Quantum simulation of the transverse Ising model

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- Spin basis Ising model

$$\hat{H} = -J \sum_i^{N_s - 1} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - h_T \sum_i^{N_s} \hat{\sigma}_i^x - J_b \sigma_{N_s}^z \sigma_1^z$$

- $J_b = +J, -J, 0$ for periodic, anti-periodic, and open boundary conditions respectively. (PBC, ABC, OBC).

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- In "particle" basis the particle number is conserved modulo 2.

$J << h_t$ with PBC or ABC

- Definitions

Unperturbed Hamiltonian

$\hat{H}_0 = -h_T \sum_i^{N_s} \sigma_i^z$

"Potential"

$\hat{V} = -J \sum_i^{N_s - 1} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x +$ boundary terms

Corrections: non-degenerate

- First order corrections:
  $\langle x|V|x\rangle = -J\langle x|(|x+1\rangle + |x-1\rangle + \sum_i |x,\ i,\ i+1\rangle) = 0$

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- Since single "particle" states are N_s-fold degenerate find transformation to lift degeneracy

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- use transformation $|k\rangle = \frac{1}{\sqrt{N_s}} \sum_x^{N_s} e^{\frac{2\pi i x k}{N_s}} |x\rangle$

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- Energies for the one "particle" states:
  $E_k^{(1)} = -h_T(N_s - 2) - 2J\cos(2\pi k/N_s)$

Time evolution

- Occupation probability $\langle\psi_j(t)|\hat{n}_l|\psi_j(t)\rangle \simeq |J^{(N_s)}_{l-j}(2Jt)|^2$

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- Define "discrete" Bessel functions:
$J^{(N_s)}_{\ell}(x) = \frac{(-i)^{\ell}}{N_s}\sum_{m=0}^{N_s-1} e^{i((\frac{2\pi m\ell}{N_s}+x\cos(\frac{2\pi m}{N_s})))}$

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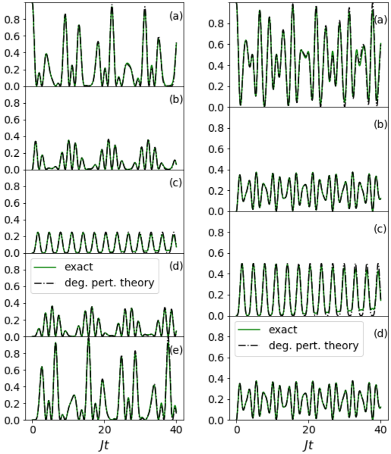
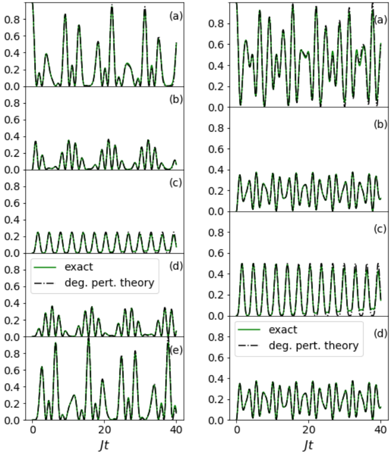
Time evolution: continued

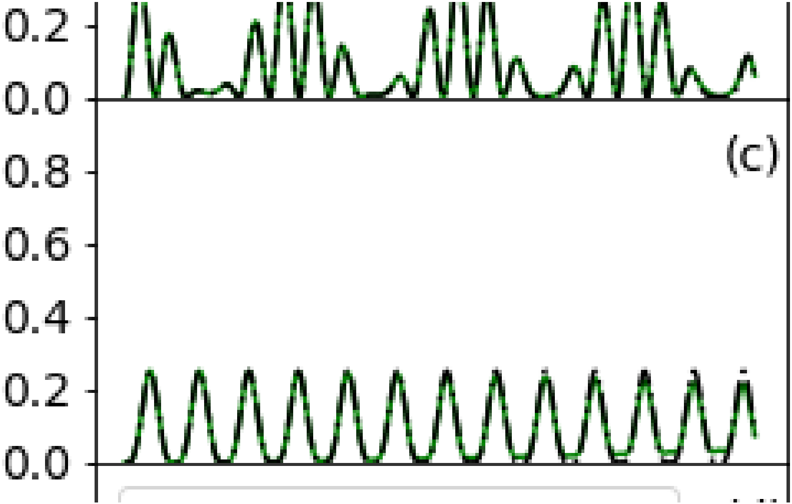
- Agreement between exact diagonalization and perturbation theory is quite close.
- 1 particle PBC left, 2 particle ABC right for 8 sites
- The quantity plotted on the y axis is $\langle n_l(t) \rangle$.



