Kähler-Dirac fermions on Euclidean dynamical triangulations

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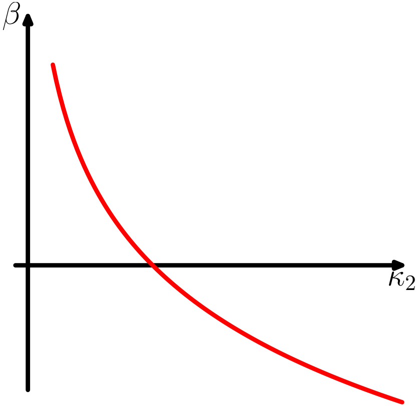
Lattice Quantum Gravity

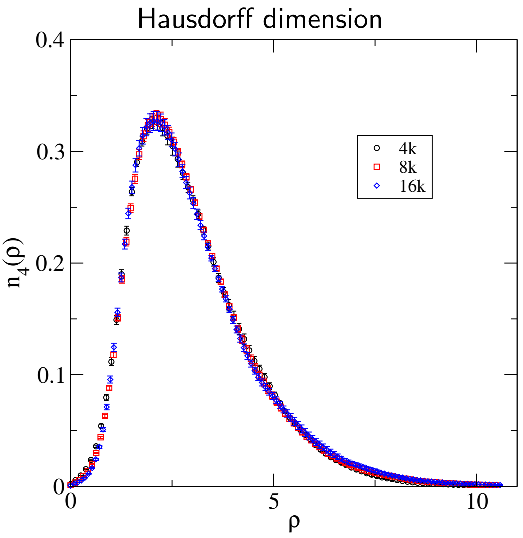
- We work with Euclidean Dynamical Triangulations (EDT) with spherical topology
- Euclidean space-time is "triangulated" with identical 4D simplices.
- We work under the asymptotic safety hypothesis
  $\rightarrow$ gravity is strongly coupled in the UV
- Ref. [Laiho et al., 2017] looked into EDT with a measure term.

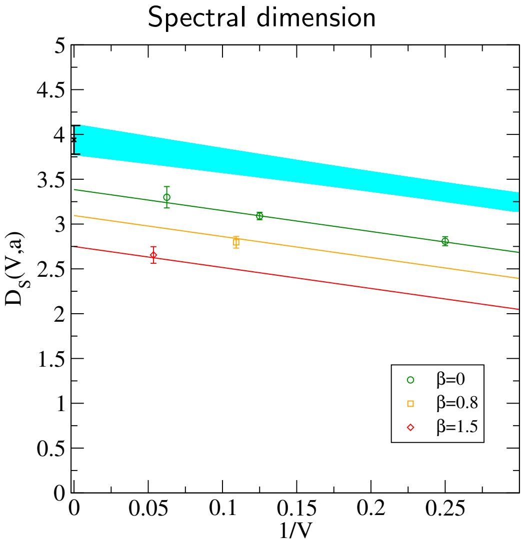


Prerequisites for valid model:

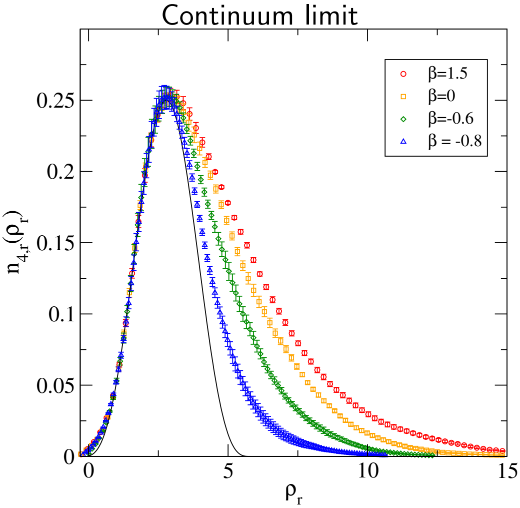
- Should have Hausdorff dimension $= 4$
- 4D de Sitter $\rightarrow$ spectral dimension
- Be able to take a continuum limit

