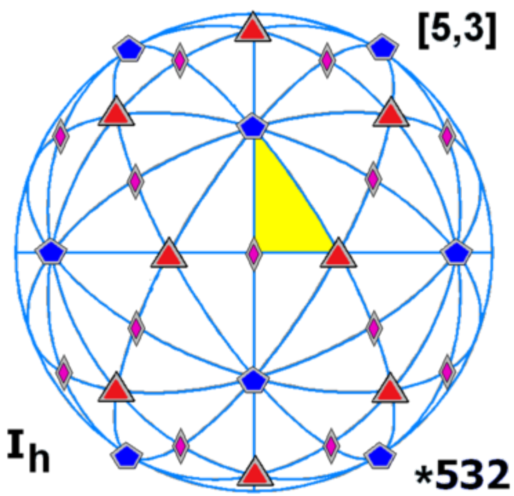
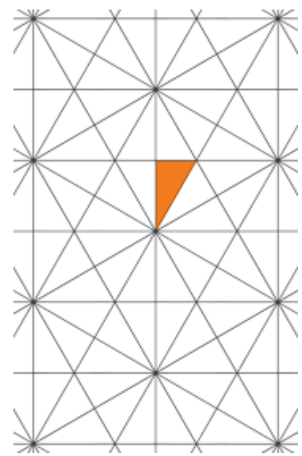
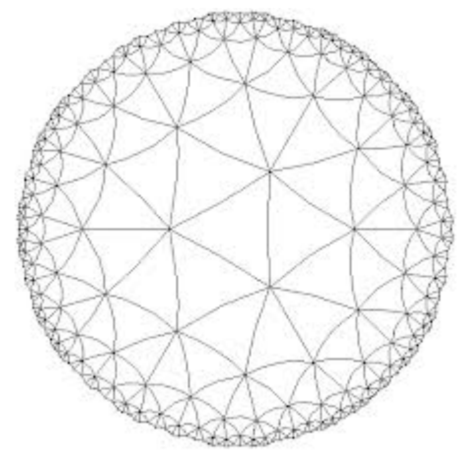
LATTICE FIELD THEORY on RIEMANN MANIFOLDS



$R > 0$


$R = 0$


$R < 0$

Rich Brower, BSM @ Syracuse May 2, 2019

with G. Fleming, A. Gasbarro, D. Howarth, T. Raben, C-I Tan, E. Weinberg

See Details in 2 publications:

https://arxiv.org/abs/1610.08587 Dirac Fermions on Simplicial Manifold

https://arxiv.org/abs/1803.08512 Phi 4th on Riemannian Manifold

Lattice Radial Quantization & BSM

$\mathbb{R} \times \mathbb{T}^3$ vs $\mathbb{R} \times \mathbb{S}^3$

$$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

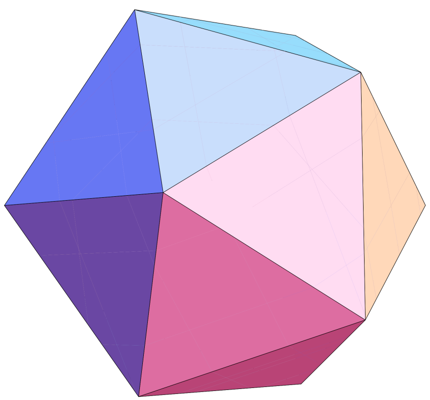
Potential advantage:

$$1 < t < aL \implies 1 < \tau = log(r) < L$$

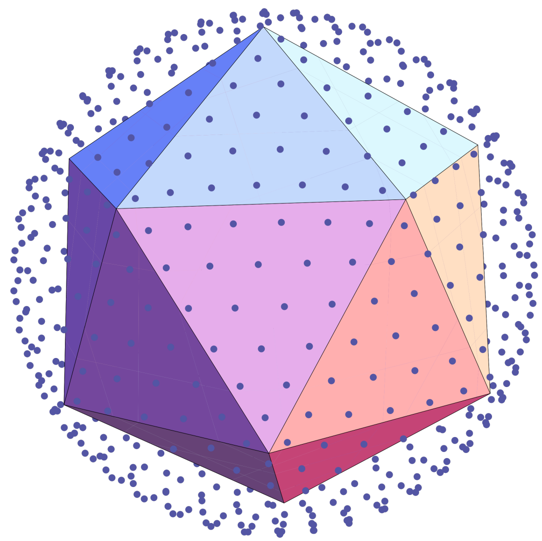
Goal for BSM theories? Begin with exact CFT in the IR and study spectral flow due to adiabatic "mass" deformation of Dimensions to Masses as the Dilatation reverts to the Hamiltonian.

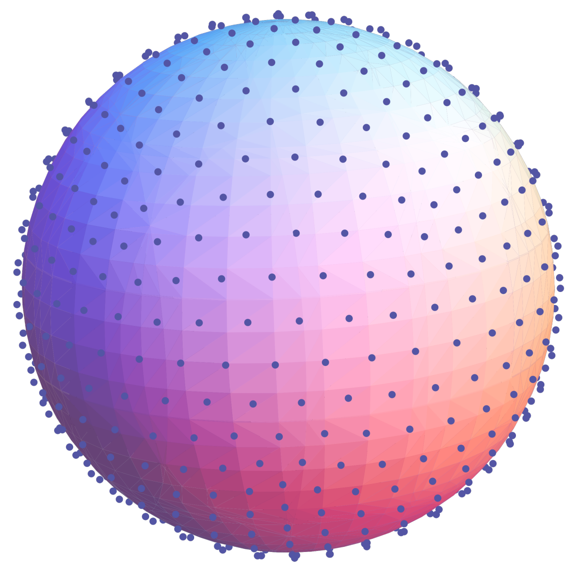
First Test: CFT on Sphere x R

$s = 1$



$s = 8$





l = 0 (A),1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of O(3)

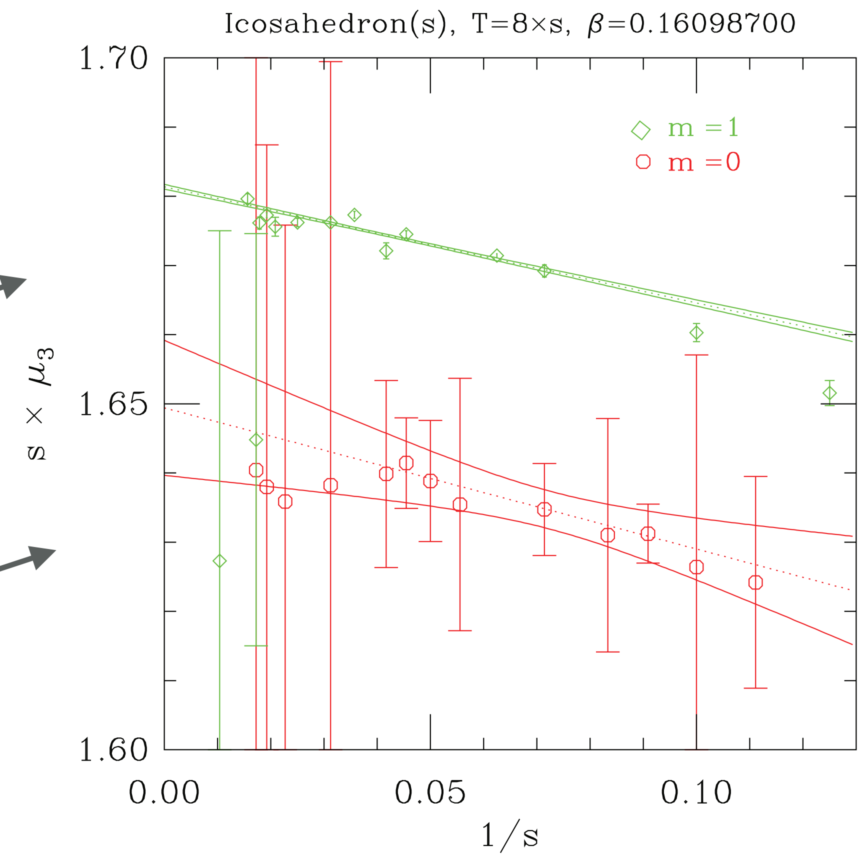
1/p + 1/q > 1/2 for regular positive curvature tessellation

Radial Quantization of Ising Model (2013)

Failure to recover O(4,1) of l = 3
with no FEM (Ising spins on Triangulated Icosahedron)

G rep

T2 rep



* R.C.Brower, G.T.Fleming and H. Neuberger, ***Lattice Radial Quantization: 3D Ising,*** *Phys.Lett.B* ***721****, 299 (2013)*