Time evolution: continued

- Agreement between exact diagonalization and perturbation theory is quite close.
- 1 particle PBC left, 2 particle ABC right for 8 sites
- The quantity plotted on the y axis is $\langle n_l(t) \rangle$.
- Deviations occur at time scales much larger than time frames of interest




0.2
0.0
(c)
0.8
0.6
0.4
0.2
0.0

Evolution operator

Definition of Evolution Operator

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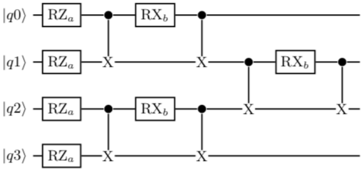
  - $U(t) \simeq e^{-it\hat{V}^{\text{even}}} e^{-it\hat{V}^{\text{odd}}} e^{-it\hat{H}_0} + \mathcal{O}(t^2)$

Quantum Circuit Full Hilbert Space (4 sites open boundary conditions)



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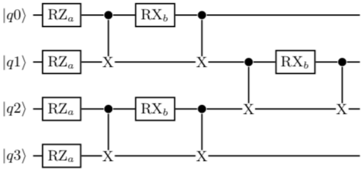
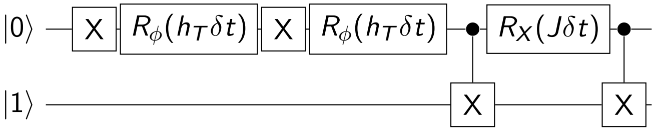


$$RZ_a = \begin{pmatrix} e^{i(\delta t)h_T} & 0 \\ 0 & e^{-i(\delta t)h_T} \end{pmatrix} \quad RX_b = \begin{pmatrix} \cos(J\delta t) & i\sin(J\delta t) \\ -i\sin(J\delta t) & \cos(J\delta t) \end{pmatrix} \tag{1}$$

$$CX = \begin{pmatrix} 1_{2x2} & 0 \\ 0 & \sigma^x \end{pmatrix} \tag{2}$$

Quantum Circuit Truncated Hilbert Space (4 sites periodic boundary conditions)



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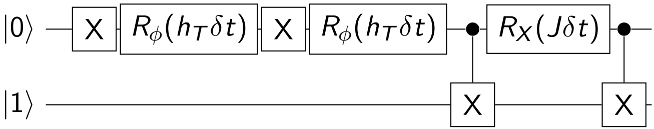
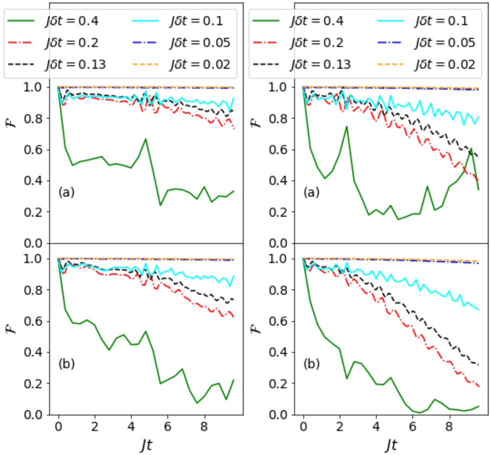


$$R_\phi(J\delta t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{iJ\delta t} \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

Trotter fidelities



- left: open boundary conditions, right: periodic boundary conditions
- top: free propagation, bottom: “scattering”

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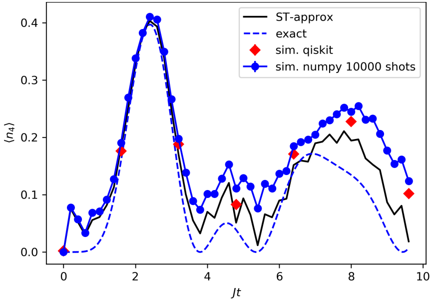
- Polynomial Ansatz: N. Kclo et al. Arxiv: 1803.03326. Noise Readout Correction: A. Kandala et al. Nature 549, 242 EP (2017)

Comparison



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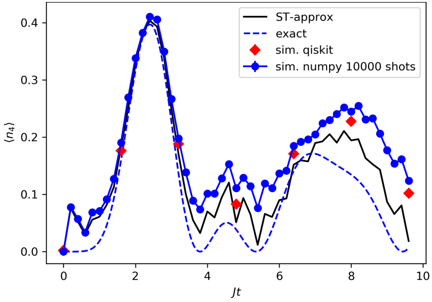

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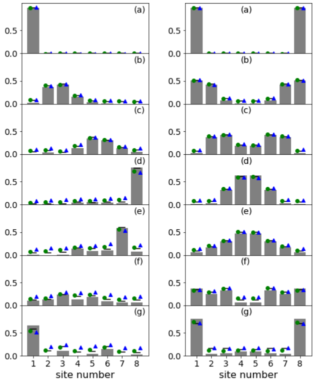
- Noise reduction methods only get us to trotter error nothing better.

