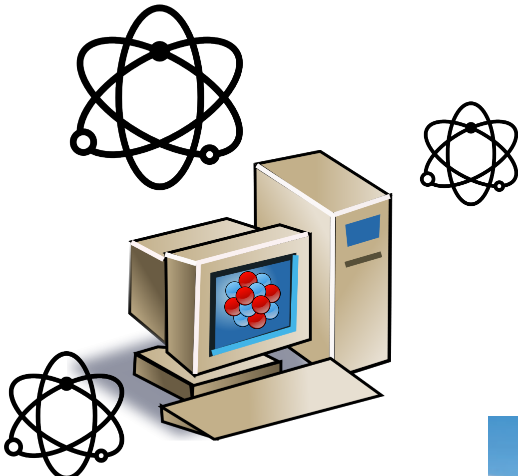
Digital Quantum Simulation of Lattice Gauge Theories



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Work done w/ David B. Kaplan (INT),
Indrakshi Raychowdhury (I.Ass. Kolkata)

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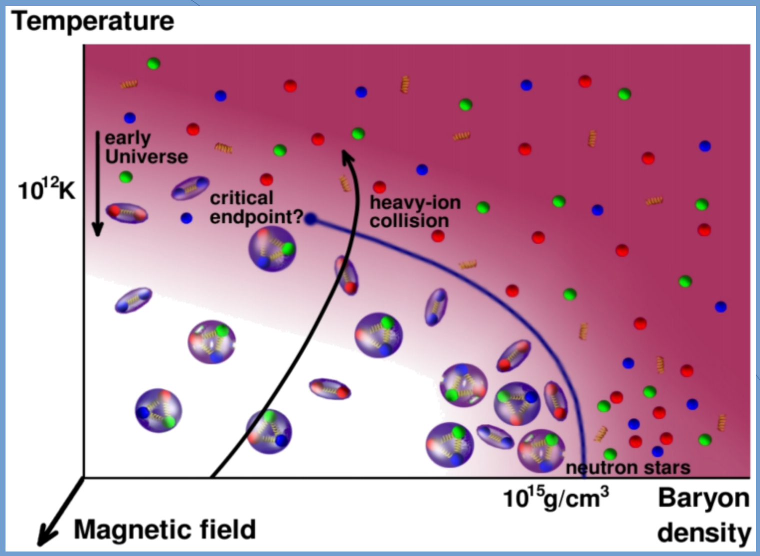
Outline

- **Big Picture**
- **Kogut-Susskind Lattice Gauge Theory**
  - Bugs
- **Gauge-Invariant Variables: U(1)**
- **Working with Non-Gauge-Invariant Variables**
- **Non-Abelian Hilbert space**
- **Summary & Future Directions**

Big Picture

Physics targets:

- **Simulation of QCD**
  - Hadronization
  - Detailed understanding of nuclear interactions
- **Complete phase diagram of QCD**
- **Nuclear equation of state**



One branch of reaching these goals:

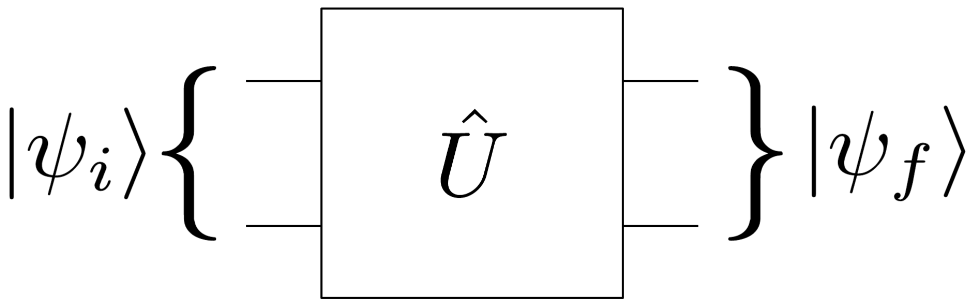
- **Quantum simulate QCD degrees of freedom**

- **Special complications present for any gauge theory simulation**

Conjectured phase diagram credit: G. Endrödi J.Phys.Conf.Ser. 503 (2014) 012009

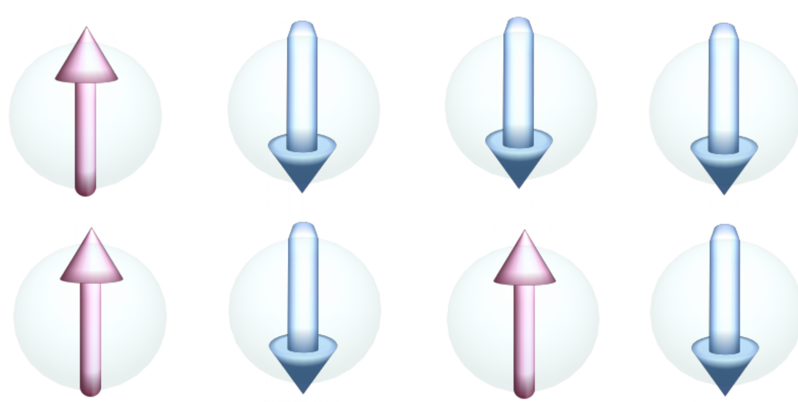
Digital Quantum Simulation

Digital quantum computers (QC):



Near-term QC architectures will have very limited capabilities

- **How to most wisely spend those qubits?**



- Unitary gates: $e^{-it\hat{H}}$ with your favorite Hamiltonian
- Want to simulate quantum field theory non-perturbatively → Lattice quantum field theory
  - QCD: a lattice gauge theory (LGT)

How to map the Hilbert space $\mathcal{H}$ and $\hat{H}$ on to the QC?