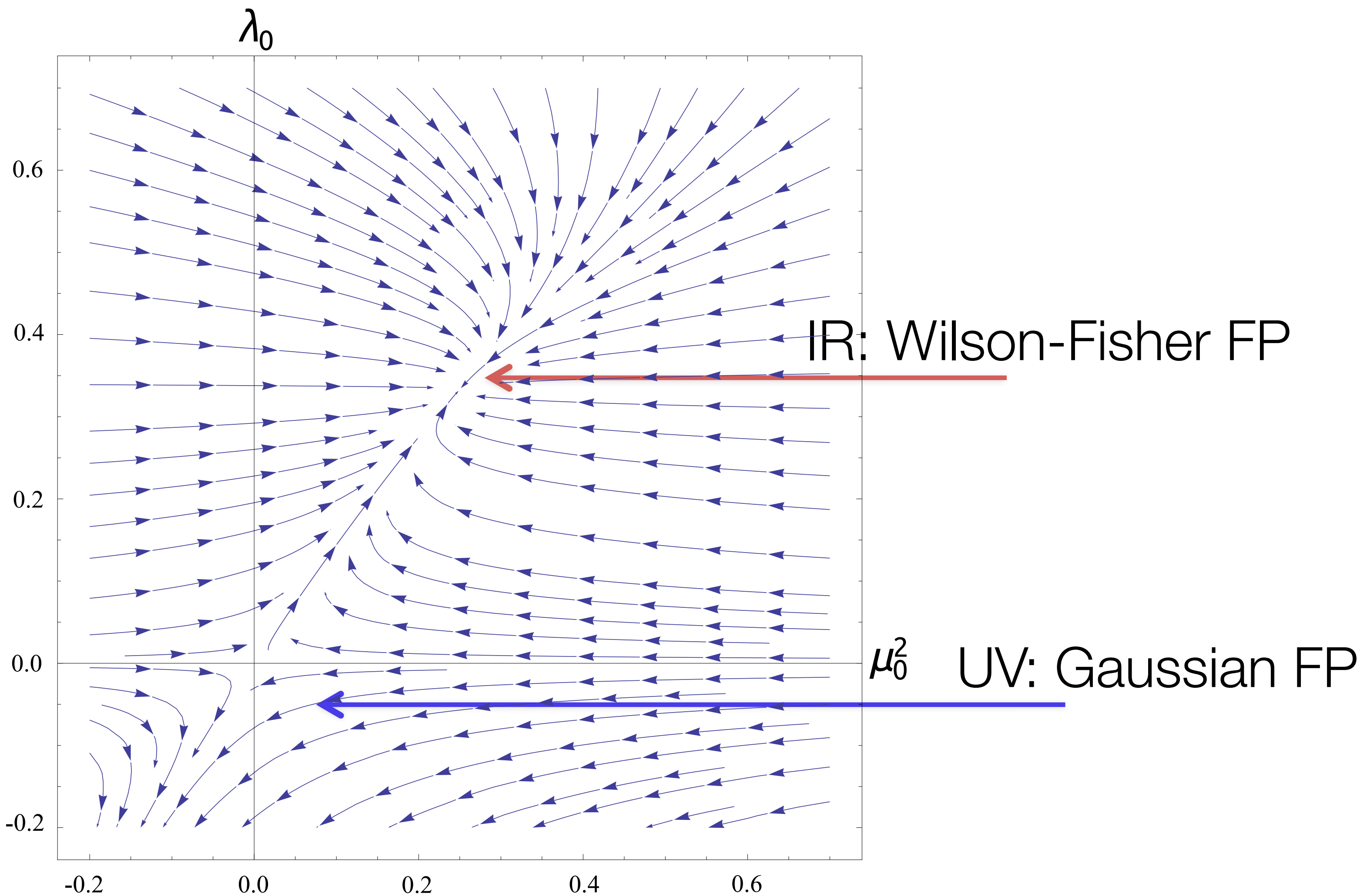
BIG QUESTION

All flat space Renormalizable QFT are Renormalizable on Smooth Riemann Manifold (see M. Luscher, H. Osborn &Literature in 1990's)

$$\{\mathbb{R}^d, \delta_{\mu\nu}\} \qquad \Longrightarrow \qquad \{\mathcal{M}, g_{\mu\nu}\}$$

Does there exist a rigorous Simplicial Lattice definition of these non-perturbative QFT on any target smooth Riemann Manifold ?

TEST CFT: PHI 4TH AT WILSON-FISHER FIXED POINT IN 2D & 3D.



$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu^2/2\lambda)^2$$

SPHERES AND CYLINDERS ARE NICE*

SPECIAL MAXIMALLY SYMMETRIC SPACES

- Conformal Field Theory are more easily studied on Sphere, Cylinders (Radial Quantization) and Hyperbolic Spaces (Gauge/Gravity Duality)

$\mathbb{S}^d$ $\qquad$ $\mathbb{R} \times S^{d-1}$ $\qquad$ $\mathbb{A}d\mathbb{S}^{d+1}$

$\mathbb{R}^d \to \mathbb{S}^d$

$$ds^2_{flat} = \sum_{\mu=1}^{d} dx^\mu dx^\mu = e^{2\sigma(x)} d\Omega^2_d \xrightarrow{Weyl} d\Omega^2_d \ .$$

$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$

$$ds^2_{flat} = \sum_{\mu=1}^{d} dx^\mu dx^\mu = e^{2\tau}(d\tau^2 + d\Omega^2_d) \xrightarrow{Weyl} (d\tau^2 + d\Omega^2_d) \ .$$

$\mathbb{R}^{d+1} \to \mathbb{A}d\mathbb{S}^{d+1}$

$$ds^2_{flat} = \sum_{\mu=1}^{d+1} dx^\mu dx^\mu \xrightarrow{Weyl} z^{-2}(dz^2 + d\vec{x}\cdot d\vec{x})$$

Constructing the Classical Simplicial Action

$$S = \frac{1}{2}\int_{\mathcal{M}} d^d x \sqrt{g(x)}\,[g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi^2(x) + \lambda\phi^4(x)]$$

Regge Calculus discretized Manifold $g_{\mu\nu}(x)$

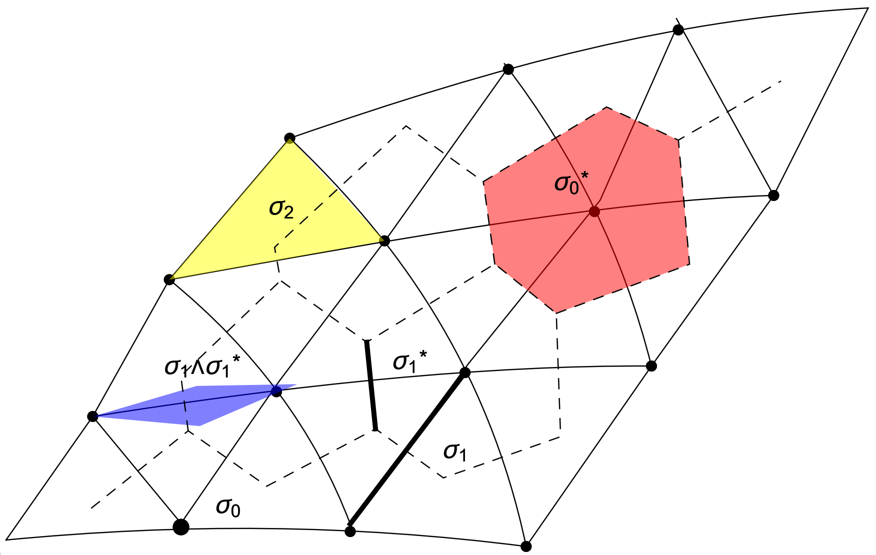
Finite Element discretized field $\phi(x)$

Classical Simplicial Action

$$S_\sigma[\phi] = \frac{1}{2}\sum_{\langle\, i,j\rangle} V_{ij}\frac{(\phi_i - \phi_j)^2}{l_{ij}^2} + \frac{1}{2}\sqrt{g_i}[m^2\phi_i^2 + \lambda\phi_i^4]$$

REGGE: Piecewise linear metric

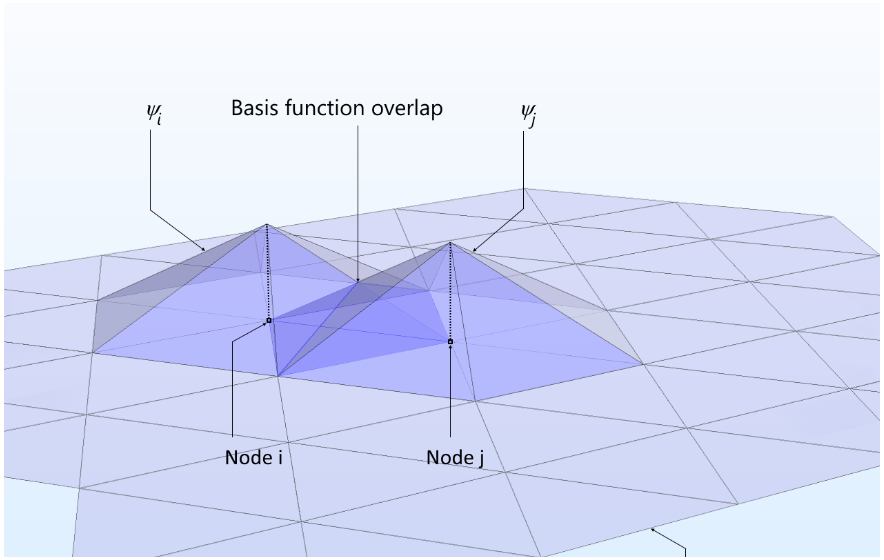
$$(\mathcal{M}, g_{\mu\nu}(x)) \leftrightarrow (\mathcal{M}_\sigma, g_\sigma = \{l_{ij}\})$$



Simplicial Complex/Delaunay Dual Complex + Regge flat metric on each Simplex

FEM: Piecewise linear fields

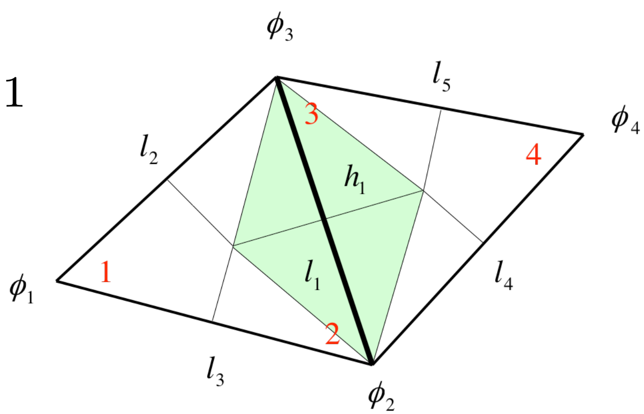
$$\phi(x) \leftrightarrow \phi = \sum_i \phi_i W_i(\xi)$$

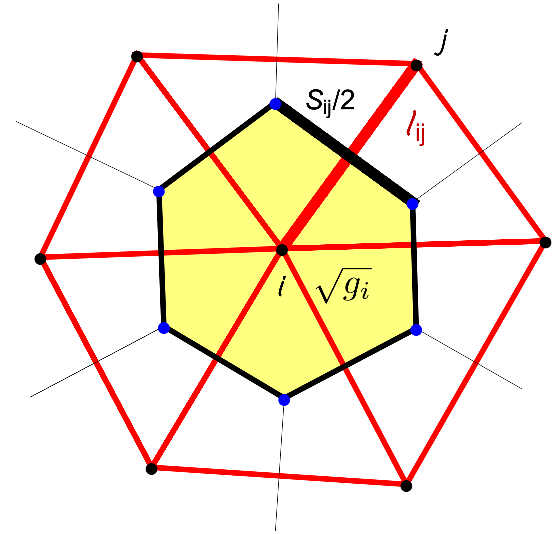


Actually fancier methods: Discrete Exterior Calculus (scalar), Spin connection (Fermion), Wilson links (gauge) , etc.