Classical Simulations of Current Trapped Ions



- OBC: left free propagation, right scattering; green expected near term digital quantum computer, blue current trapped ion. $J\delta t = 0.2$. Frames are separated by $Jt = 1.6$.

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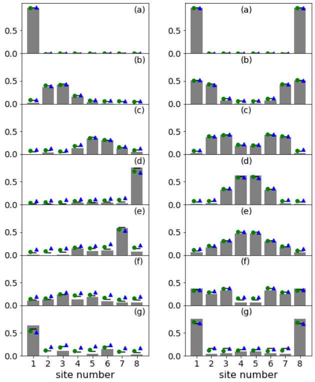

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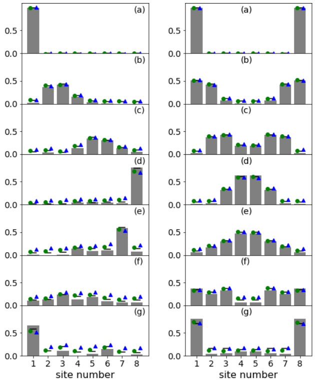

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- results between exact and simulation deviate when Jt increases (we expect this)
- results generally consistent with ST approximation

Classical Simulations of Current Quantum Computers

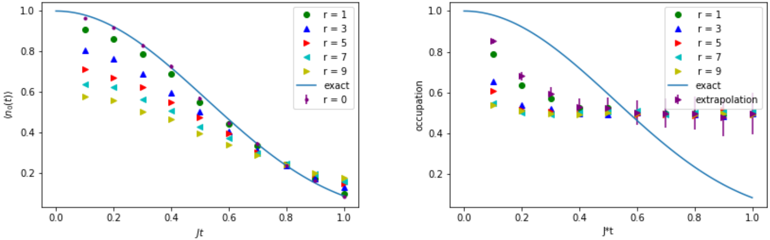

Figure: left truncated Hilbert space, right full hilbert space

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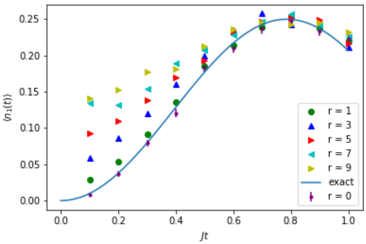

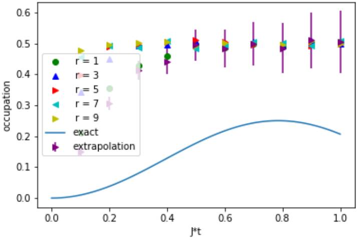

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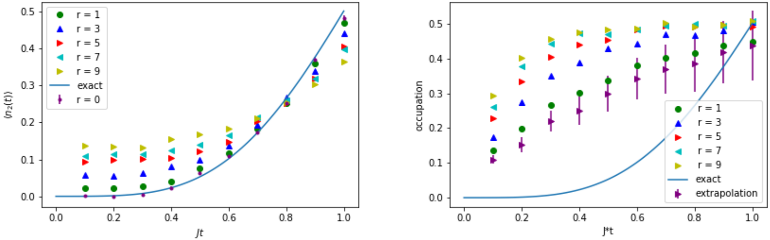

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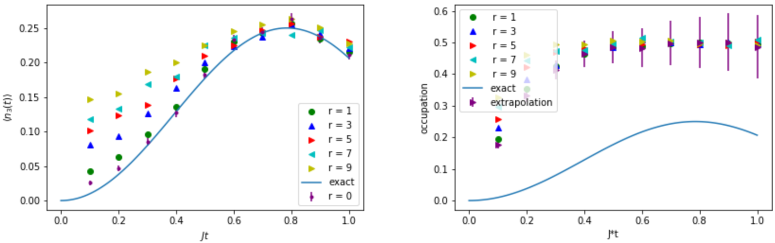