Lattice Quantum Gravity





Lattice Quantum Gravity



Coupling to matter

- Coupling lattice gravity to scalar matter has been done [de Bakker and Smit, 1994, Hamber and Williams, 1994, de Bakker and Smit, 1997, Ambjørn et al., 2011]
- Coupling lattice gravity to gauge fields has also been explored [Renken et al., 1994, Bilke et al., 1998]
- Lattice fermions on curved lattice spacetimes are less understood [Jurkiewicz et al., 1988, Ambjørn and Varsted, 1991, Brower et al., 2017]
- Here we consider the most natural[1] version of fermion fields to couple to lattice gravity.
  - $\rightarrow$ **Kähler-Dirac fermions**
- We considered the "quenched" approximation
  - $\rightarrow$ A fluctuating background with no back-reaction from the fermions

[1] In the sense that construction is easy.

Why Kähler-Dirac fermions?

- Old friends → staggered fermions in flat-space hypercubic lattices from lattice gauge theory
- No need to devise a spin-connection
- No need to invent a lattice action
- Should be 4× copies of Dirac Fermions in continuum, infinite volume, small curvature limit.

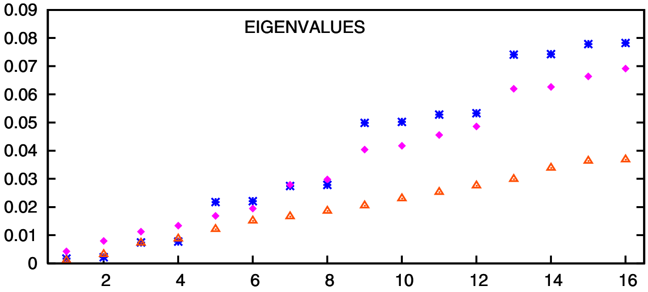


Fig. 2 [Follana et al., 2005]

Kähler-Dirac fermions in the continuum

- Simplest view is from the Laplace-de Rham operator:

$$\nabla^2 = (\mathrm{d} - \delta)^2 = -(\mathrm{d}\,\delta + \delta\,\mathrm{d})$$

  with d the exterior derivative, and δ its adjoint.
- Following Dirac's intuition, the Kähler-Dirac operator is

$$D = (\mathrm{d} - \delta)$$

- It's anti-Hermitian
- The spin-connection (and covariant derivative for that matter) are hidden inside d and δ.

Kähler-Dirac fermions in the continuum

Consider the action of $(\gamma^\mu \partial_\mu + m)$ on ψ.

- promote ψ to a 4×4 matrix.
- the γs form a basis for these.
- $\psi = f_0 + f_\mu \gamma^\mu + \frac{1}{2} f_{\mu\nu} \gamma^\mu \gamma^\nu + \dots$
- Any 4×4 matrix would work.
- This gives four identical copies of the Dirac equation.

$$\psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}$$

e.g.

$$\psi_{10} = \psi_2 = f_0 \mathbb{1}_{10} + f_\mu \gamma^\mu_{10} + \frac{1}{2} f_{\mu\nu} \gamma^\mu_{1a} \gamma^\nu_{a0} + \dots$$

Kähler-Dirac fermions in the continuum

- Kähler-Dirac fields are a combination of p-forms

$$\omega = f_0 + f_\mu dx^\mu + \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu + \dots$$

$$\psi = f_0 + f_\mu \gamma^\mu + \frac{1}{2} f_{\mu\nu} \gamma^\mu \gamma^\nu + \dots$$

- $\gamma^\mu \partial_\mu$ acts identically on ψ as $\mathrm{d} - \delta$ on ω.
- However this operator is still valid for any curved space-time
- Unfortunately, in curve spaces:

