \) is the quantum number

Note \( E_1, E_2, \ldots, E_n \) are called eigenvalues of the potential.

For each \( n \), can have a \( \psi_n(x) + \phi_n(t) \), corresponding to an eigenfunction for each eigenvalue. For each eigenfunction we have a wave function.

Can have a linear combination (superposition) of these states -

\[ \Psi(x,t) = a \Psi_1(x,t) + b \Psi_2(x,t) \]

where \( a \) and \( b \) are coefficients and are normalized so the overall probability is 100%.

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\[ \Psi(x,t) = a \Psi_1(x,t) + b \Psi_2(x,t) \]

Substituting this function into the Schrödinger equation,

\[ -\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} + V\Psi - i\hbar \frac{\partial\Psi}{\partial t} = 0 \]

Since both \( \Psi_1 \) and \( \Psi_2 \) satisfy the Schrödinger equation, both brackets must vanish.

\[ a \left[ -\frac{\hbar^2}{2m} \frac{d^2\Psi_1}{dx^2} + V\Psi_1 - i\hbar \frac{\partial\Psi_1}{\partial t} \right] + b \left[ -\frac{\hbar^2}{2m} \frac{d^2\Psi_2}{dx^2} + V\Psi_2 - i\hbar \frac{\partial\Psi_2}{\partial t} \right] \equiv 0 \]
We can write the wave function
\[ \Psi(x, t) = \sum_{n=1}^{\infty} a_n e^{-iE_n t / \hbar} \psi_n(x) \]
and the complex conjugate as
\[ \Psi^*(x, t) = \sum_{l=1}^{\infty} a_l^* e^{+iE_l t / \hbar} \psi_l^*(x) \]

Multiplying the two together we get:
\[ \Psi^*(x,t)\Psi(x,t) = \sum_{n=1}^{\infty} a_n^* a_n \psi_n^*(x)\psi_n(x) + \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} a_l^* a_n \psi_l^*(x)\psi_n(x) e^{-i(E_n-E_l)t / \hbar} \]

Note we see that in general the probability density depends on time. Means to locate the particle depends on the time at which the measurement is made.
For the special case of a particle whose associated wavefunction is one of the functions corresponding to a single eigenvalue \( E_n \), then

\[
\Psi^*(x,t)\Psi(x,t) = \sum_{n=1}^{\infty} a^*_n a_n \psi^*_n(x)\psi_n(x)
\]

Probability is independent of time. These are called stationary states or eigenstates of potential \( V(x) \). Also, the probability interpretation demands that any wave function must be *normalized*.

It is clear that the normalization can be achieved at any instant of time, by proper choice of \( a_n \), but it has to be satisfied at all times.
Math Properties cont.

\[ \int_{-\infty}^{\infty} \psi_n^*(x) \psi_n(x) \, dx = 1 \]

\[ \int_{-\infty}^{\infty} \psi_l(x) \psi_n(x) \, dx = 0 \quad l \neq n \]

Because of this property – product of one eigenfunction of the set times the complex conjugate of a different eigenfunction of the set vanishes, the eigenfunctions are said to be orthogonal.

Let’s try the product of eigenfunctions for the infinite square well.
\[ \Psi(x, t) = \psi(x)\phi(t) = \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi x}{a} \right)e^{-iEt/\hbar} \]

\[
\int \sin mx \sin nx \, dx = \frac{\sin(m - n)x}{2(m - n)} - \frac{\sin(m + n)x}{2(m + n)}
\]

\[
\int \sin^2 nx \, dx = -\frac{\cos nx \sin nx}{2} + \frac{nx}{2}
\]

\[
k = \frac{n\pi}{a}
\]

When evaluated at 0 and at \(a\) for the infinite square well only the squared eigenfunctions are retained.
Clicker question 1  Set frequency to DA

Q. What atomic element comes last alphabetically?

A. Beryllium
B. Copper
C. Arsenic
D. Barium
E. Zirconium
Clicker question 2

Q. What acronyms do physicists use to explain “Weakly interacting Massive Particles” and “Massive Compact Halo Objects”? 

A. Whales and Marlines
B. Wimps and Machos
C. Willies and Massives
D. None of the above

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probability density = $\Psi^* \Psi = A^* e^{-i(kx - Et / \hbar)} A e^{i(kx - Et / \hbar)}$