Hamiltonian Lattice Gauge Theory I

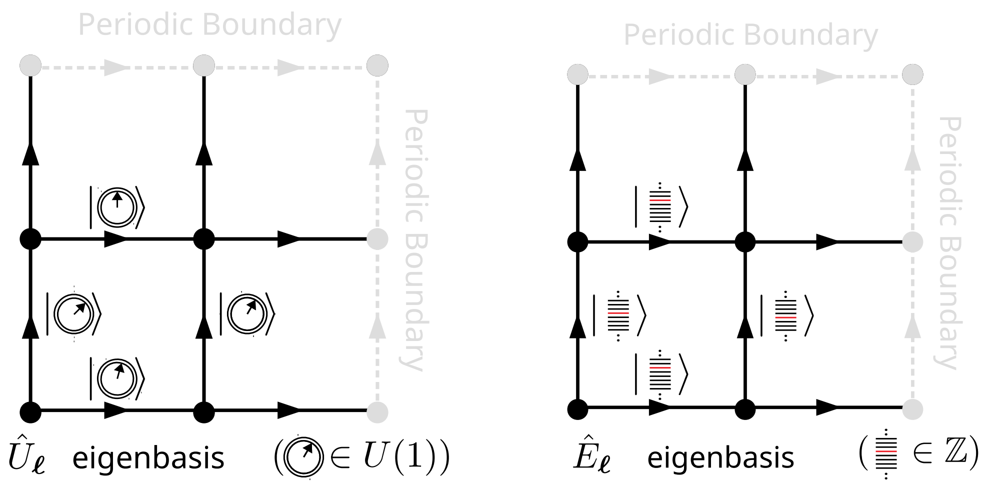
LGT Hilbert space, operator algebra:

- U(1)

$$\langle \phi | q \rangle = \frac{1}{\sqrt{2\pi}} e^{i\phi q}$$

group element or "coordinate" basis

representation or "momentum" basis



$$[E, U] = U$$

Canonical (same-link) commutation relation

$$U |q\rangle = |q+1\rangle$$

"*U* raises *E*"

$$\hat{H}_E = \frac{g^2}{2} \sum_{n,i} \hat{E}^2_{n,i} \qquad \hat{H}_B = -\sum_n \frac{1}{2g^2} \mathrm{Re}(\hat{U}_{n,\square})$$

Hamiltonian Lattice Gauge Theory II

LGT Hilbert space, operator algebra:

- Non-Abelian Lie group

Gauge transformations: $\hat{U}_{n,i} \to \Omega_n \hat{U}_{n,i} \Omega^\dagger_{n+e_i}$

$$[E^a_{L/R}, E^b_{L/R}] = i f^{abc} E^c_{L/R}$$
$$[E^a_R, U] = U T^a$$
$$[E^a_L, U] = -T^a U$$

Left, right electric fields to generate the independent left, right rotations.

$$\hat{H}_E = \frac{g^2}{2} \sum_{n,i} \hat{E}^a_{n,i} \hat{E}^a_{n,i} \qquad \hat{H}_B = -\sum_n \frac{1}{2g^2} \mathrm{tr}(\hat{U}_{n,\Box} + \hat{U}^\dagger_{n,\Box})$$

$$\langle g | j, m, m' \rangle = \sqrt{\frac{d_j}{|G|}} D^{(j)}_{m,m'}(g)$$

group element

representation state

"*U* mixes representations"

$$U_{m,m'} |j, M, M'\rangle =$$
$$C_+ |j + 1/2, M + m, M' + m'\rangle$$
$$+ C_- |j - 1/2, M + m, M' + m'\rangle$$

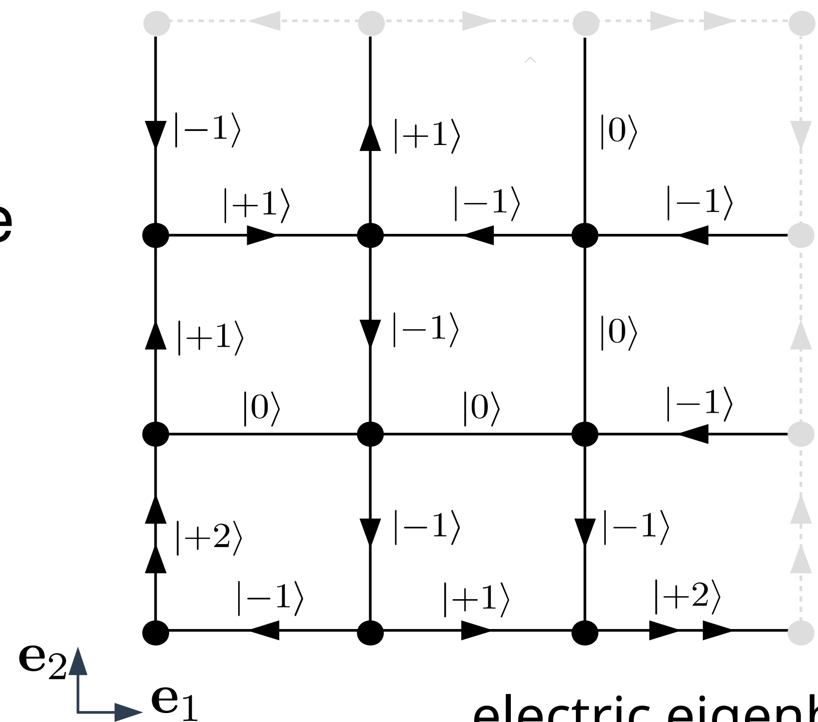
Hamiltonian Lattice Gauge Theory III

Plus Gauss law constraints

Gauss law ↔ gauge invariance

U(1) $\nabla \cdot \mathbf{E} - \rho = 0$

$\hat{\mathcal{G}}_n$



electric eigenbasis

SU(N) $\nabla \cdot \mathbf{E}^a - \rho^a = 0$

(also $E^a_L E^a_L = E^a_R E^a_R$)

lattice discrete "divergence"

$\hat{\mathcal{G}}^a_n$

Bugs in This Framework

- Qubits <u>wasted</u> on unphysical states

- Quantum noise <u>will</u> create components along unphysical directions
- In practice, gauge invariance could suffer systematic errors from <u>approximated evolution</u>

Gauge-Invariant Variables I

Q: Build gauge invariance in from start?

