A qubit model for U(1) lattice gauge theory

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Quantum computers offer advantages such as:

- quantum parallelism,
- access to real-time dynamics.

We are using classical simulators to explore an implementation of U(1) gauge fields.

outline

- from action to Hamiltonian
- electric field eigenstates
- energy eigenvalues
- the Hamiltonian expressed in Pauli operators
- real-time propagation 
- real-time collisions 

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from action to Hamiltonian

review:Kogut,Rev.Mod.Phys.51,659(1979)

The Euclidean lattice action is a sum over all plaquettes,

$$S = -\frac{\beta}{2}\sum_P \left(U_P + U_P^*\right)$$

where $\beta = 1/g^2$ and $U_P = e^{i\theta_\mu(n)}e^{i\theta_\nu(n+\hat{\mu})}e^{-i\theta_\mu(n+\hat{\nu})}e^{-i\theta_\nu(n)}$.

To convert this into a Hamiltonian, let $\theta_\mu(n) = agA_\mu(n)$. Choosing the temporal gauge ($\theta_t(n) = 0$), and sliding the temporal lattice spacing to zero brings us to the Hamiltonian for compact U(1) gauge theory:

$$H = \sum_n \left(\frac{a^3}{2}\vec{E}^2(n) - \frac{\beta}{a}\sum_{i=2}^{3}\sum_{j=1}^{i-1}\cos\left(\theta_i(n) + \theta_j(n+\hat{i}) - \theta_i(n+\hat{j}) - \theta_j(n)\right)\right)$$

where $\vec{E} = \partial_t \vec{A}$ is the electric field.

electric field eigenstates

review:Kogut,Rev.Mod.Phys.51,659(1979)

The conjugate momentum for $\vec{\theta}$ (name it $\vec{L}$) is proportional to $\vec{E}$.
Angles constrained to $[0, 2\pi)$ produce quantized conjugate momentum eigenvalues:

$$\ell_i(n) = \ldots, -2, -1, 0, 1, 2, \ldots$$

An immediate consequence is that link variables ($U_j = e^{i\theta_j}$) are ladder operators because

$$[U_j, L_j] = -U_j \quad \text{and} \quad [U_j^\dagger, L_j] = U_j^\dagger$$

The Hamiltonian can now be written as

$$H = \frac{1}{2a\beta}\left(\sum_j L_j^2 - \beta^2 \sum_P (U_P + U_P^\dagger)\right)$$

Let's rescale energies by dropping the overall factor of $\frac{1}{2a\beta}$ from now on.

the simplest example

A 2×2 lattice with Dirichlet boundary conditions has only one plaquette so

$$H = L_1^2 + L_2^2 + L_3^2 + L_4^2 - \beta^2(U_1 U_2 U_3^\dagger U_4^\dagger + U_1^\dagger U_2^\dagger U_3 U_4)$$

Gauss's law requires L_i^2 is the same for all 4 links.
Local gauge transformations allow 3 links to be rotated to the identity.
This brings us to