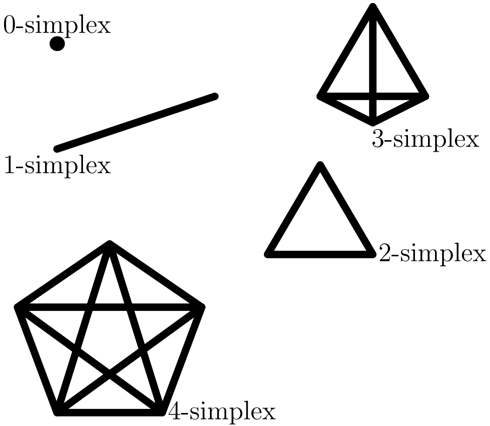
  Kähler-Dirac operator $\neq$ Dirac operator.
- Nevertheless, when the curvature is negligible:

  Kähler-Dirac operator $=$ Dirac operator[4]

Kähler-Dirac fermions on the lattice

Lattice gravity

The lattice is built from simplices of various dimensions.



- (co)Homology theory tells us that there is a straightforward transcription to the lattice:

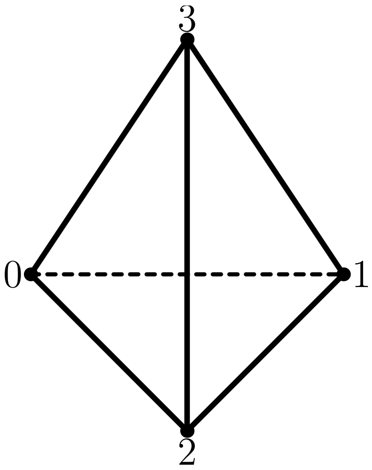
$$\mathrm{d} \mapsto \bar{\mathrm{d}}, \quad \delta \mapsto \bar{\delta}$$

- $\bar{\delta}$ is the simplex boundary operator
- Given a set of simplex vertices $\{0,1,2,3\}$:

$$\bar{\delta}\{0,1,2,3\} = \{1,2,3\} - \{0,2,3\} + \{0,1,3\} - \{0,1,2\}$$

for example.
- $\bar{\mathrm{d}}$ is its transpose.

$$p\text{-forms} \mapsto p\text{-simplices}$$



Kähler-Dirac fermions on the lattice

- With this transcription, the continuum results still hold on the lattice:

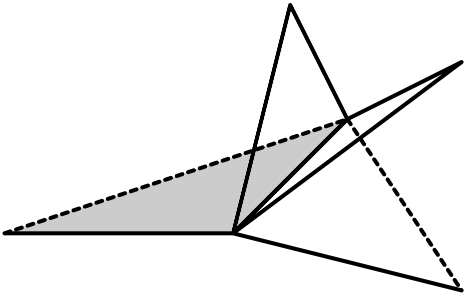
$$\bar{\nabla}^2 = (\bar{\mathrm{d}} - \bar{\delta})^2 = -(\bar{\mathrm{d}}\bar{\delta} + \bar{\delta}\bar{\mathrm{d}})$$

  is the lattice Laplacian, however in all simplex sectors, not just the 4- and 0-simplex sectors.

- The lattice Kähler-Dirac operator

$$\bar{D} = \bar{\mathrm{d}} - \bar{\delta}$$

  then carries all the same information as the lattice Laplace-de Rham operator.



Results

We looked at:

- the eigenvalues of the Kähler-Dirac matrix
  - $\rightarrow$ restore spectrum degeneracy in $V \rightarrow \infty$, continuum limit.
- correlations
  - $\rightarrow$ see meson masses tend towards degeneracy in chiral, $V \rightarrow \infty$, continuum limit.
- condensates
  - $\rightarrow$ remnant chiral symmetry should be unbroken in chiral, $V \rightarrow \infty$ limit.

Eigenvalues

We can add a real mass term and consider the spectral decomposition of the Kähler-Dirac operator

$$\bar{D} + m = \sum_{n=1}^{N} (i\lambda_n + m)|n\rangle\langle n|$$

Consider a similarity transform for transposition for $\bar{D}$, Γ,

$$\Gamma \bar{D} \Gamma^{-1} = \bar{D}^T = -\bar{D}$$

Then the Γ anti-commutes with $\bar{D}$ and ensures the eigenvalues come in complex conjugate pairs.