= $A^* A$

Probability flux = $\frac{\hbar k}{m} A^* A$

= $v$ probability density
Step Potential

Think of a charged particle moving along x-axis of a system of two electrodes held at different voltages.

\[ V(x) = \begin{cases} V_0, & x > 0 \\ 0, & x < 0 \end{cases} \]
Take a particle of mass m and energy $E < V_0$
Solving the Step Potential- Use TISE

\[ \psi(x) = A e^{iK_1x} + B e^{-iK_1x}, \quad x < 0 \]

where

\[ K_1 = \sqrt{2mE/\hbar} \]

\[ -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V_0 \psi(x) = E \psi(x), \quad x > 0 \]

\[ \psi(x) = C e^{K_2x} + D e^{-K_2x}, \quad x > 0 \]

where

\[ K_2 = \sqrt{2m(V_0 - E)/\hbar} \]

and \( E < V_0 \)

To verify this, calculate

\[ \frac{d^2\psi(x)}{dx^2} = CK_2^2 e^{K_2x} + D(-K_2)^2 e^{-K_2x} \]

\[ = K_2^2 \psi(x) = \frac{2m(V_0 - E)}{\hbar^2} \psi(x) \]

Arbitrary constants \( A, B, C, D \) must satisfy boundary conditions ie \( C = 0 \)
Continuity of wave function at x=0 yields:

\[ D(e^{-K_1 \chi})_{x=0} = A(e^{iK_1 \chi})_{x=0} + B(e^{-iK_1 \chi})_{x=0} \]

or \[ D = A + B \] (1)

Continuity of derivative of wave function at x=0 yields

\[ -K_2 D(e^{-K_1 \chi})_{x=0} = iK_1 A(e^{iK_1 \chi})_{x=0} - iK_1 B(e^{-iK_1 \chi})_{x=0} \]

or

\[ \frac{iK_2}{K_1} D = A - B \] (2)

Adding (1) and (2) gives

\[ A = \frac{D}{2} \left(1 + \frac{iK_2}{K_1}\right) \]

Subtracting (1) and (2) gives

\[ B = \frac{D}{2} \left(1 - \frac{iK_2}{K_1}\right) \]
\[
\frac{D}{2} (1 + iK_2/K_1)e^{iK_1x} + \frac{D}{2} (1 - iK_2/K_1)e^{-iK_1x}, \quad x \leq 0
\]
\[
\psi(x) = De^{-K_1x},
\]
\[
\psi(x) = De^{-K_1x},
\]
\[
\Psi(x, t) = \frac{Ae^{-iEt/\hbar} e^{iK_1x} + Be^{-iEt/\hbar} e^{-iK_1x}}{De^{-iEt/\hbar} e^{-K_1x}}, \quad x \leq 0
\]
\[
\Psi(x, t) = \frac{Ae^{-iEt/\hbar} e^{iK_1x} + Be^{-iEt/\hbar} e^{-iK_1x}}{De^{-iEt/\hbar} e^{-K_1x}}, \quad x \geq 0
\]
\[
S(x,t) = \frac{i\hbar}{2m} \left[ \Psi^*(x,t) \frac{\partial \Psi(x,t)}{\partial x} - \Psi(x,t) \frac{\partial \Psi^*(x,t)}{\partial x} \right]
\]
\[
S(x,t) = \frac{vA^*A - vB^*B}{2m}
\]
\[
S(x,t) = \frac{vA^*A - vB^*B}{2m}, \quad x < 0 \quad \text{where} \quad v = \frac{\hbar K_1}{m}
\]
\[
R = \frac{vB^*B}{vA^*A} = \frac{B^*B}{A^*A} = \frac{(1 - iK_2/K_1)^*(1 - iK_2/K_1)}{(1 + iK_2/K_1)^*(1 + iK_2/K_1)}
\]
\[
R = \frac{(1 + iK_2/K_1)(1 - iK_2/K_1)}{(1 - iK_2/K_1)(1 + iK_2/K_1)} = 1
\]