$$H = 4L^2 - \beta^2(L^+ + L^-)$$

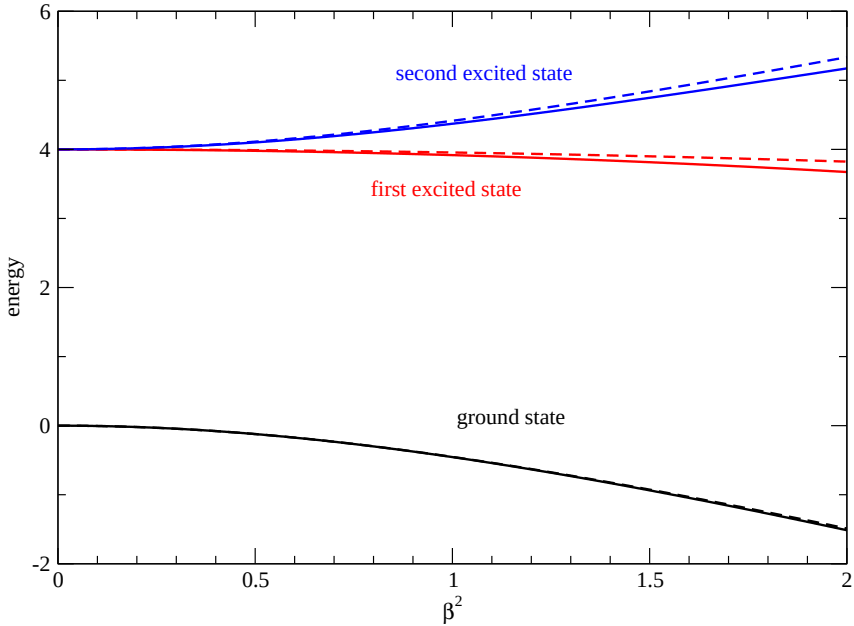
where $L^\pm$ are the ladder operators. Truncating to $-4 \le \ell \le 4$ gives

$$H = \begin{pmatrix} 64 & -\beta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta^2 & 36 & -\beta^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta^2 & 16 & -\beta^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta^2 & 4 & -\beta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta^2 & 0 & -\beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta^2 & 4 & -\beta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta^2 & 16 & -\beta^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta^2 & 36 & -\beta^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta^2 & 64 \end{pmatrix}$$

energy eigenvalues for the 2×2 lattice

The 3 lowest energies for a single plaquette

Dashed curves use $-1 \le l \le 2$. Solid curves use any larger bounds.



writing the Hamiltonian with Pauli operators

For a qubit-inefficient alternative, see Byrnes&Yamamoto,PRA73,022328(2006)

To run someday on a quantum computer (and today on a classical simulator), the states can be mapped to qubits, and operators expressed as Pauli products.

2 qubits per gauge link

ℓ	state
2	$\|00\rangle$
1	$\|01\rangle$
0	$\|10\rangle$
-1	$\|11\rangle$

$$\begin{aligned} L^+ &= \sigma_0^+ + \sigma_0^- \sigma_1^+ \\ L^- &= \sigma_0^- + \sigma_0^+ \sigma_1^- \\ L &= \tfrac{1}{2}\left(1 + \sigma_0^z + 2\sigma_1^z\right) \end{aligned}$$

3 qubits per gauge link

ℓ	state
4	$\|000\rangle$
3	$\|001\rangle$
2	$\|010\rangle$
1	$\|011\rangle$
0	$\|100\rangle$
-1	$\|101\rangle$
-2	$\|110\rangle$
-3	$\|111\rangle$

$$\begin{aligned} L^+ &= \sigma_0^+ + \sigma_0^- \sigma_1^+ + \sigma_0^- \sigma_1^- \sigma_2^+ \\ L^- &= \sigma_0^- + \sigma_0^+ \sigma_1^- + \sigma_0^+ \sigma_1^+ \sigma_2^- \\ L &= \tfrac{1}{2}\left(1 + \sigma_0^z + 2\sigma_1^z + 4\sigma_2^z\right) \end{aligned}$$