DISCRETE EXTERIOR CALCULUS (DEC) FOR BETRAMI-LAPLACE OPERATOR

$$A_{23} = h_1 l_1$$





$$\text{FEM:} \quad A_{23}\frac{(\phi_2 - \phi_3)^2}{l_{12}^2}$$

$$\langle\, \sigma_n | d\omega \rangle = \langle\, \partial\sigma_n | \omega \rangle$$

$$\Longrightarrow$$

$$*d * d\phi_i = * \frac{1}{|\sigma_0^*(i)|}\int_{\sigma_0^*} d[*(\phi_i - \phi_j)/l_{ij}] = \frac{1}{\sqrt{g_i}} \sum_{j \in \langle\, i,j \rangle} \frac{V_{ij}}{l_{ij}} \frac{\phi_i - \phi_j}{l_{ij}}$$

DEC implement discrete exterior Hodge * to Dual lattice and Stokes Theorem etc
(See also classic papers by Christ, Friedberg and Lee. NP 1982)

FEM/REGGE TOOL BOX

Geometry of Simplicial Complex is VERY useful. Get Discrete Exterior Calculus

$$\partial\sigma_n(i_0 i_1 \cdots i_n) = \sum_{k=0}^{n} (-1)^k \sigma_{n-1}(i_0 i_1 \cdots \widehat{i}_k \cdots i_n) \; ,$$

$$|\sigma_n \wedge \sigma_n^*| = \frac{n!(d-n)!}{d!} |\sigma_n||\sigma_n^*|$$

$$\int_{\sigma_{k+1}} \mathbf{d}\omega(x) = \int_{\partial\sigma_{k+1}} w(x) \quad \to \quad \langle\, \mathbf{d}\omega_k, \sigma_{k+1} \rangle = \langle\, \omega_k, \partial\sigma_{k+1} \rangle$$

NOTE: Only Kahler Dirac & Staggered Fermions follows form Exterior Calculus

SUMMARY OF SIMPLICIAL FIELDS

J = 0

$$S_{scalar} = \frac{1}{2} \sum_{\langle i,j \rangle} \frac{V_{ij}}{l_{ij}^2} (\phi_i - \phi_j)^2 \ , \qquad l_{ij}^2 = |\sigma_1(ij)|^2$$

J = 1/2

$$S_{Wilson} = \frac{1}{2} \sum_{\langle i,j \rangle} \frac{V_{ij}}{l_{ij}} (\bar{\psi}_i \hat{e}_a^{j(i)} \gamma^a \Omega_{ij} \psi_j - \bar{\psi}_j \Omega_{ji} \hat{e}_a^{i(j)} \gamma^a \psi_i)$$

J = 1

$$S_{gauge} = \frac{1}{2g^2 N_c} \sum_{\triangle_{ijk}} \frac{V_{ijk}}{A_{ijk}^2} Tr[2 - U_{\triangle_{ijk}} - U^\dagger_{\triangle_{ijk}}]$$

FFdual

$$\epsilon^{ijkl} Tr[U_{\triangle_{0ij}} U_{\triangle_{0kl}}] \simeq V_{ijkl} \epsilon^{\mu\nu\rho\sigma} Tr[F_{\mu\nu}(0) F_{\rho\sigma}(0)]$$

$$U_{\triangle_{ijk}} = U_{ij} U_{jk} U_{ki} \qquad A_{ijk} = |\sigma_2(ijk)| \qquad V_{ijk} = |\sigma_2(ijk) \wedge \sigma_2^*(ijk)|$$

$$U_{0ij} = U_{0i} U_{ij} U_{j0} \quad , \quad U^\dagger_{0ij} = U_{0j} U_{ji} U_{i0} \qquad V_{ij} = |\sigma_1(ij) \wedge \sigma_1^*(ij)|$$

Dirac ON SIMPLICIAL MANIFOLD

$$S = \frac{1}{2}\int d^D x \sqrt{g}\bar{\psi}[\mathbf{e}^\mu(\partial_\mu - \frac{i}{4}\boldsymbol{\omega}_\mu(x)) + m]\psi(x)$$

$\mathbf{e}^\mu(x) \equiv e^\mu_a(x)\gamma^a$ Vierbein & Spin connection*

$\boldsymbol{\omega}_\mu(x) \equiv \omega^{ab}_\mu(x)\sigma_{ab}$, $\sigma_{ab} = i[\gamma_a, \gamma_a]/2$

(1) New spin structure “knows” about intrinsic geometry

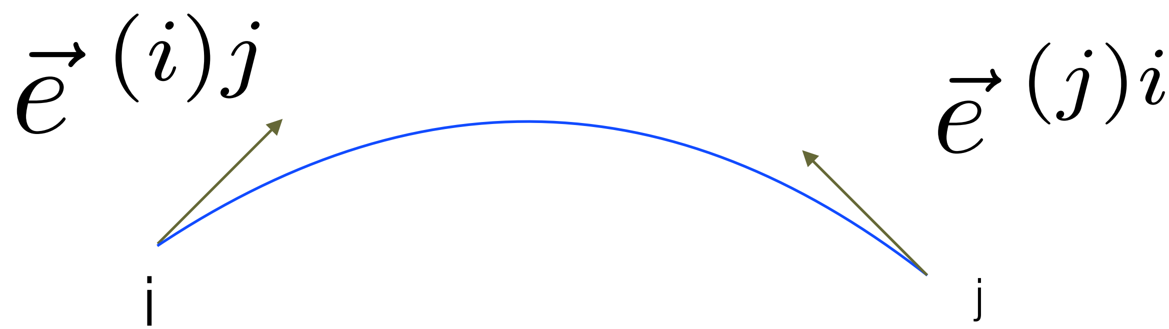
(2) Need to avoid simplex curvature singularities at sites.

(3) Spinors rotations (Lorentz group) is double of O(D).

* Must satisfy the Tetrad Hypothesis

$$\omega^{ab}_\mu = \frac{1}{2}e^{\nu[a}(e^{b]}_{\nu,\mu} - e^{b]}_{\mu,\nu} + e^{b]\sigma}e^c_\mu e_{\nu c,\sigma}).$$

$$S_{naive} = \frac{1}{2}\sum_{\langle i,j\rangle}\frac{V_{ij}}{l_{ij}}[\bar{\psi}_i\vec{e}^{\,(i)j}\cdot\vec{\gamma}\Omega_{ij}\psi_j - \bar{\psi}_j\Omega_{ji}\vec{e}^{\,(i)j}\cdot\vec{\gamma}\psi_i] + \frac{1}{2}mV_i\bar{\psi}_i\psi_i$$



Simplicial Tetrad Hypothesis

$$e_a^{(i)j}\gamma^a\Omega_{ij} + \Omega_{ij}e_a^{(j)i}\gamma^a = 0$$

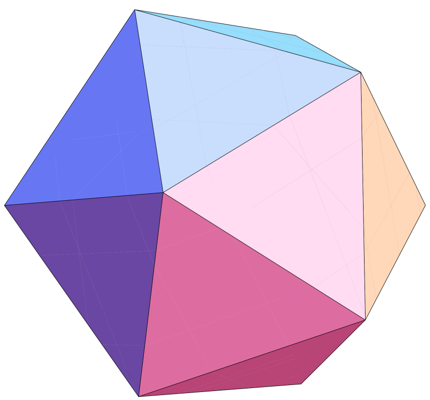
Gauge Invariance under Spin(D) transformations

$$\psi_i \to \Lambda_i\psi \quad , \quad \bar{\psi}_j \to \bar{\psi}_j\Lambda_j^\dagger \quad , \quad \mathbf{e}^{(i)j} \to \Lambda_i\mathbf{e}^{(i)j}\Lambda_i^\dagger \quad , \quad \Omega_{ij} \to \Lambda_i\Omega_{ij}\Lambda_j^\dagger$$

Note generalization to Domain Wall straight forward. Add an extra flat direction. Limit of extra dimension is overlap Fermion.

Test CFT on Sphere

$s = 1$



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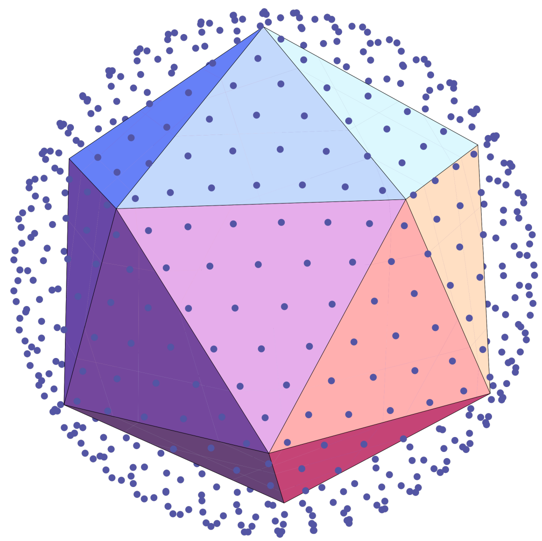
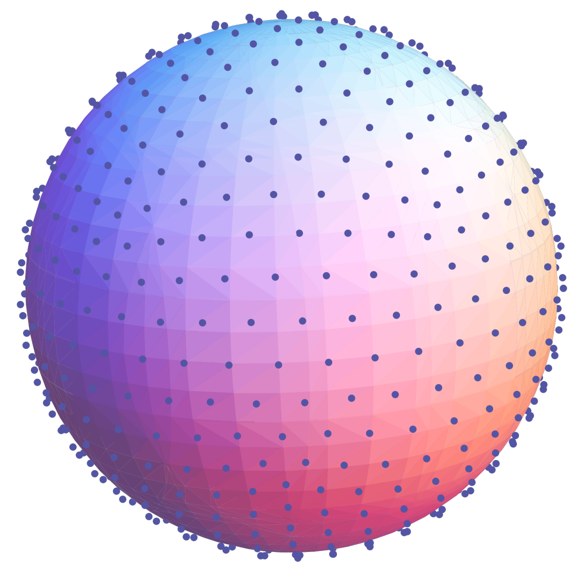