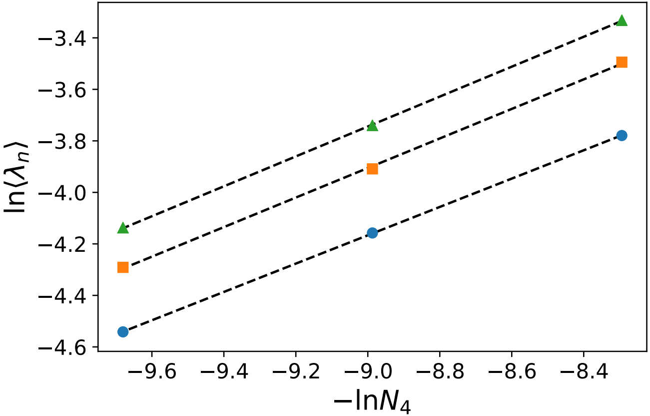
$$\Gamma = \bigoplus_{p=0}^{4} (-1)^p \quad \Longrightarrow \quad e^{i\Gamma\theta} \text{ is a } U(1) \text{ symmetry}$$

Eigenvalues

We looked at the finite-size scaling of the lowest eigenvalues.

$$\langle \lambda_n \rangle \sim \left(\frac{1}{N_4}\right)^{p_n}$$

- We have multiple volumes for $\beta = 0$ and $\beta = 1.5$ and can extract p in those cases.

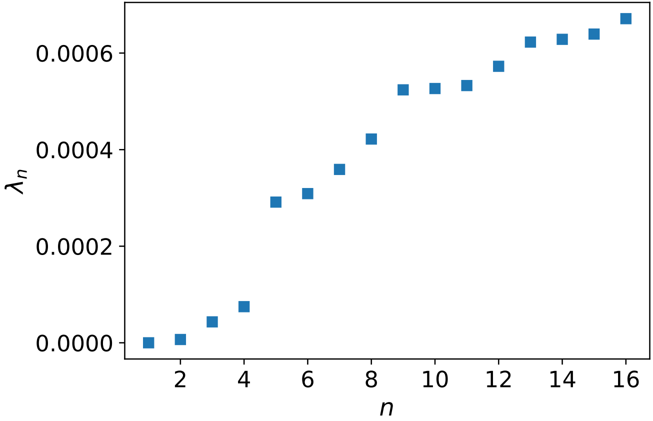


Eigenvalues

- Fitting to an ansatz

$$\langle \lambda_n \rangle = \frac{a_n}{N_4^{p_n}} + b_n$$

  we can extract the infinite volume value.
- For $\beta = 0$ we see a hint of clustering in the first 16 eigenvalues.

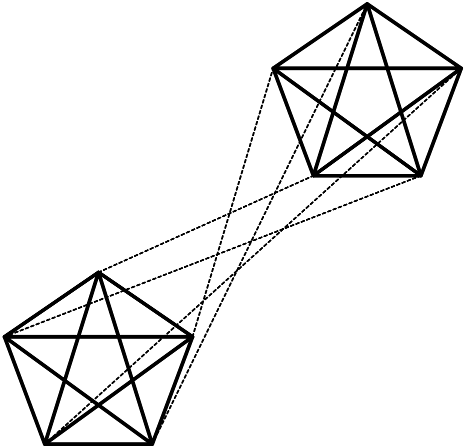


Correlations

- The inverse Kähler-Dirac operator can be written cleanly as

$$K^{-1} = (\bar{D} + m)^{-1} = \frac{K}{K^2}$$

- This matrix records the correlations between simplices
- There are a few possible (legitimate) definitions of distance for the simplices.
- We used smearing over the 4-simplex for its simplicity