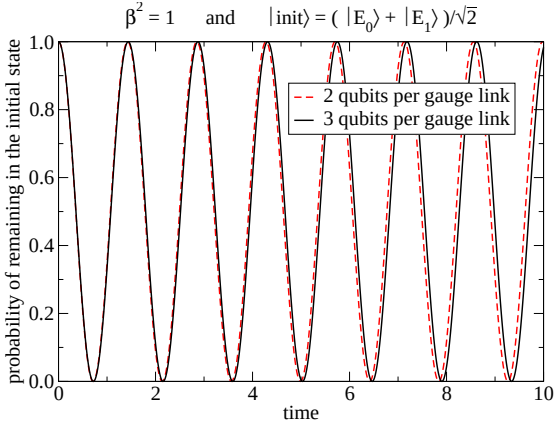
time evolution

- With 3 qubits per gauge link, we have

$$
\begin{aligned}
H \;=\; & 22 + 2\sigma_0^z + 4\sigma_1^z + 8\sigma_2^z + 4\sigma_0^z\sigma_1^z + 8\sigma_0^z\sigma_2^z + 16\sigma_1^z\sigma_2^z \\
& -\beta^2 \left(\sigma_0^x + \frac{1}{2}(\sigma_0^x\sigma_1^x + \sigma_0^y\sigma_1^y) + \frac{1}{4}(\sigma_0^x\sigma_1^x\sigma_2^x - \sigma_0^y\sigma_1^y\sigma_2^x + \sigma_0^y\sigma_1^x\sigma_2^y + \sigma_0^x\sigma_1^y\sigma_2^y) \right)
\end{aligned}
$$

- Time evolution is computed by using the 2nd-order Suzuki-Trotter formula,

$$e^{-i(A+B)t} = e^{-iAt/2} e^{-iBt} e^{-iAt/2} + O(t^3)$$



Finding the ground state

Use the variational principle. Peruzzo et al, Nature Communications 5, 4213 (2014)

Our 2-qubit Hamiltonian can be written as

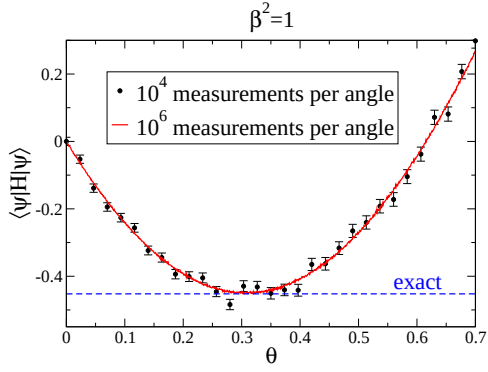
$$H = 4 + 12P_{\uparrow\uparrow} - 4P_{\downarrow\uparrow} - \beta^2(2P^x_{\uparrow\uparrow} + P^x_{\downarrow\uparrow} - P^{xx}_{\uparrow\downarrow} - P^{xx}_{\downarrow\uparrow} - P^{yy}_{\uparrow\downarrow} - P^{yy}_{\downarrow\uparrow})$$

where

$$P_i = |\langle i|\psi\rangle|^2, \quad P^x_i = \left|\left\langle i|R^y(-\tfrac{\pi}{2},0)\psi\right\rangle\right|^2, \quad P^{xx}_i = \left|\left\langle i|R^y(-\tfrac{\pi}{2},0)R^y(-\tfrac{\pi}{2},1)\psi\right\rangle\right|^2,$$
$$P^{yy}_i = \left|\left\langle i|R^x(\tfrac{\pi}{2},0)R^x(\tfrac{\pi}{2},1)\psi\right\rangle\right|^2.$$

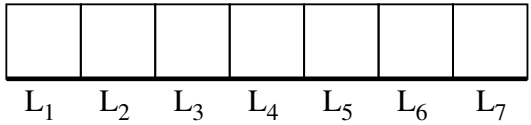
A simple trial state is $|\psi\rangle = 0\,|\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}}\sin\theta\,|\uparrow\downarrow\rangle + \cos\theta\,|\downarrow\uparrow\rangle + \frac{1}{\sqrt{2}}\sin\theta\,|\downarrow\downarrow\rangle$ which can be built as follows:

- Begin with $|\uparrow\uparrow\rangle$.
- Apply X(1) to get $|\downarrow\uparrow\rangle$.
- Apply $R^y(2\theta, 0)$ to get $\cos\theta\,|\downarrow\uparrow\rangle + \sin\theta\,|\downarrow\downarrow\rangle$.
- Apply $CR^y(-\frac{\pi}{2}, 0, 1)$ to get $|\psi\rangle$.