- **Start simple:**
  - Pure U(1) in d=2+1
  - Consider strong-coupling vacuum (SCV): $|\Omega\rangle \equiv \otimes_{\boldsymbol{\ell}} |0\rangle_{\boldsymbol{\ell}}$

<u>Ask:</u> What does *H* **do**? (States connected to SCV?)

$$\begin{aligned} \hat{H} &= \hat{H}_E + \hat{H}_B \ , \\ \hat{H}_B &= \frac{1}{2a_s}\left[\frac{1}{\tilde{g}_s^2}\sum_{\boldsymbol{p}}\left(2 - \hat{P}_{\boldsymbol{p}} - \hat{P}_{\boldsymbol{p}}^\dagger\right)\right] , \\ \hat{H}_E &= \frac{1}{2a_s}\left[\tilde{g}_t^2 \sum_{\boldsymbol{\ell}} \hat{\mathcal{E}}_{\boldsymbol{\ell}}^2\right] \ . \end{aligned}$$

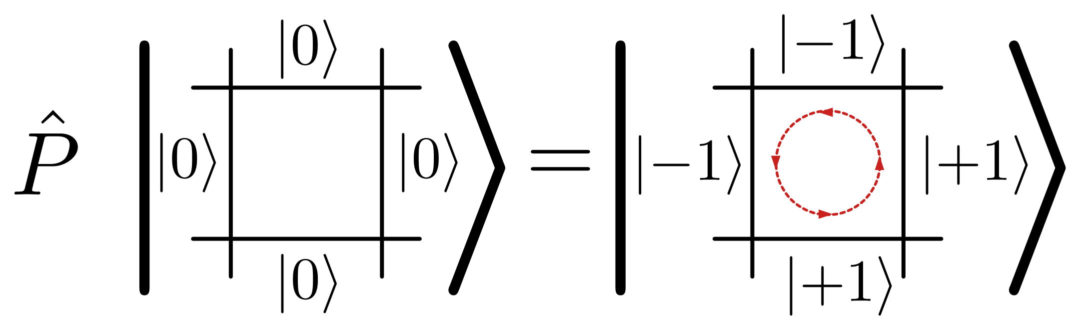
DB Kaplan & JRS, 1806.08797

See also work by Y. Meurice, J. Unmuth-Yockey et al.

Gauge-Invariant Variables II

What does *H* do in electric basis?

- $\hat{H}_E \supset \hat{\mathcal{E}}_{\boldsymbol{\ell}}$: **just multiplies by eigenvalue**
- $\hat{H}_B \supset \hat{P}_{\boldsymbol{p}}$: **excite electric flux loops**
- That's really it.



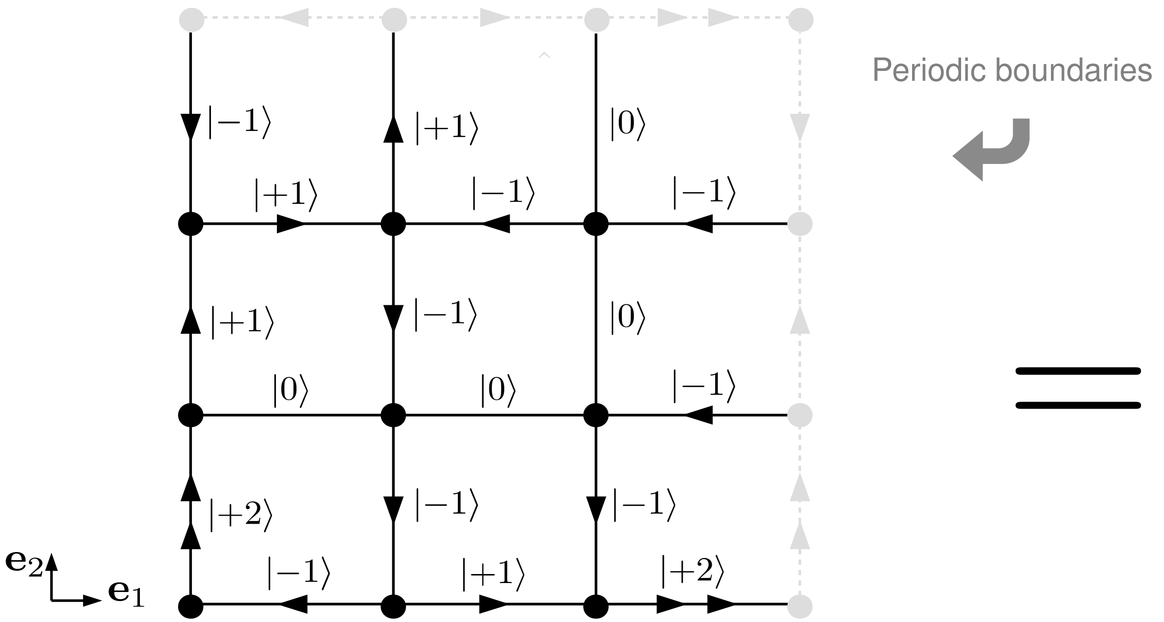
$\Rightarrow$ **Basis for $\mathcal{H}_{\text{phys}}$ is generated by acting with plaquettes on SCV**