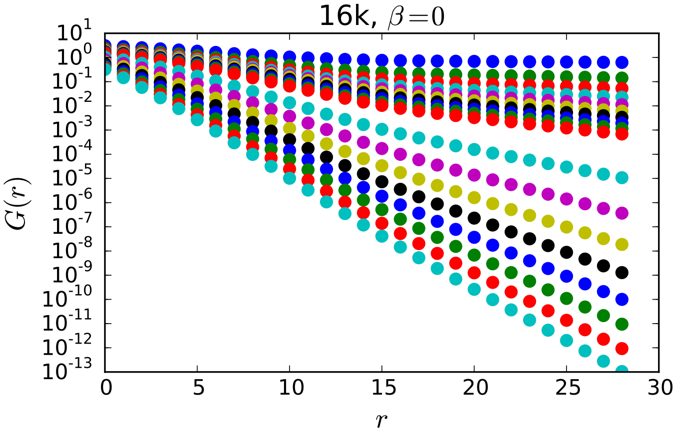


Correlations

- We fit to the ansatz that at long distances:

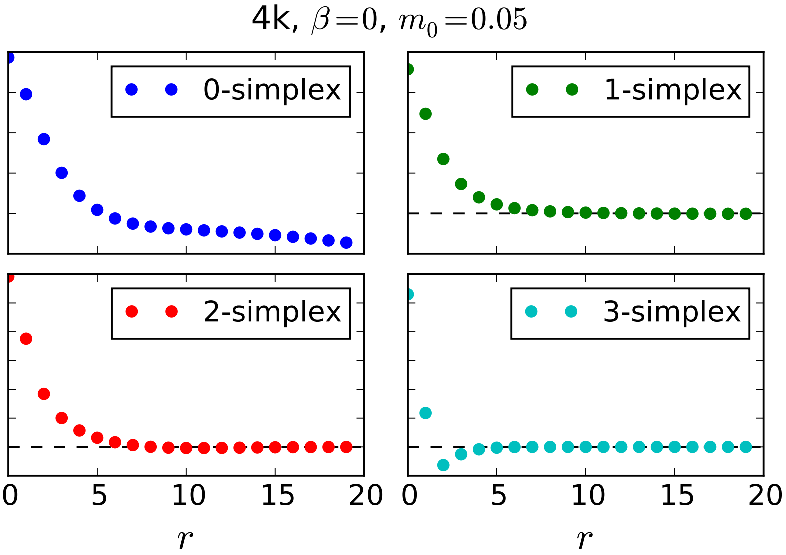
$$G(r) \sim e^{-mr} \text{ and } e^{-mr}/r^{\alpha}$$

- we find this empirically for a wide range of probe masses.
- We find the meson propagators also have this form.



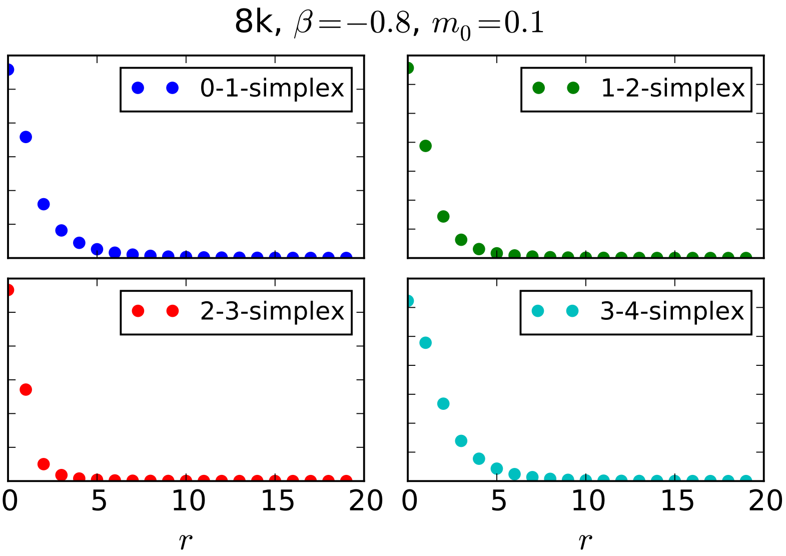
Correlations

- There are nine types of correlations
- Diagonal blocks are given by the inverse Laplacian
- These are p to p-simplex correlators
- off-diagonal blocks correlate p- and $p \pm 1$-simplices