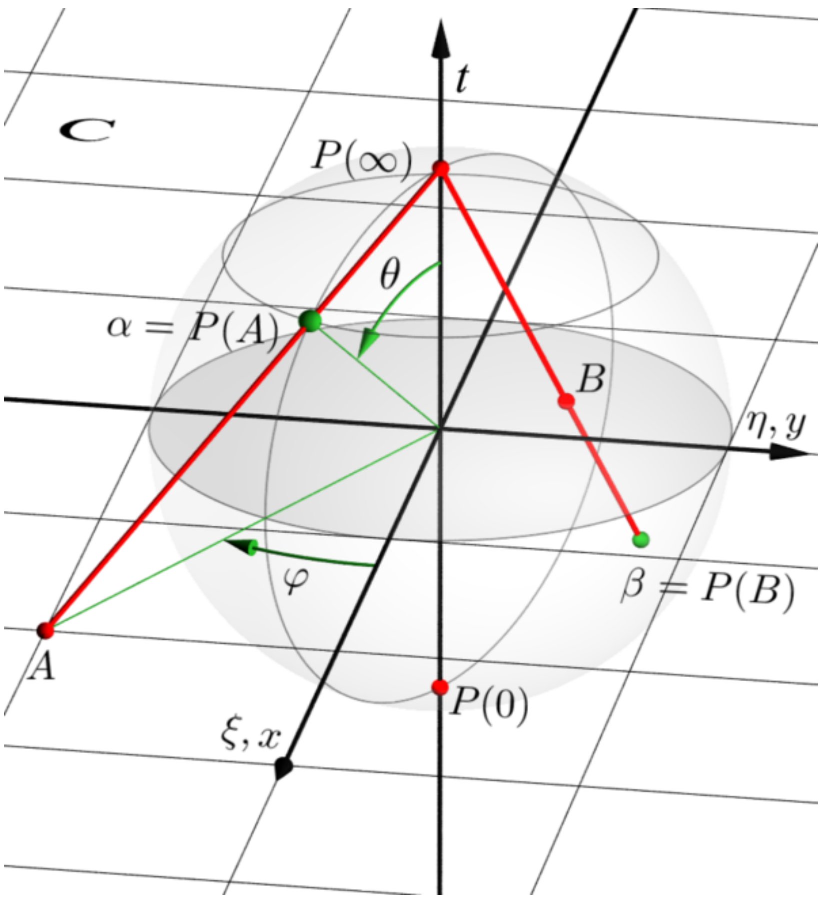


l = 0 (A),1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of O(3)

1/p + 1/q > 1/2 for regular positive curvature tessellation

TEST 2D ISING/PHI 4TH ON THE RIEMANN SPHERE

Stereographic project of Complex Plane:



Conformally Invariant
Cross Ratios are "Preserved"

$$\xi = \tan(\theta/2)e^{-i\phi} = \frac{x+iy}{1+z}$$

$$\xi = \xi_1 + i\xi_2$$

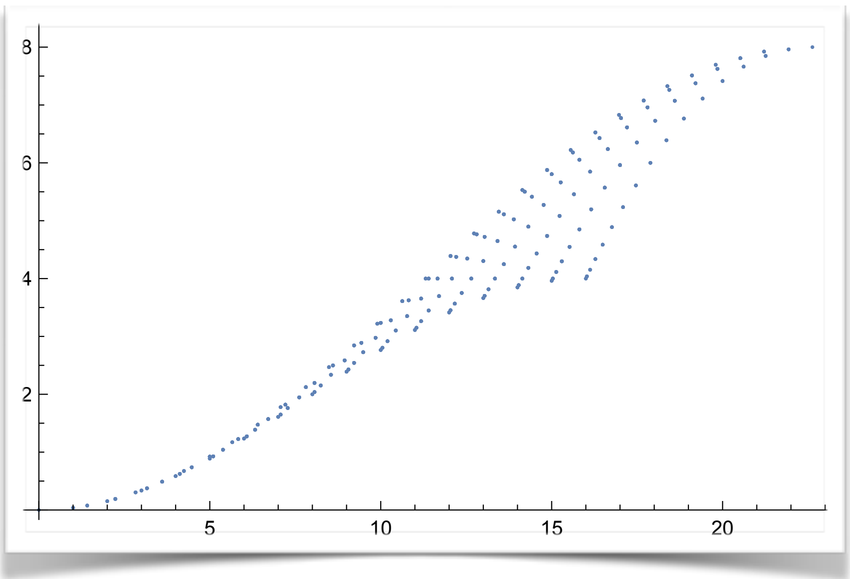
$$\vec{r} = (x, y, z) \qquad \vec{r}\cdot\vec{r} = 1$$

$$|\vec{r}_1 - \vec{r}_2|^2 = 2 - 2\cos(\theta_{12})$$

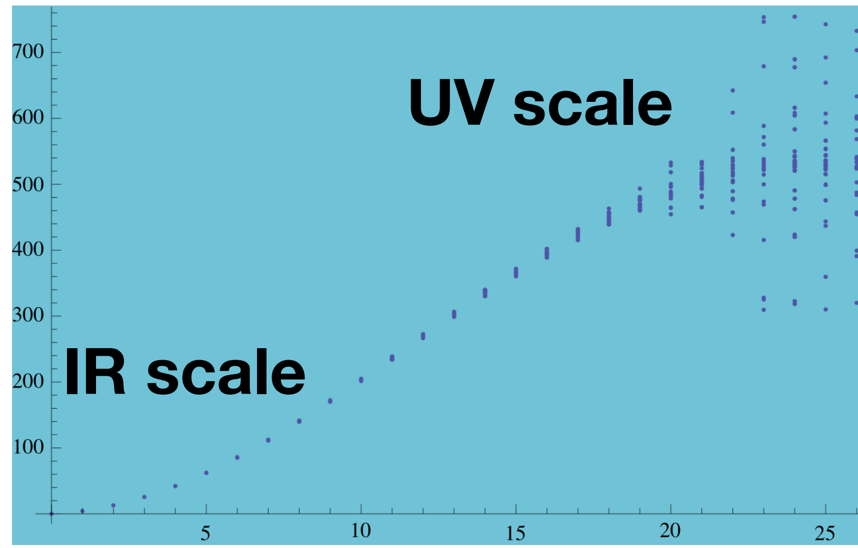
$$\sqrt{u} = \frac{|r_1 - r_2||r_3 - r_4|}{|r_1 - r_4||r_2 - r_3|}$$

$$\sqrt{v/u} = \frac{|r_1 - r_2||r_3 - r_4|}{|r_1 - r_3||r_2 - r_4|}$$

Restoring Isometries for ON A SIMPLICIAL COMPLEX

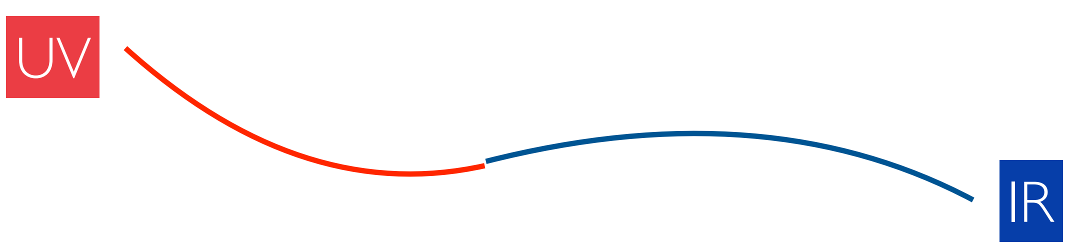


Hypercubic Lattice

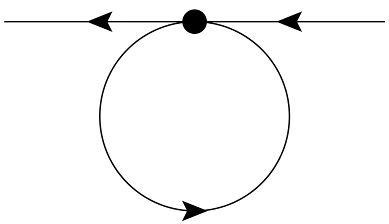


Simplicial Sphere

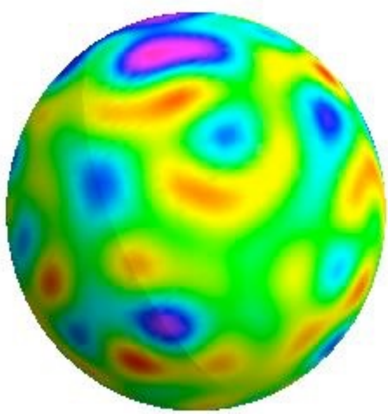
Is Renormalized perturbation theory at the UV fixed point enough to uniquely/correctly define the IR Quantum Field Theory?



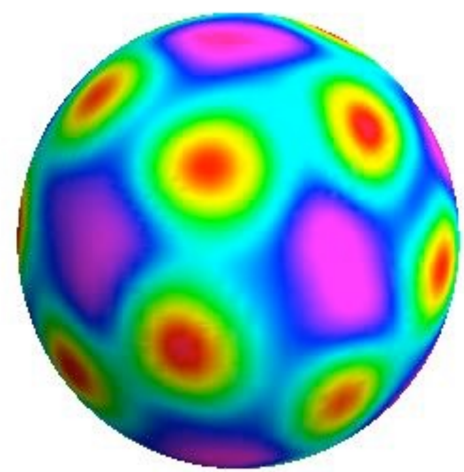
UV DIVERGENCE BREAKS ROTATIONS



$$\delta m^2 = \lambda \langle \phi(x)\phi(x) \rangle \to \frac{1}{K_{xx}}$$



one configuration



average of config.