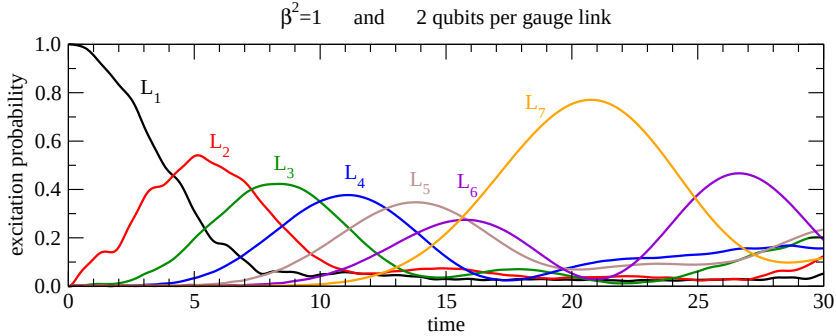
a row of plaquettes

Consider 7 plaquettes in a row:



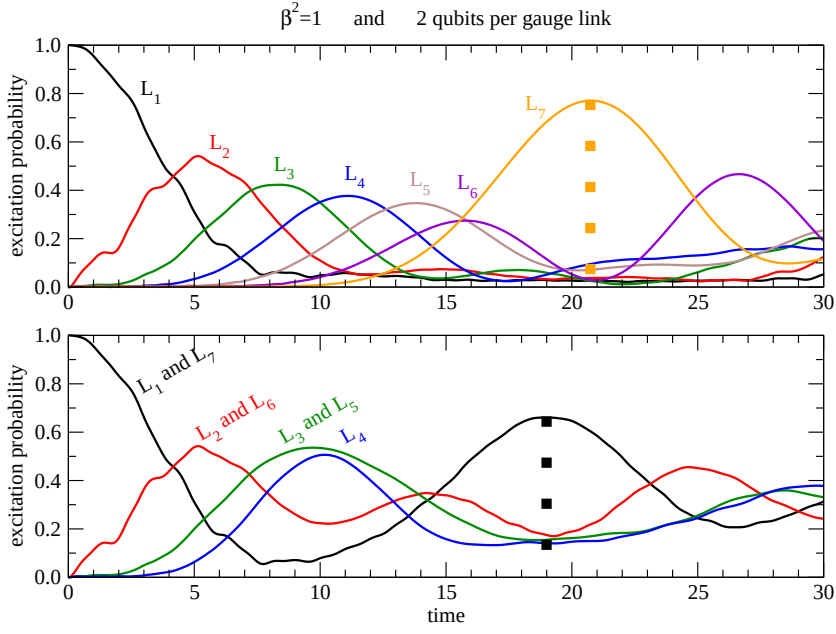
The Hamiltonian is $H = \sum_{i=1}^{7} \left(4L_i^2 - \beta^2 (L_i^+ + L_i^-) \right) - 2 \sum_{i=1}^{6} L_i L_{i+1}$

Begin with L_1 in the 1st excited state and all others in the ground state.
Compute the probability to be above the ground state.