Gauge-Invariant Variables III

Integer scalar field

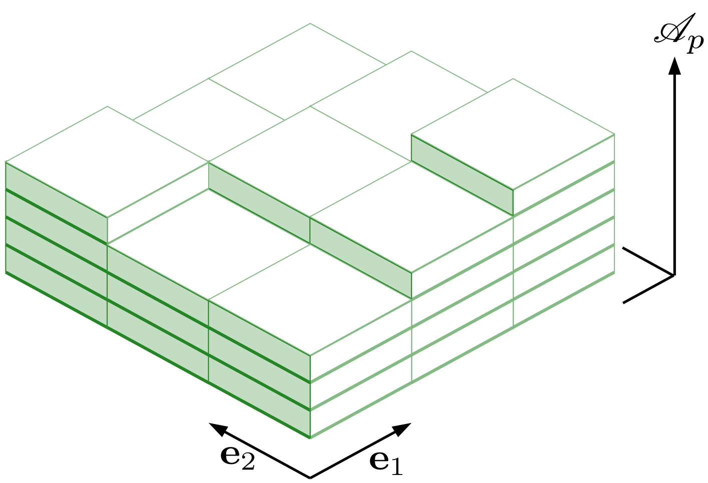
$$|\mathscr{A}_L\rangle \equiv \prod_{\boldsymbol{p}} \left(\hat{P}_{\boldsymbol{p}}\right)^{\mathscr{A}_{\boldsymbol{p}}} |\Omega\rangle$$

Plaquette powers, $\mathscr{A}_{\boldsymbol{p}}$: Encoding for valid $\mathcal{E}_{\boldsymbol{\ell}}$ configurations... or *now quantum numbers*



$\hat{E}_{\boldsymbol{\ell}}$ eigenbasis

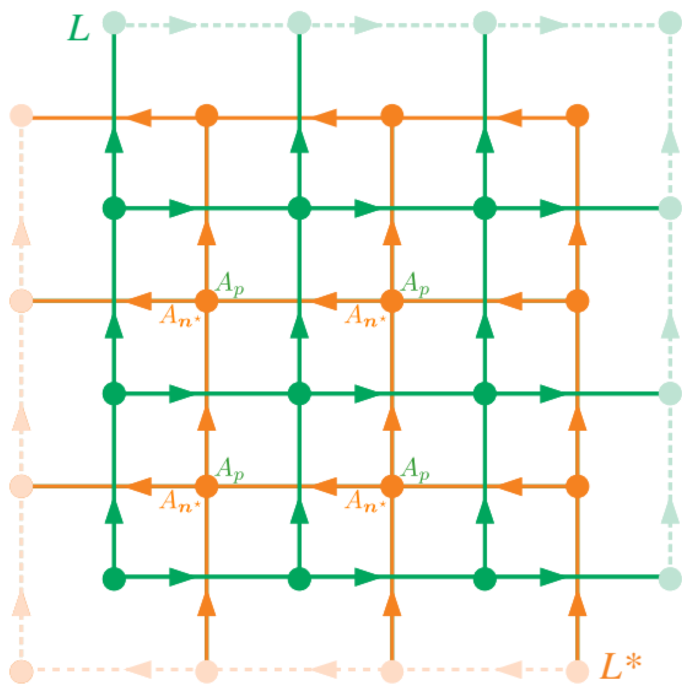
$=$



Number of plaquette operator applications

U(1) Punchline

- **d=2+1 E&M is dual to a scalar field theory (known) – dual uses less variables**
  - Naturally emerges from gauge invariant building blocks
  - Resulting *H* describes the same physics more concisely



Limitations

- $\oint d^2x \; B = 0$ **is not automatic**
  - **Enforcing for 2+1 (w/ periodic BC) impractical for large volumes**
- **Not obvious how generalize to matter, non-Abelian gauge groups**

Want to preserve: Local Hilbert spaces, Hamiltonian built from local operators, local constraints

Living with Gauss's law

How to live with local Gauss law constraints?

Immediate issues:

- **Preparing gauge invariant initial states**
- **Protecting digital quantum simulation from unphysical errors**
- **Is digitized time evolution actually gauge invariant?**

These highlight a more basic problem.

How to even validate a state without collapsing it?

Oracles for U(1) Gauge Invariance I

Cannot measure $\mathbf{E}$, ρ without collapsing $|\Psi\rangle$

→ Measure $\nabla\cdot\mathbf{E} - \rho$; project to $\mathcal{H}_{\text{phys}} / \mathcal{H}_{\text{unphys}}$

JRS, Phys. Rev. A **99**, 042301 (2019)