One LOOP Counter Term

$$\Delta m_i^2 = 6\lambda \left[K^{-1}\right]_{ii} \simeq \frac{\sqrt{3}}{8\pi}\lambda \log(1/m_0^2 a_i^2) = \frac{\sqrt{3}}{8\pi}\lambda \log(N_s) + \frac{\sqrt{3}}{8\pi}\lambda \log(a^2/a_i^2)$$

Exact Continuum Divergence

Local Cut-off Scheme Dependence

$$\delta\mu_i^2 = -6\lambda(\left[K^{-1}\right]_{ii} - \frac{1}{N_s}\sum_{j=1}^{N_s}\left[K^{-1}\right]_{jj})$$

Now Binder Cumulant Converges

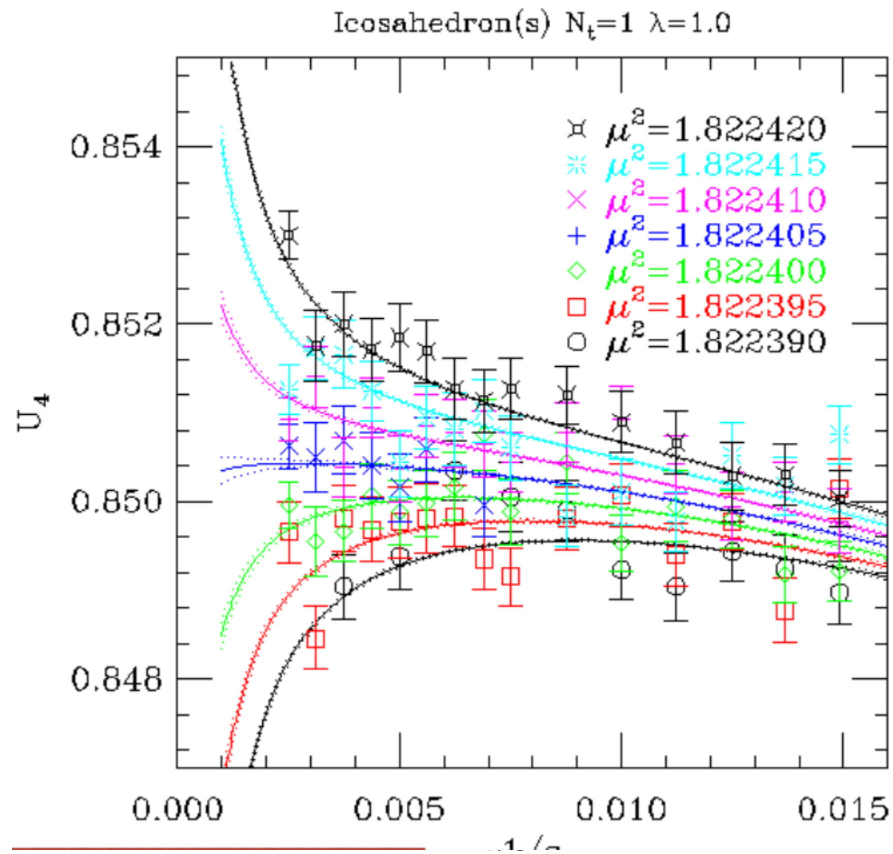
$$U_{2n}(\mu^2, \lambda, s) = U_{2n,cr} + a_{2n}(\lambda)[\mu^2 - \mu^2_{cr}]s^{1/\nu} + b_{2n}(\lambda)s^{-\omega}$$

FIT $U_{4,\mathrm{cr}} = 0.85020(58)(90)$

THEORY $U_4^* = 0.8510207(63)$

FIT $U_{6,\mathrm{cr}} = 0.77193(37)(90)$

THEORY $U_6^* = 0.773144(21)$



$\mu^2_{cr} = 1.82240070(34)$

Simultaneous fit for s up 800:
E.G. **6,400,002** Sites on Sphere

$dof = 1701 \quad , \quad \chi^2/dof = 1.026$

EXACT CORRELATOR FOR C = 1/2 CFT ON 2D SPHERE

2 pt function

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \to \frac{1}{|1 - cos\theta_{12}|^{\Delta}} \qquad \Delta = \eta/2 = 1/8$$

4 pt function

$$g(u,v) = \frac{\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle}{\langle \phi(x_1)\phi(x_3)\rangle \; \langle \phi(x_2)\phi(x_4)\rangle}$$

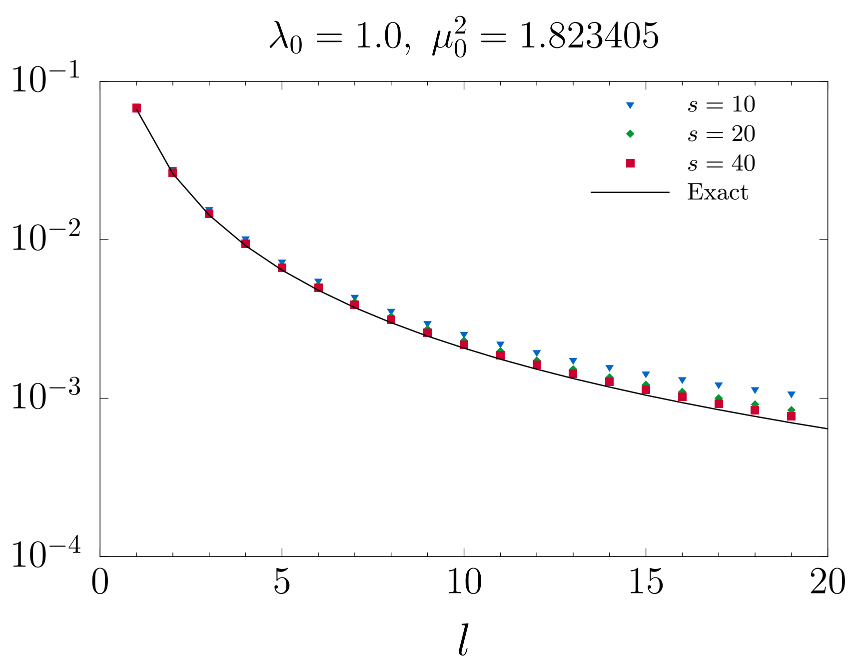
$$= \frac{1}{2|z|^{1/4}|1-z|^{1/4}}[|1+\sqrt{1-z}| + |1-\sqrt{1-z}|]$$

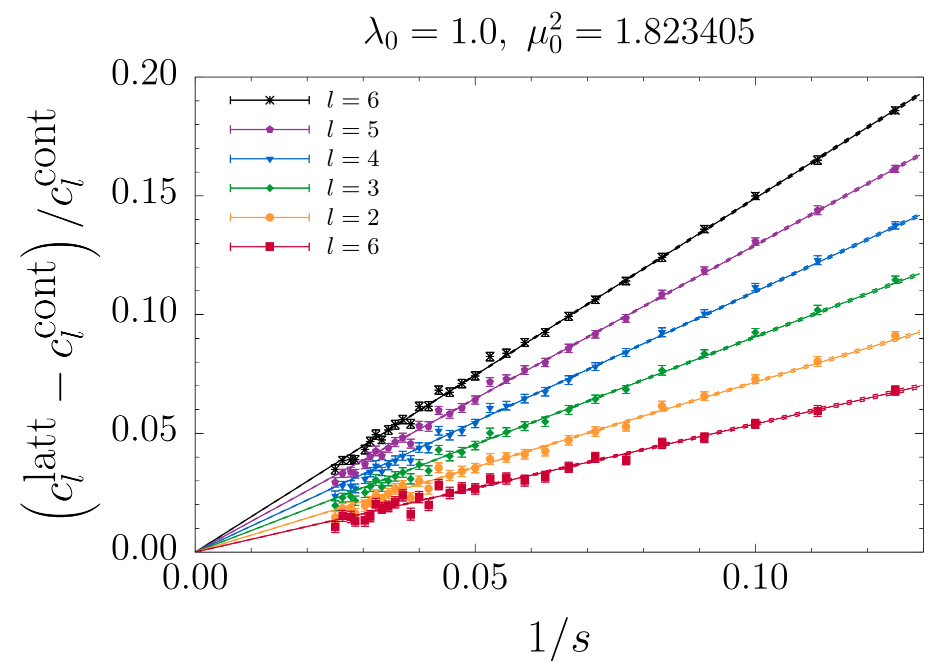
$$u = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 x_{24}^2} \quad , \quad v = \frac{r_{14}^2 r_{32}^2}{r_{13}^2 r_{24}^2} \qquad u = |z|^2 \qquad v = |1-z|^2$$

Critical Binder Cumulant

$$M = \sum_x \phi(x)$$

$$U_4^* = \frac{3}{2}[1 - \frac{\langle M^4\rangle}{3\langle M^2\rangle \; \langle M^2\rangle}] = 0.85102$$



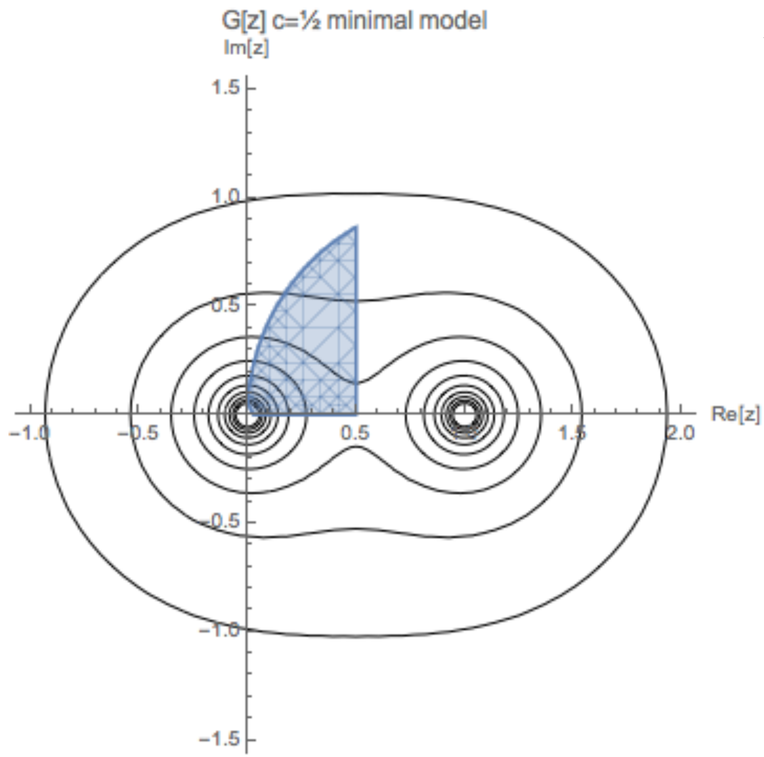


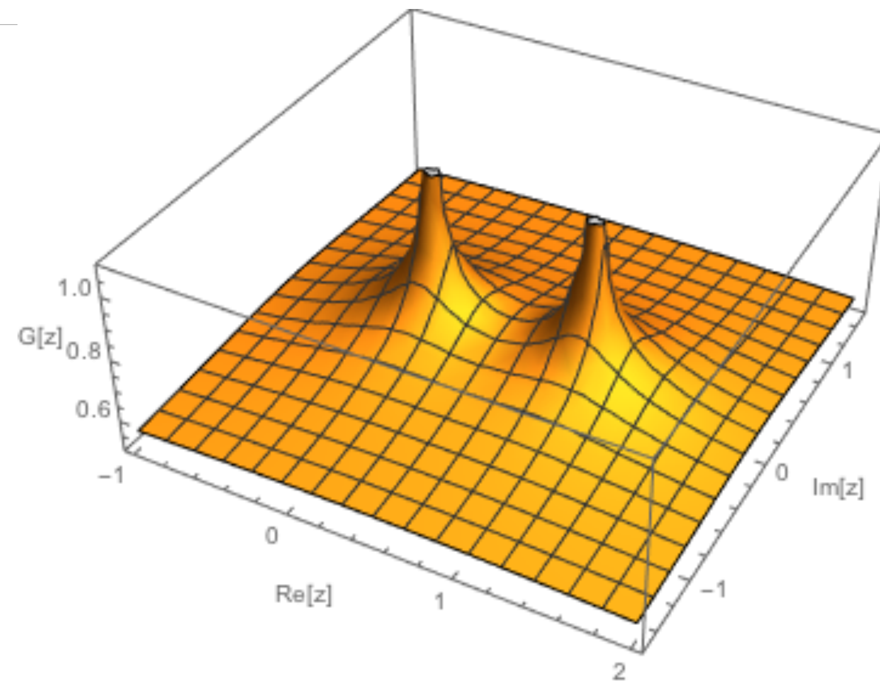
$$\int_{-1}^{1} dz \left(\frac{2}{1-z}\right)^{1/8} P_l(z)$$

$$\Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

$$\Longrightarrow \frac{16}{7}, \frac{16}{35}, \frac{48}{161}, \frac{816}{3565}, \frac{12240}{64883}, \frac{493680}{3049501}, \frac{33456}{234577}, \frac{55760}{435643}, \frac{3602096}{30930653}, \frac{20129360}{187963199}, \frac{541373840}{5450932771} \cdots$$