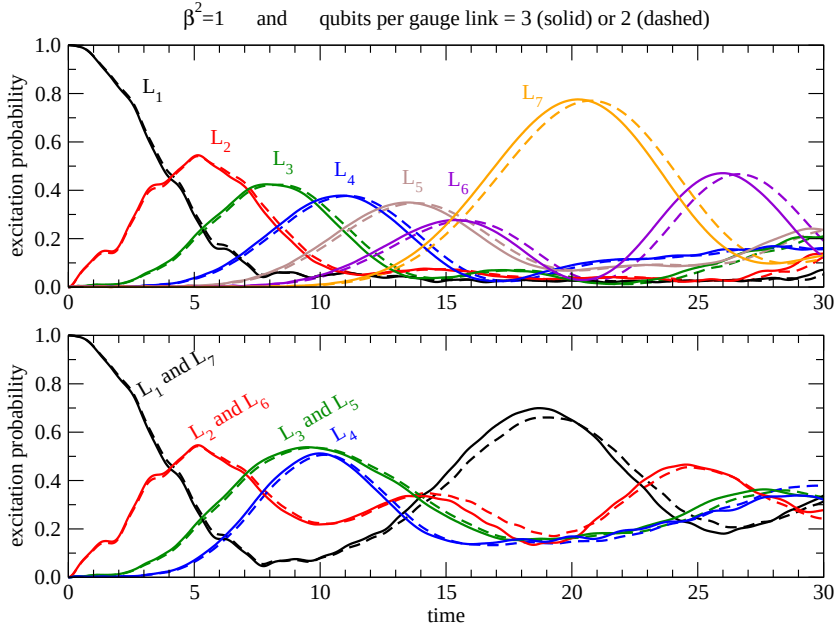
attractive or repulsive?

Compare a single excitation (upper plot) to colliding excitations (lower plot).



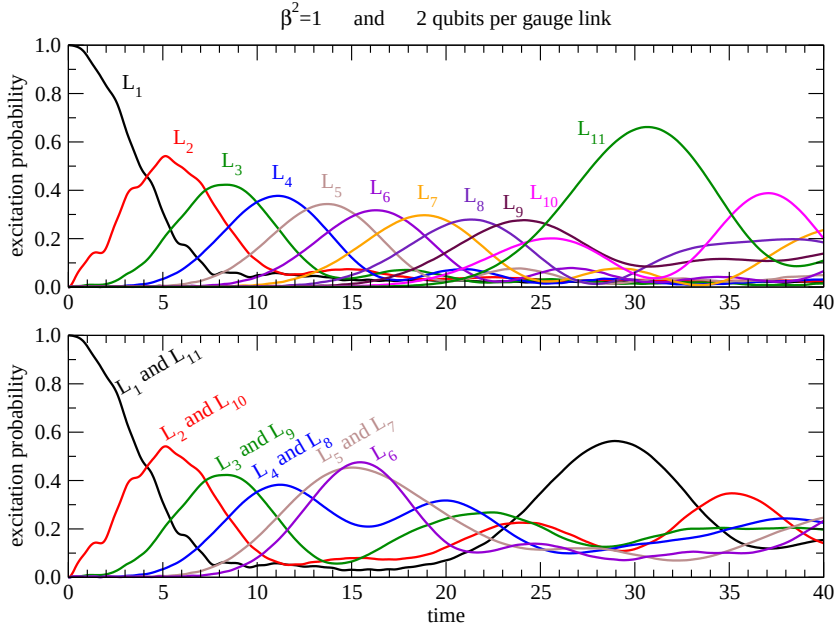
as before, but with 3 qubits per gauge link

Compare a single excitation (upper plot) to colliding excitations (lower plot).