Internally like a quantum Grover search

|phys> → –|phys>

|unphys> → |unphys>

- **Discretize continuous errors, protect against (mitigate) bit-flip errors**

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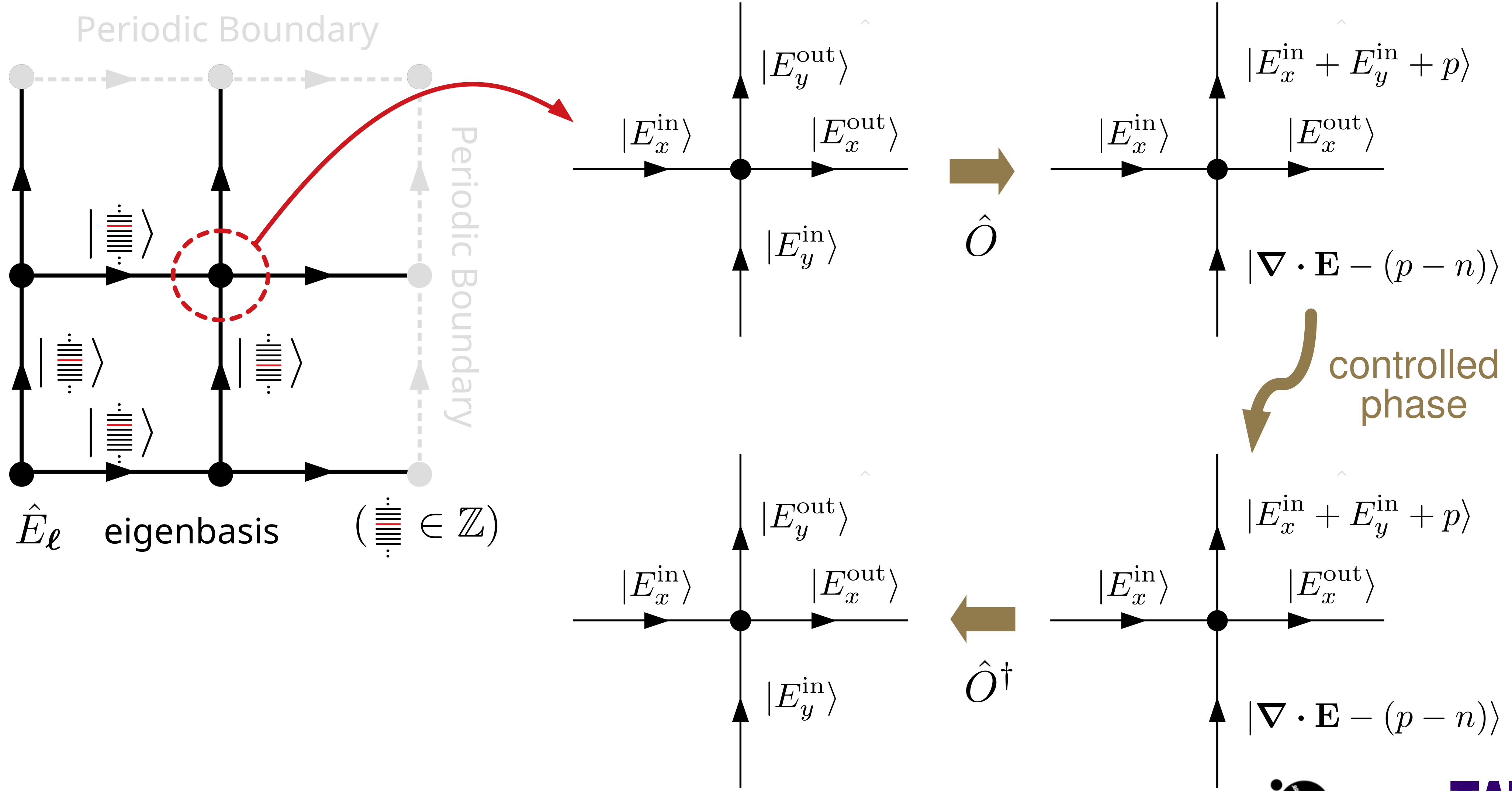
Oracles for U(1) Gauge Invariance II

Basic idea



Non-Abelian LGT Hilbert space I

Previous slides: Challenges existing already with Abelian

New complications for non-Abelian? (here SU(2))

- Truncation on basis states

$$|j, m, n\rangle \,, \quad 0 \leq j \leq J \qquad\qquad d_j = (2j+1)^2$$

$$\Rightarrow \dim(\mathcal{H}) = (8/3)(J+1/2)(J+3/4)(J+1) \neq 2^n$$

Not ideal for qubits

- Gauge invariance

$$[\mathcal{G}^a, \mathcal{G}^b] \neq 0 \qquad (\mathcal{G}^a \equiv \nabla \cdot \mathbf{E}^a - \rho^a)$$

→I. e., constraints not simultaneously diagonalizable
→Simply probing physicality involves change of basis

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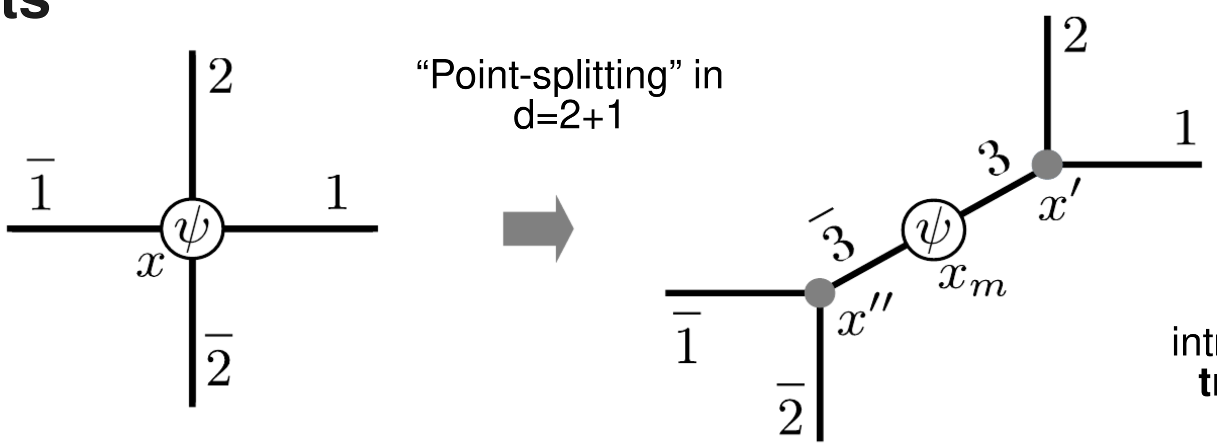
Non-Abelian LGT Hilbert space II

- Situation does not bode well, before even *thinking about* state preparation, time evolution, or measurements
- Enter: Schwinger boson formulation of LGT *
  - Angular momentum basis ↔ Many bosonic harmonic oscillators ("prepotential" formulation)
  - Newer loop reformulation ** **solves all non-Abelian constraints**