OPE Expansion: $\phi \times \phi = \mathbf{1} + \phi^2$ or $\sigma \times \sigma = \mathbf{1}$ -





$$G_s(r,\theta) \propto 1 + \lambda_\epsilon^2 \, g_{\epsilon,0}(r,\theta) + \lambda_T^2 \, g_{T,2}(r,\theta)$$

$\lambda_T^2 = \frac{\Delta_\sigma^2 d^2 |z|^{d-2}}{C_T (d-1)^2} \to \frac{1}{16 C_T}$ for $d = 2$, $g_{T,2}(z) = -3\left(1 + \frac{1}{z}\left(1 - \frac{z}{2}\right)\log(1-z)\right) + \text{c.c.}$

Fit TO OPE EXPANSION

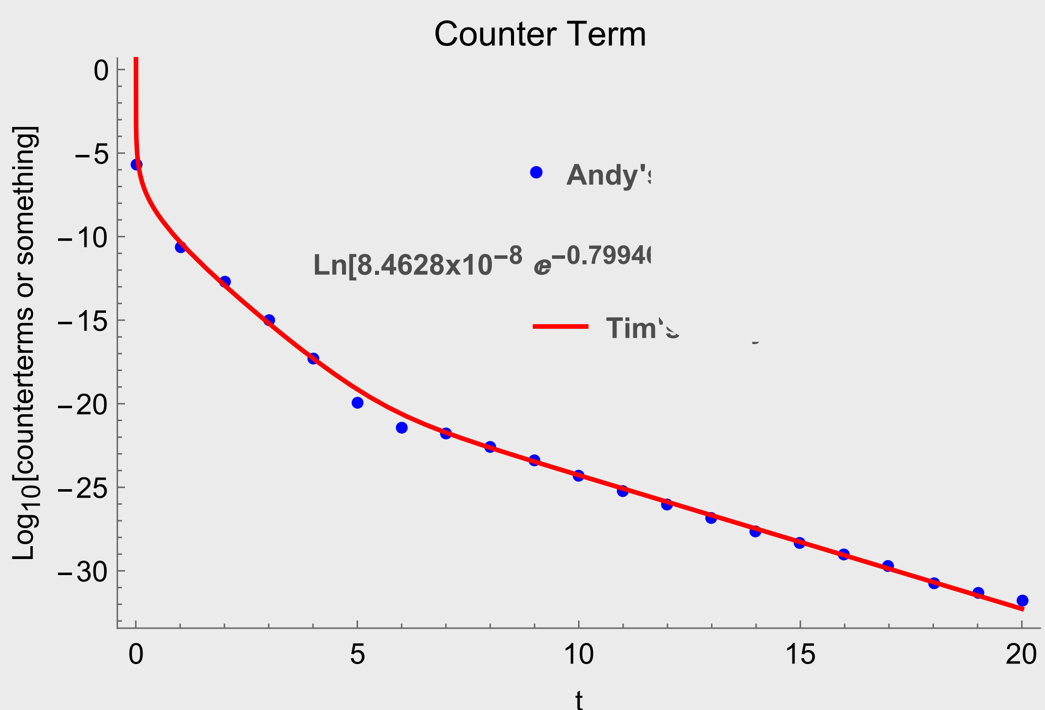
μ^2	s	$r_{\min} \leq r \leq r_{\max}$	norm	Δ_ϵ	λ_ϵ^2	c
1.82241	9	$0.25 \leq r \leq 0.75$	0.2900	1.075	0.2536	0.4668
1.82241	9	$0.30 \leq r \leq 0.70$	0.2901	1.075	0.2533	0.4704
1.82241	9	$0.35 \leq r \leq 0.65$	0.2902	1.077	0.2533	0.4738
1.82241	9	$0.40 \leq r \leq 0.60$	0.2902	1.016	0.2427	0.4747
1.82241	18	$0.25 \leq r \leq 0.75$	0.2051	1.068	0.2563	0.4866
1.82241	18	$0.30 \leq r \leq 0.70$	0.2051	1.056	0.2544	0.4878
1.82241	18	$0.35 \leq r \leq 0.65$	0.2051	1.050	0.2535	0.4904
1.82241	18	$0.40 \leq r \leq 0.60$	0.2051	1.046	0.2526	0.4884
1.82241	36	$0.25 \leq r \leq 0.75$	0.1457	1.031	0.2528	0.4926
1.82241	36	$0.30 \leq r \leq 0.70$	0.1458	1.026	0.2519	0.4932
1.82241	36	$0.35 \leq r \leq 0.65$	0.1458	1.018	0.2508	0.4931
1.82241	36	$0.40 \leq r \leq 0.60$	0.1458	1.007	0.2486	0.4933

WHAT'S NEXT?

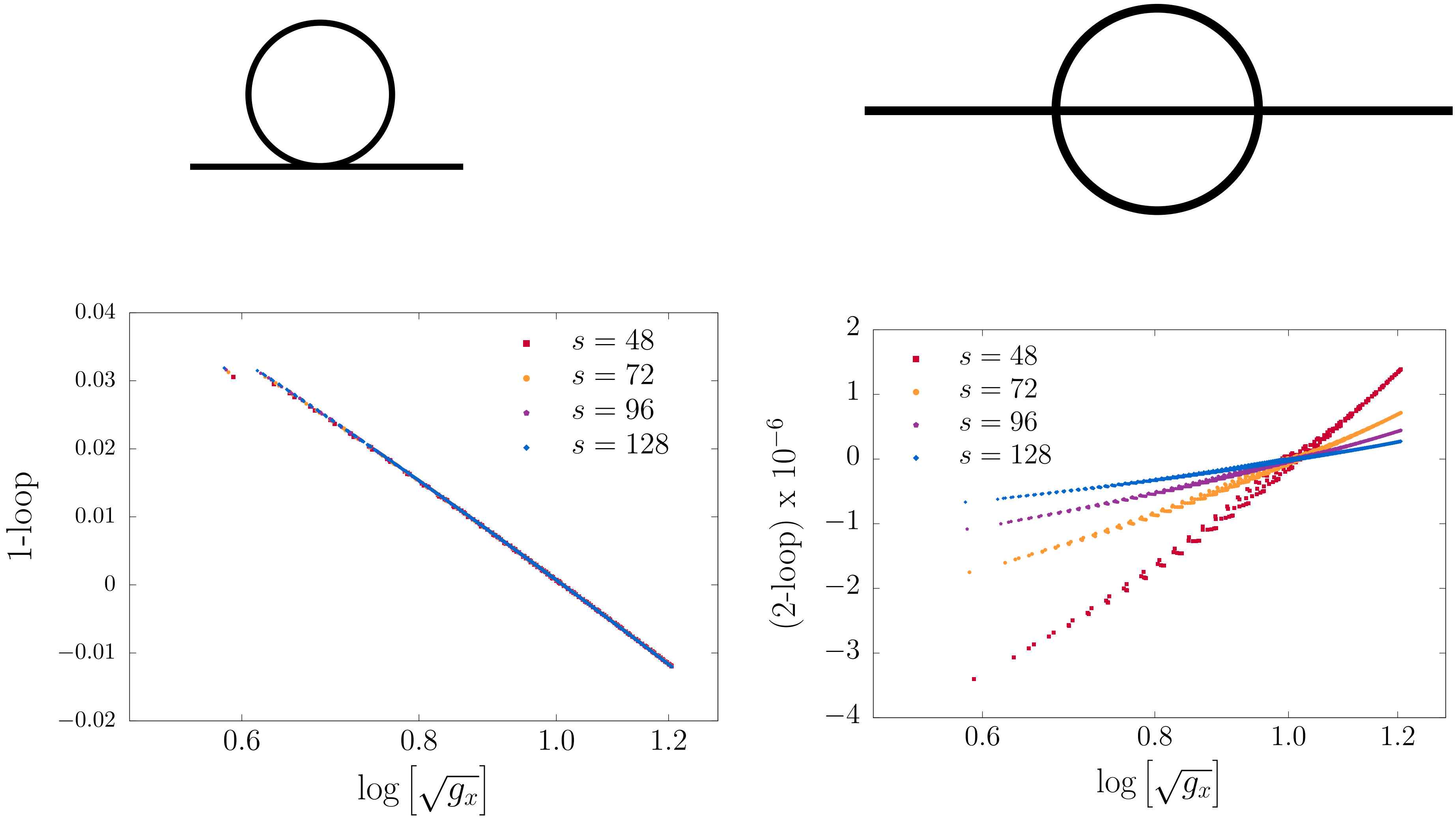
- 3D Radial Phi 4th Theory
- 4D Radial Gauge theory: Mass deformation for BSM
- Hyperbolic Disk dual to 1D CFT! SYK Model
- Writing Generic Lattice Code for GPUs
- Too many options. Need other to join in! It is fun.