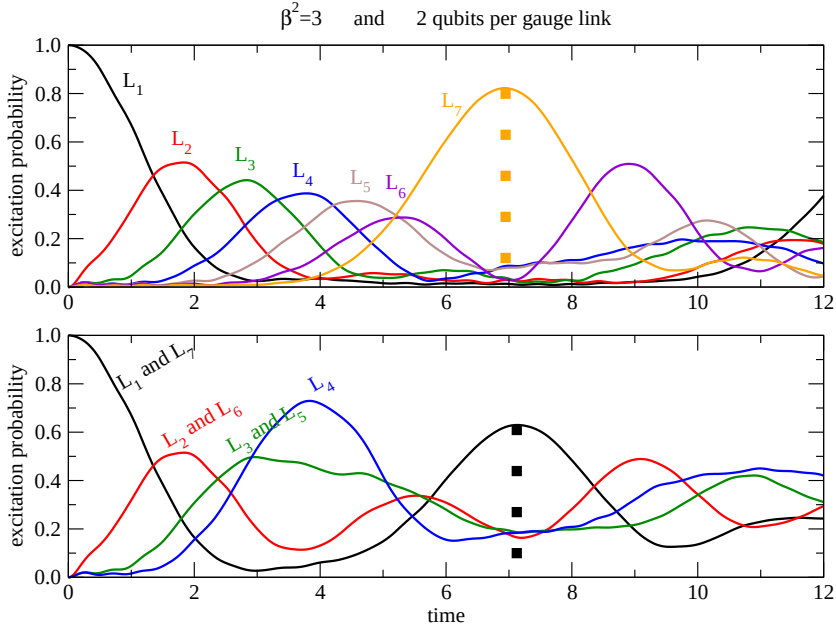
as before, but with a longer lattice

Compare a single excitation (upper plot) to colliding excitations (lower plot).



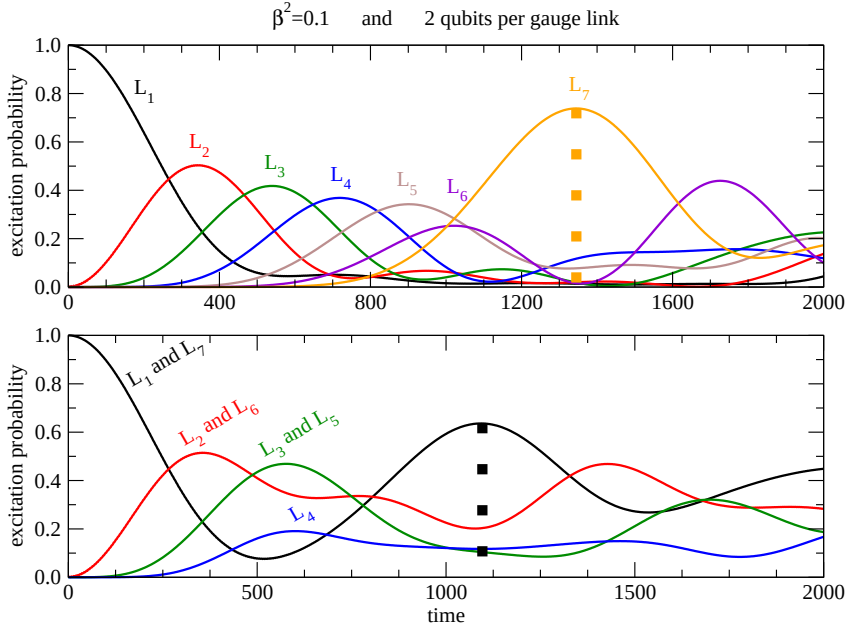
as before, but with larger β^2

Compare a single excitation (upper plot) to colliding excitations (lower plot).