JRS & I. Raychowdhury, 1812.07554

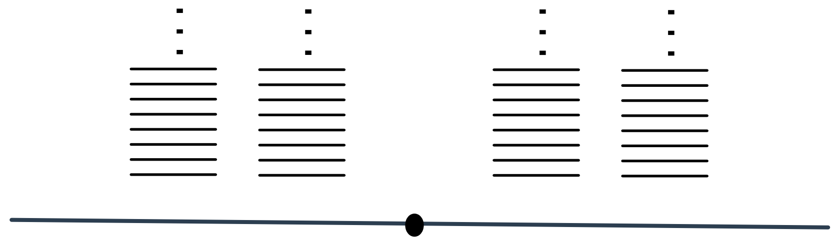


Virtual links introduced to yield **trivalent** lattice

* See papers by e.g. Mathur, Anishetty, Raychowdhury, Sreeraj
** I. Raychowdhury, 1804.01304

SU(2)-invariant variables

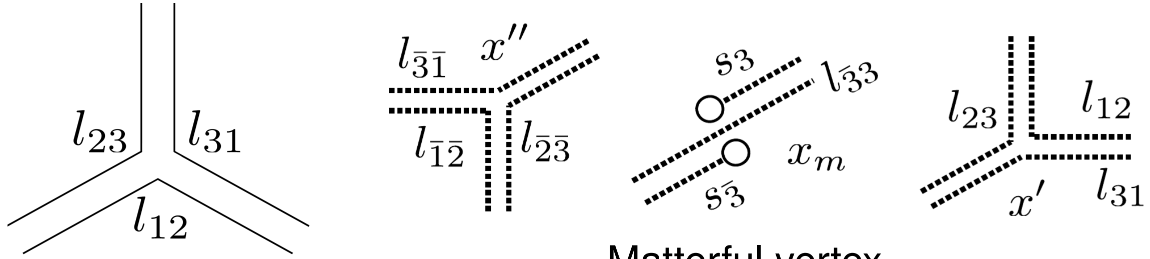
Kogut-Susskind $\quad |j, m, m'\rangle \,, \quad 0 \leq m, m' \leq j$



Schwinger bosons / prepotentials $\quad |n_{L,1}, n_{L,2}, n_{R,1}, n_{R,2}\rangle \,, \quad n_L + n_R = 2j$

Loop-String

- Gauge-invariant quantum numbers follow from Schwinger boson formulation
- Fundamental quarks
- Matter always locally reduced to a 1D problem

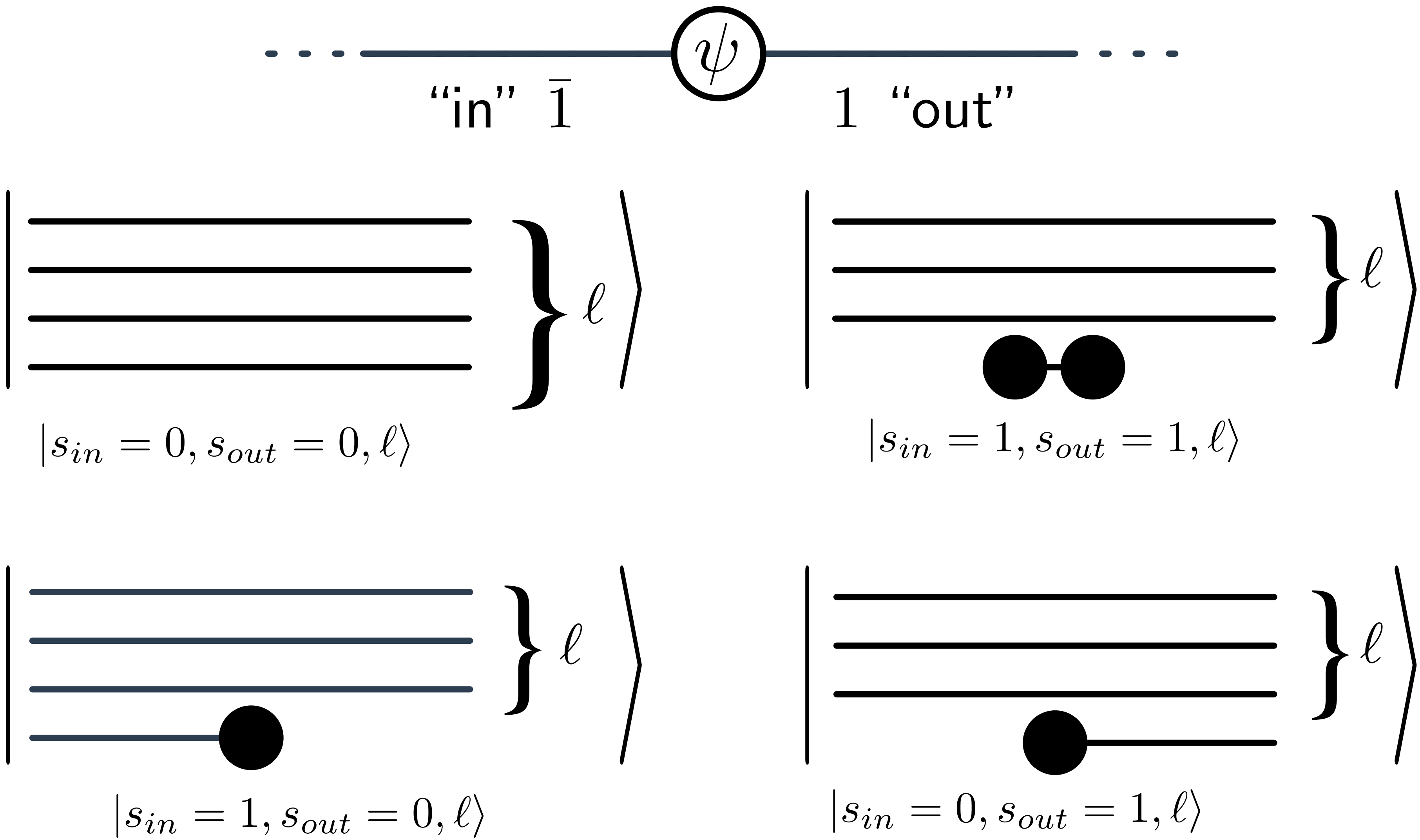


Matterless vertices

Matterful vertex

Example operators: $\quad L^{++}_{1\bar{1}} = b^\dagger \epsilon a^\dagger \quad S^{++}_{1} = \psi^\dagger \epsilon a^\dagger$

On-(matter)site Hilbert space



All SU(2)-invariant bilinears act on these states like simple ladder operators.