Counter term in 3D

$$\delta\mu^2_{CT}(x,y) \;\sim\; \lambda_0 c_x \delta_{xy} + \lambda_0^2 e^{-6|x-y|/a}$$



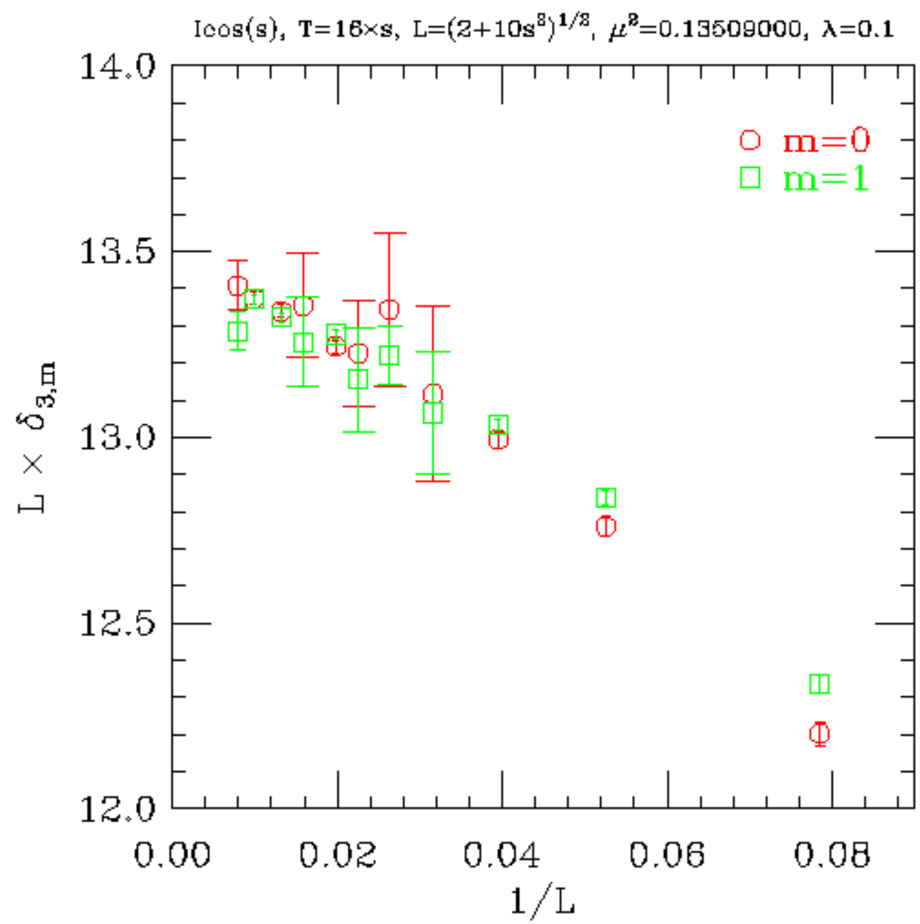
One Loop Counter Term vs Two Loop Convergence



PRELINARY DATA ON QFE L = 3 Spectrum
Hope to show recovery O(4,1) for l = 3?

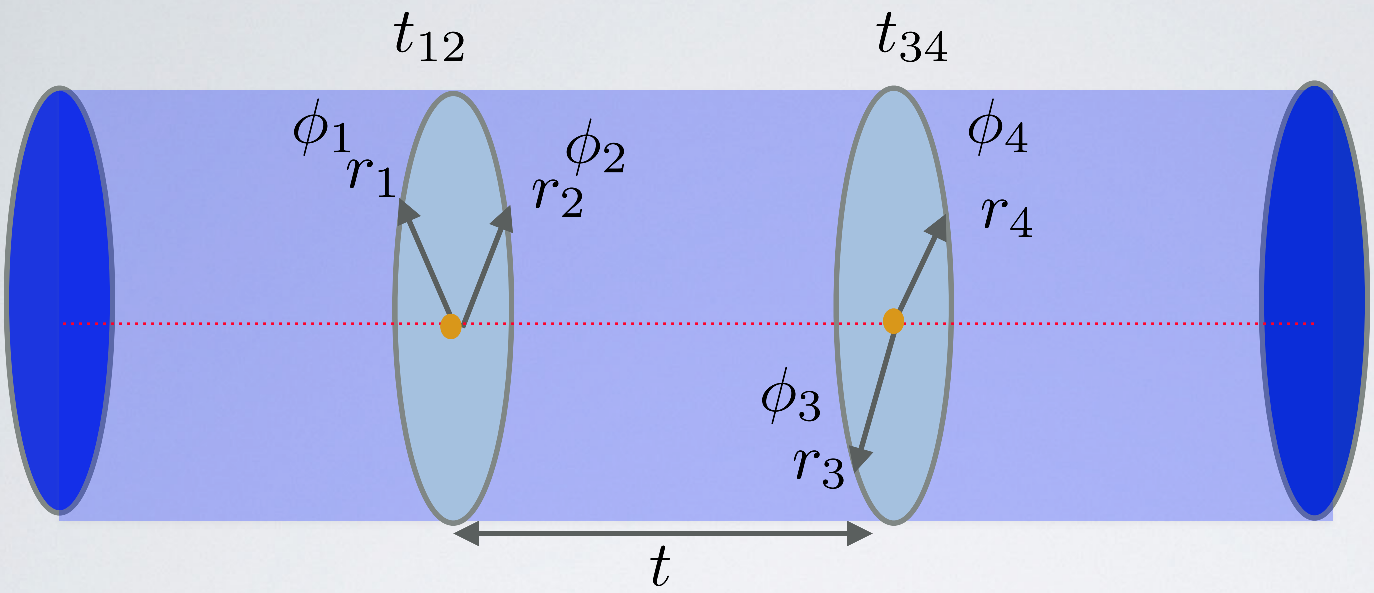
G rep

T2 rep



NEED MUCH MORE DATA!

4PT FUNCTION AND CENTRAL CHARGE



$$\cosh(\tau) = \frac{1+\sqrt{v}}{\sqrt{u}}$$

$$\cos(\alpha) = \frac{1-\sqrt{v}}{\sqrt{u}}$$

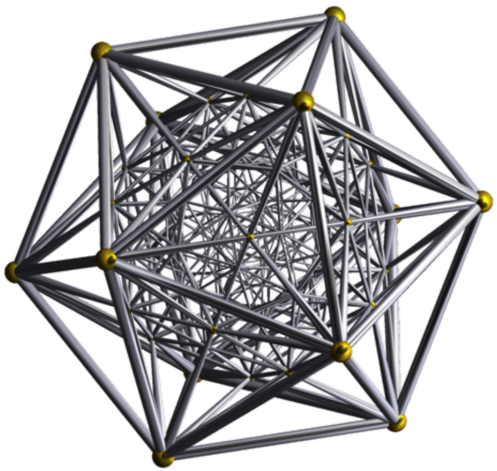
$$g(\tau,\alpha) = \langle 0 || r_1 - r_2|^{2\Delta_\sigma} \phi_1 \phi_2 \, e^{-tD} \, |r_3 - r_4|^{2\Delta_\sigma} \phi_3 \phi_4 | 0 \rangle$$

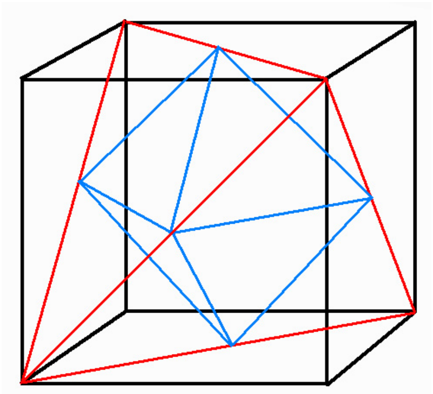
$$= \sum_{\Delta_l, l=0,2,4,\cdots} \langle \, |r_1 - r_2|^{\Delta_l} |r_3 - r_4|^{\Delta_l} \rangle \, \lambda^2_{\Delta_l} \, e^{-\Delta_l t} \, P_l(\cos\alpha)$$

where $\cos(\alpha) \simeq \frac{(r_1 - r_2)\cdot(r_3 - r_4)}{|r_1 - r_2||r_3 - r_4|}$ $\quad e^{-\tau} \simeq |r_1 - r_2||r_3 - r_4| e^{-t}$

3 Spheres and 4D Radial Simplicial Lattices

$$\mathbb{S}^3 \quad \Longrightarrow \quad \mathbb{R} \times \mathbb{S}^3$$





Aristotle' s 2% Error!

Fast Code Domains of Regular 3D Grids on Refinement

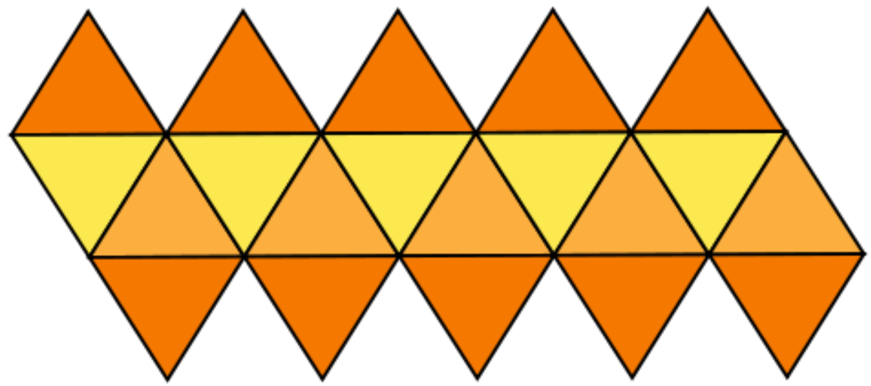
$$(2\pi - 5ArcCos[1/3])/(2\pi) = 0.0204336$$

The full symmetry group of the 600-cell is the Weyl group of H_4. This is a group of order 14400. It consists of 7200 rotations and 7200 rotation-reflections. The rotations form an invariant subgroup of the full symmetry group.