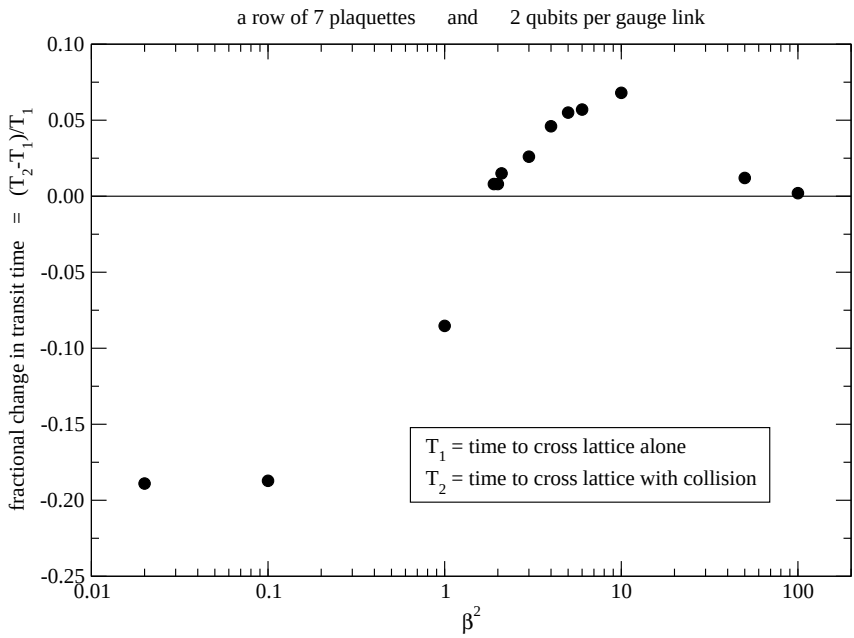
as before, but with smaller β^2

Compare a single excitation (upper plot) to colliding excitations (lower plot).



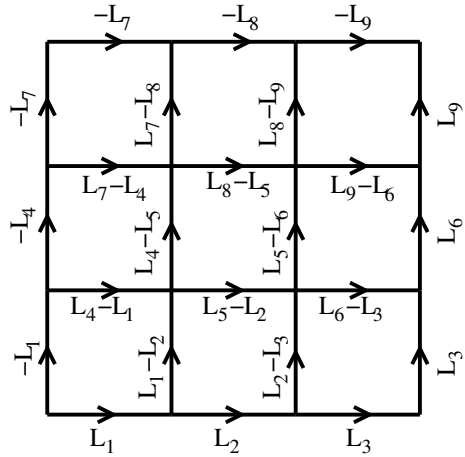
an overview of the β^2 dependence

(This plot is made from simple eyeball estimates of peak locations on graphs.)



a 2D lattice of plaquettes

Consider 9 plaquettes in a square:



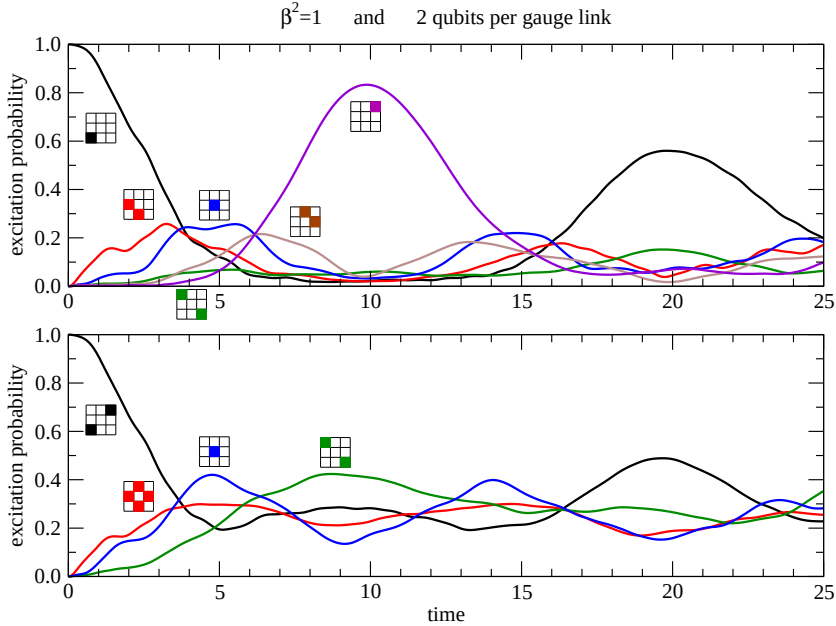
Gauss's law leaves just one independent gauge link per plaquette.