Advantages of loop-string quantum numbers

- Remnant constraints are Abelian ($E_L^a E_L^a = E_R^a E_R^a$ no longer automatic); **they commute**
- So techniques from U(1) can be ported over for wave function validation
  - Now known how to validate SU(2) LGT wave functions
- Much greater similarity with U(1) in "how *H* acts on states" and simpler to understand.
  - Borrow quantum algorithms?

The framework we propose is a very promising starting point for digital quantum simulation (more to come)

Summary and Future Directions

- **Consequences of LGT basis choice are far-reaching in DQS**
  - **Very long, interesting road to QCD**
- **Must confront unphysical errors**
  - **Routines given to validate wave functions in U(1), SU(2) theories**
- **SU(2) + loop-string formulation is promising for DQS**
  - **Structurally closer to U(1)**
  - **Clearer path to SU(3)?**

[Anishetty & Sreeraj 1903.07956]

Ongoing work:

- **Explore applications of Gauss subroutines**
- **Time evolution**

See also talk by H. Lamm

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FIN

Thank you for your attention!

Questions?

Other helpful colleagues at INT: N. Klco, A. Roggero, M. Savage

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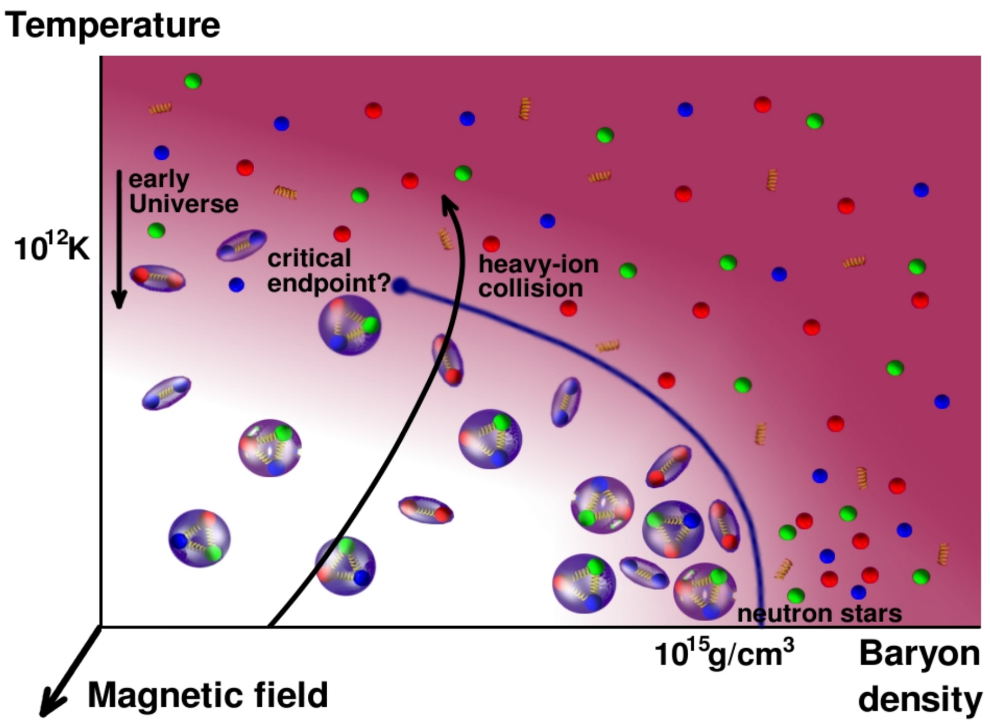
Phase diagram graphic credit

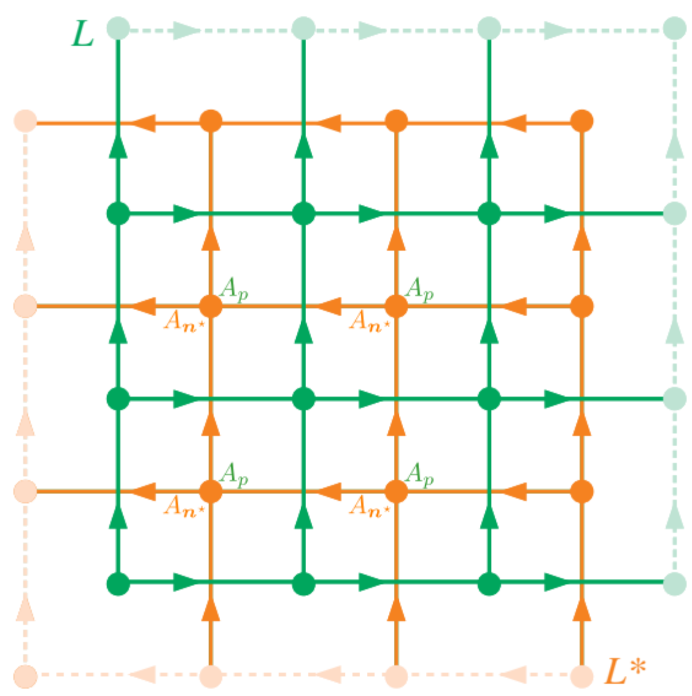


QCD phase diagram: Overview of recent lattice results - Scientific Figure on ResearchGate. Available from: https://www.researchgate.net/figure/Conjectured-QCD-phase-diagram_fig1_261701898 [accessed 23 Jan, 2019]