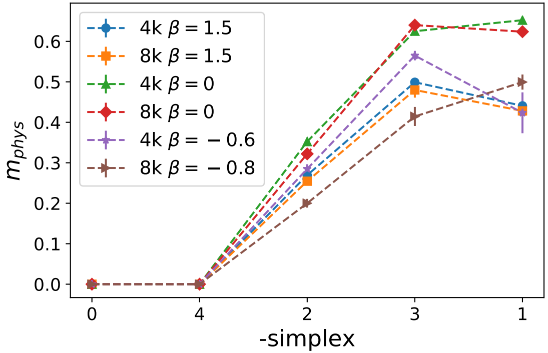


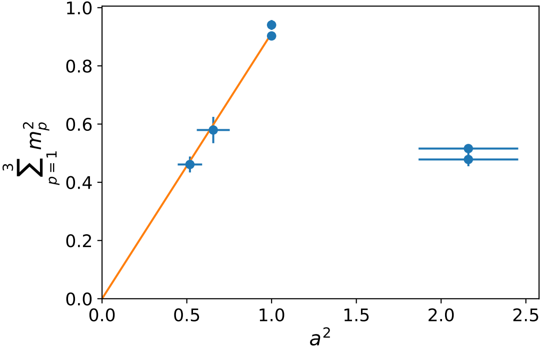
Pion-like correlators

- Off-diagonal correlators of Pion-like mesons.
- $\langle \omega_x \Gamma \bar{\omega}_x \omega_y \Gamma \bar{\omega}_y \rangle \rightarrow \langle (\bar{\omega}_x \omega_y)(\bar{\omega}_x \omega_y)^\dagger \rangle$
- Γ anti-commutes with D ($\bar{D}$)
- All-in-all there are nine pion-like mesons one can make directly from the *p*-simplex sectors.



Pion-like correlators





Condensates

We can also consider the diagonal of $(\bar{D}+m)^{-1}$. This is the bilinear condensate.

- We used Z_2 stochastic noise to extract the diagonal [Dong and Liu, 1994]

$$\langle \eta_i \eta_j \rangle = \delta_{ij}, \quad \langle \eta_i \rangle = 0$$

$$(\bar{D}+m)X = \eta \implies \langle \eta X \rangle = \langle \eta (\bar{D}+m)^{-1} \eta \rangle = (\bar{D}+m)^{-1}$$

- We used mini-ensembles of stochastic vectors, and averaged over EDT configs.
- We considered

$$\mathrm{Tr}[(\bar{D}+m)^{-1}] = \mathrm{Tr}[\langle \eta X \rangle] \sim \sum_x \langle \bar{\omega}_x \omega_x \rangle$$

and

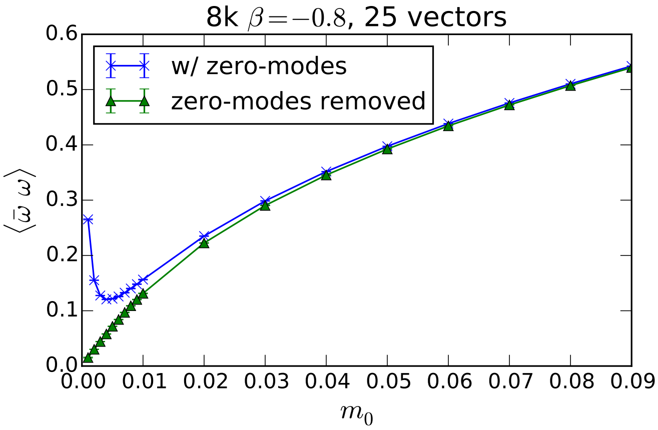
$$\mathrm{Tr}[\langle \eta X \rangle^2] \sim \sum_x \langle (\bar{\omega}_x \omega_x)^2 \rangle$$