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Phase shifts

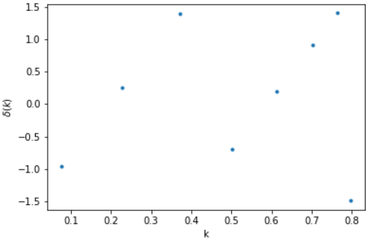

Figure: $j = 0.02$, $h_t = 1$, impurity on 1 site of $\delta h_t = 1.0$

- for $J << h_t$ the Ising model is far away from the continuum limit

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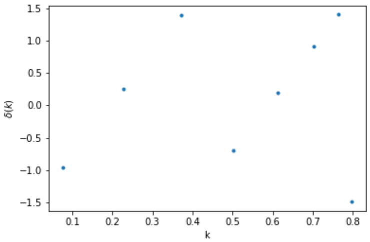

Figure: $j = 0.02$, $h_t = 1$, impurity on 1 site of $\delta h_t = 1.0$

- for $J << h_t$ the Ising model is far away from the continuum limit
- examining discrete quantum mechanics a better step to look at phase shifts

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- Where $k = \sqrt{2 * m * E}$.

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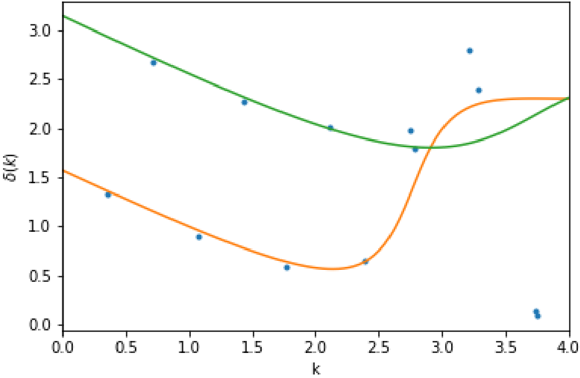
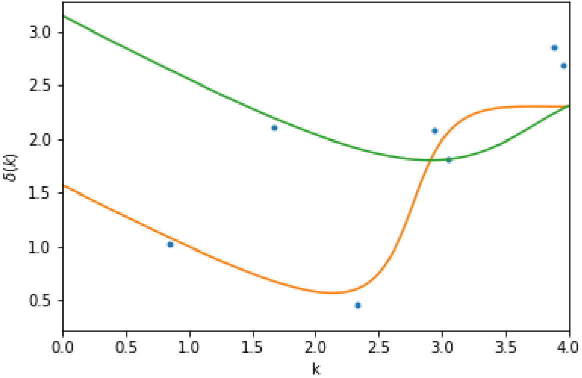
- the phase shifts can be extracted using the fact that the scattered states are given by: $|\psi^{\pm}\rangle = |k\rangle \pm C^{\pm}(k)| - k\rangle$, where $C^{\pm} = 1/\sqrt{2}(t(k) \pm r(k))$ and $\delta^{\pm}(k)) = ln(C^{\pm}(k))/(2i)$

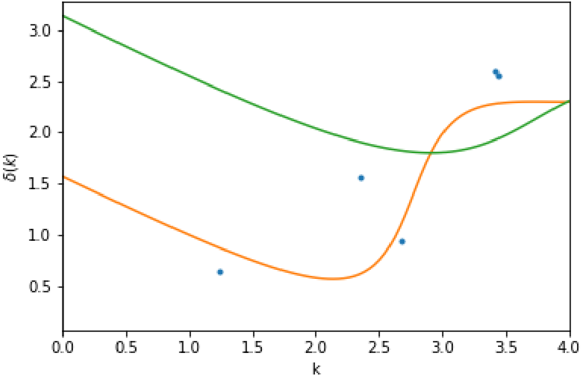
Spectroscopy for $L = 10, \; N = 32, \; w = 2, \; m = 1$



Spectroscopy for $L = 5, \; N = 16, \; w = 2, \; m = 1$



Spectroscopy for $L = 4,\ N = 8,\ w = 2,\ m = 1$



Variational Methods

- Minimize the ansatz $\langle\psi^{\pm}(\theta,\phi)|V|\psi^{\pm}(\theta,\phi)\rangle$, where $|\psi^{\pm}(\theta,\phi)\rangle = \cos(\theta)|k\rangle + e^{i\phi}\sin(\theta)|-k\rangle$.

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- extensions could include modes slightly higher or lower momenta states.

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- See arxiv:1901.0544 for further details

Thanks

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