Physical Hilbert Space Generation

Notice

- Plaquettes $\boldsymbol{p} \sim$ dual sites $\mathbf{n}^\star$.
  $\Rightarrow \mathscr{A}_{\boldsymbol{p}}$ is scalar field $\mathscr{A}_{\mathbf{n}^\star}$ on $L^\star$.
- $E_{\boldsymbol{\ell}}$ on a link $\sim$ difference $\Delta \mathscr{A}_{\mathbf{n}^\star}$ along a dual link



Formalize this:

1.) Identify E-field states with plaquette powers

$$\prod_{\boldsymbol{p}} \left(\hat{P}_{\boldsymbol{p}}\right)^{\mathscr{A}_{\boldsymbol{p}}}\Bigg|_{\mathscr{A}_{\boldsymbol{p}}=\mathscr{A}_{\mathbf{n}^\star(\boldsymbol{p})}} |\Omega\rangle \quad \overset{\text{identify}}{\longleftrightarrow} \quad \otimes_{\mathbf{n}^\star} |\mathscr{A}_{\mathbf{n}^\star}\rangle \qquad \{|\mathscr{A}_{\mathbf{n}^\star}\rangle : \mathscr{A}_{\mathbf{n}^\star} \in \mathbb{Z}\}$$

Physical Hilbert Space Repackaging

2.) Define identical local orthonormal bases $\{|\mathscr{A}_{\mathbf{n}^\star}\rangle\}$ that diagonalize

$$\hat{\mathscr{U}}_{\mathbf{n}^\star} \equiv \sum_{\mathscr{A}_{\mathbf{n}^\star}=-\infty}^{\infty} |\mathscr{A}_{\mathbf{n}^\star}\rangle\, e^{i\xi\mathscr{A}_{\mathbf{n}^\star}} \langle\mathscr{A}_{\mathbf{n}^\star}| \ .$$

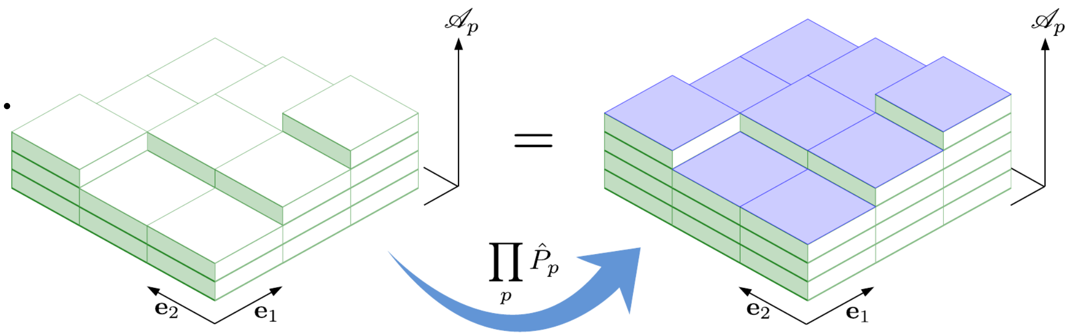
3.) Global basis states

$$|\mathscr{A}_{L^\star}\rangle \equiv \otimes_{\mathbf{n}^\star} |\mathscr{A}_{\mathbf{n}^\star}\rangle$$

4.) (Local) raising operators:

$$\hat{\mathscr{Q}}_{\mathbf{n}^\star} \equiv \sum_{\mathscr{A}_{\mathbf{n}^\star}=-\infty}^{\infty} |\mathscr{A}_{\mathbf{n}^\star}+1\rangle \langle\mathscr{A}_{\mathbf{n}^\star}| \ .$$

Redundancy:



Since

$$\prod_{\boldsymbol{p}} \left(\hat{P}_{\boldsymbol{p}}\right) = \hat{1} \ ,$$

must impose

$$\prod_{\mathbf{n}^\star} \hat{\mathscr{Q}}_{\mathbf{n}^\star} |\mathscr{A}_{L^\star}\rangle = |\mathscr{A}_{L^\star}\rangle \ .$$

This is magnetic Gauss law.

Reformulation in Dual Variables

Original		Dual
plaquette, $\boldsymbol{p}$	$\leftrightarrow$	site, $\mathbf{n}^\star$
plaquette operator, $\hat{P}_{\boldsymbol{p}}$	$\leftrightarrow$	site raising operator, $\hat{\mathscr{Q}}_{\mathbf{n}^\star}$
link, $\boldsymbol{\ell}$	$\leftrightarrow$	(perpendicular) link, $\boldsymbol{\ell}^\star$
field square, $\mathcal{E}_{\boldsymbol{\ell}}^2$	$\leftrightarrow$	field laplacian, $\hat{\mathscr{U}}_{\mathbf{n}^\star}^\dagger \partial_i^+ \partial_i^- \hat{\mathscr{U}}_{\mathbf{n}^\star}$

We have $\langle \mathscr{A}'_L | \hat{H} | \mathscr{A}_L \rangle = \langle \mathscr{A}'_{L^\star} | \mathscr{H} | \mathscr{A}_{L^\star} \rangle$ for the dual Hamiltonian

$$\hat{\mathscr{H}} = \frac{1}{2a_s} \sum_{\mathbf{n}^\star} \Bigg[\frac{1}{\tilde{g}_s^2} \left(2 - \hat{\mathscr{Q}}_{\mathbf{n}^\star} - \hat{\mathscr{Q}}_{\mathbf{n}^\star}^\dagger \right) - \frac{\tilde{g}_t^2}{\xi^2} a_s^2 \hat{\mathscr{U}}_{\mathbf{n}^\star}^\dagger \partial_i^+ \partial_i^- \hat{\mathscr{U}}_{\mathbf{n}^\star} \Bigg] , \qquad (D = 2)$$

(subject to magnetic Gauss).