Condensates

- In the un-normalized case we know the small-mass scaling:

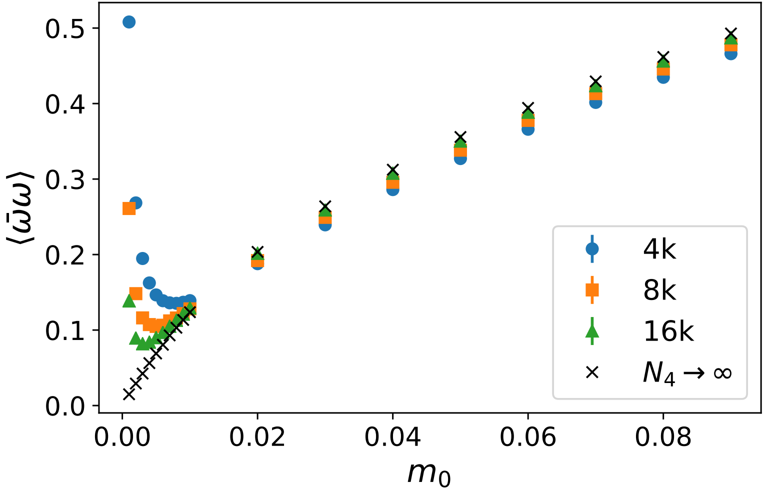
$$\langle \bar{\omega}\omega \rangle = \frac{2}{m_0} + 2m_0 \sum_n \frac{1}{\lambda_n^2 + m_0^2}$$

  there are two zero modes which scale $\sim 1/m_0$.
- With the zero-modes removed, the Γ-related symmetry is unbroken spontaneously in the chiral limit.



Condensates

for the $\beta = 0$ ensembles:

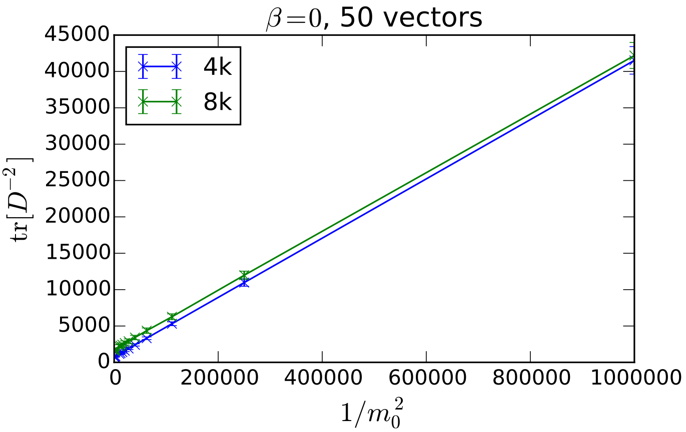


Condensates

- For the four-fermion condensate, it should go $\sim 1/m_0^2$ for small m_0.
- In fact, by naïve power counting, it should contribute to Z.

$$\rightarrow \det[\bar{D} + m_0] \sim m_0^2$$
$$\implies \det[\bar{D} + m_0](\bar{\omega}\omega)^2 \sim 1$$

- This condensate should contribute to the mass of the fermions, and chiral symmetry is unbroken.



Conclusions & future work

- Kähler-Dirac fermions can be put on dynamical triangulations straightforwardly
- We expect the large-volume, small-curvature limit is similar to Dirac fermions
- As we approach the continuum limit we see degeneracy restoring (four copies of Dirac fermions)
- remnant Chiral symmetry is unbroken
- Vanishing discretization effects, and promising phenomenology lend support to asymptotically safe gravity.
- *To do:* Simulations with dynamical fermions

Thank you!

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