The Hamiltonian is $H = \sum_{i=1}^{9} \left(4L_i^2 - \beta^2 (L_i^+ + L_i^-) \right) - 2 \sum_{i=1}^{6} L_i L_{i+3}$

$-2(L_1 L_2 + L_2 L_3 + L_4 L_5 + L_5 L_6 + L_7 L_8 + L_8 L_9)$

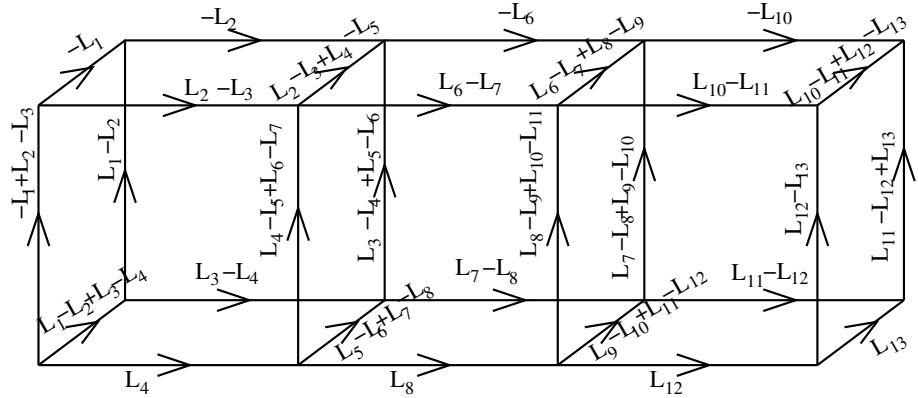
excitations begin in opposing corners

Compare a single excitation (upper plot) to colliding excitations (lower plot).



a 3D lattice of plaquettes

Consider 3 cubes in a row:



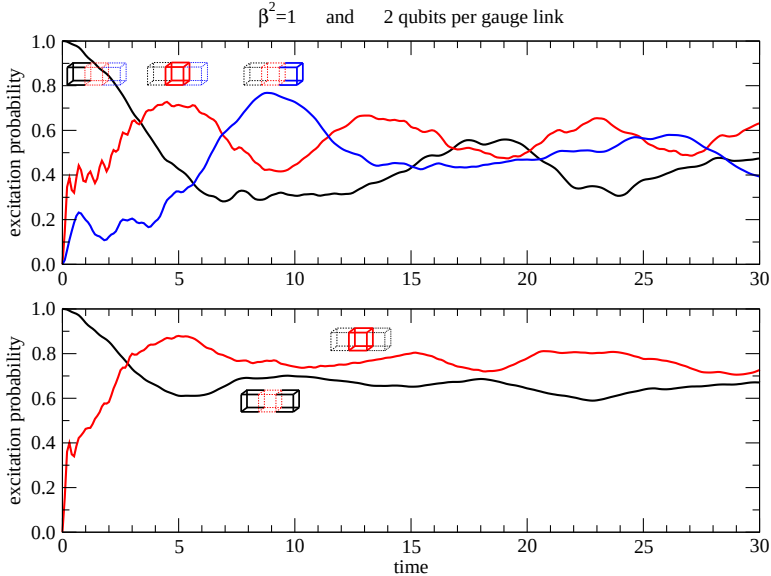
Gauss's law leaves fewer than one independent gauge link per plaquette.
This means the Hamiltonian must contain link products in both E and B terms.
The B terms are more expensive (i.e. more Pauli factors).

The diagram shown here minimizes the number of link products in B terms.
Other definitions of the L_i typically lead to vastly more Pauli factors in H.

excited cubes

A single-plaquette excitation will quickly disperse on this lattice.
Try exciting an entire cube. . .

single excitation (upper plot), colliding excitations (lower plot)



. . . but this also shows rapid dispersal.

summary

We have developed a qubit implementation of compact U(1) gauge theory.
The Hamiltonian is written in terms of Pauli matrices.

A quantum variational eigensolver allows calculation of eigenvalues and eigenstates.

Propagation of an excitation is demonstrated for planar and $2\times2\times N$ lattices.

On a row of plaquettes, colliding excitations are observed easily.
On more general lattices, colliding excitations decohere more quickly.

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