GPU DATA PARALLEL CODE: REGULAR DOMAIN REFINEMENTS

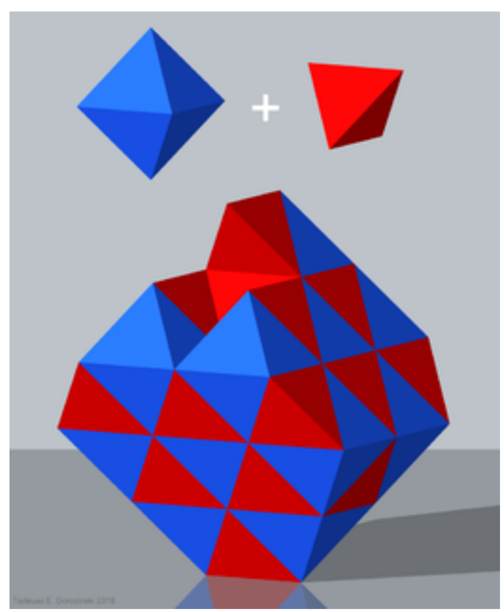
$\mathbb{S}^2$ Refinement $\Longrightarrow$

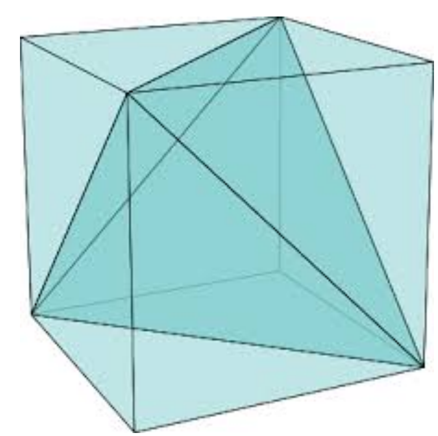
5 Regular Cartesian Triangle Graphs



$\mathbb{S}^3$ Refinement $\Longrightarrow$

Tetrahedral-octahedral honeycomb refinement

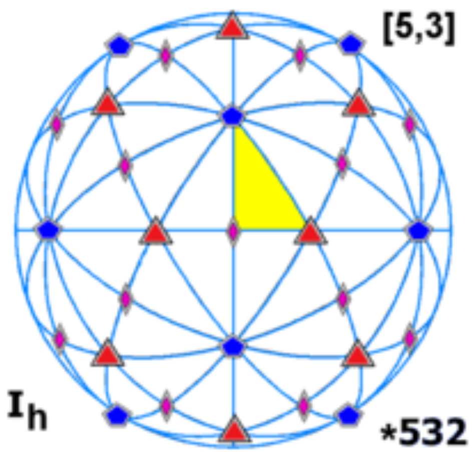




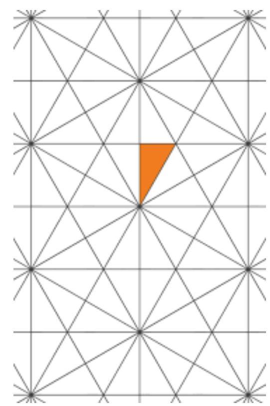
DISCRETE ISOMETRIES & THE TRIANGLE GROUP

$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} \begin{cases} > \pi & \text{Postive curvature} \\ = \pi & \text{Zero curvature} \\ < \pi & \text{Negative Curvature} \end{cases}$$

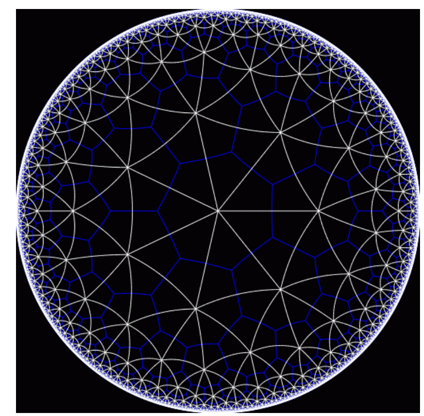
Tesselate by x, y, z refection
give p,q,r rotations: S = xy, T = yz, U = zx



$(2, 3, 5)$
120 element
Icosahedral in O(3)



$(2, 3, 6)$
Triangle Lattice
on Euclidean $\mathbb{R}^2$

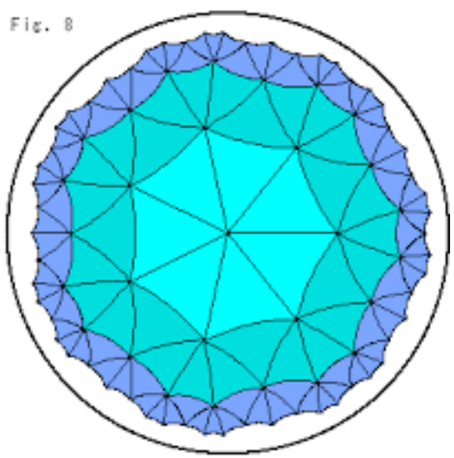


$(2, 3, 7)$
Subgroup of Modular
Group on $\mathbb{H}^2$

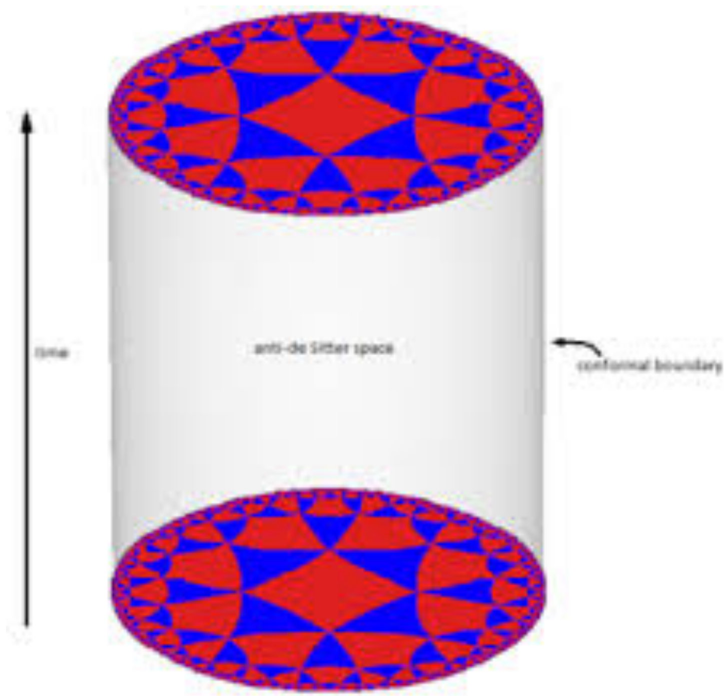
https://en.wikipedia.org/wiki/(2,3,7)_triangle_group

Hyperbolic (e.g. Poincare Disk) and Global AdS

$$\mathbb{H}^d \quad \rightarrow \quad \mathbb{A}dS^d$$

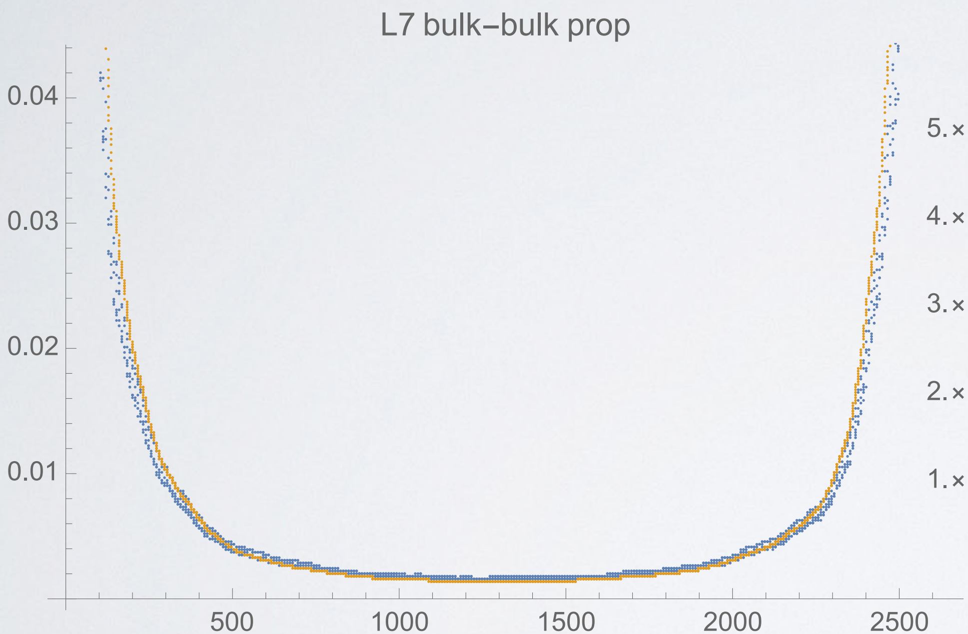


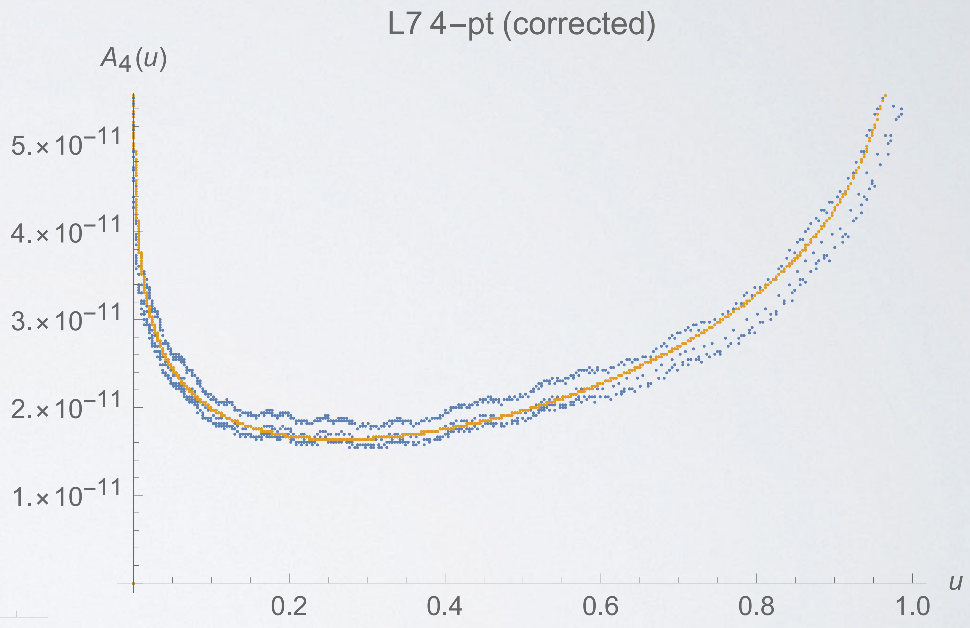
1/p + 1/q < 1/2



Triangle Group Tessellation: Preserve Finite subgroup of the Modular Group

PERTURBATION ON THE HYPERBOLIC DISK H2

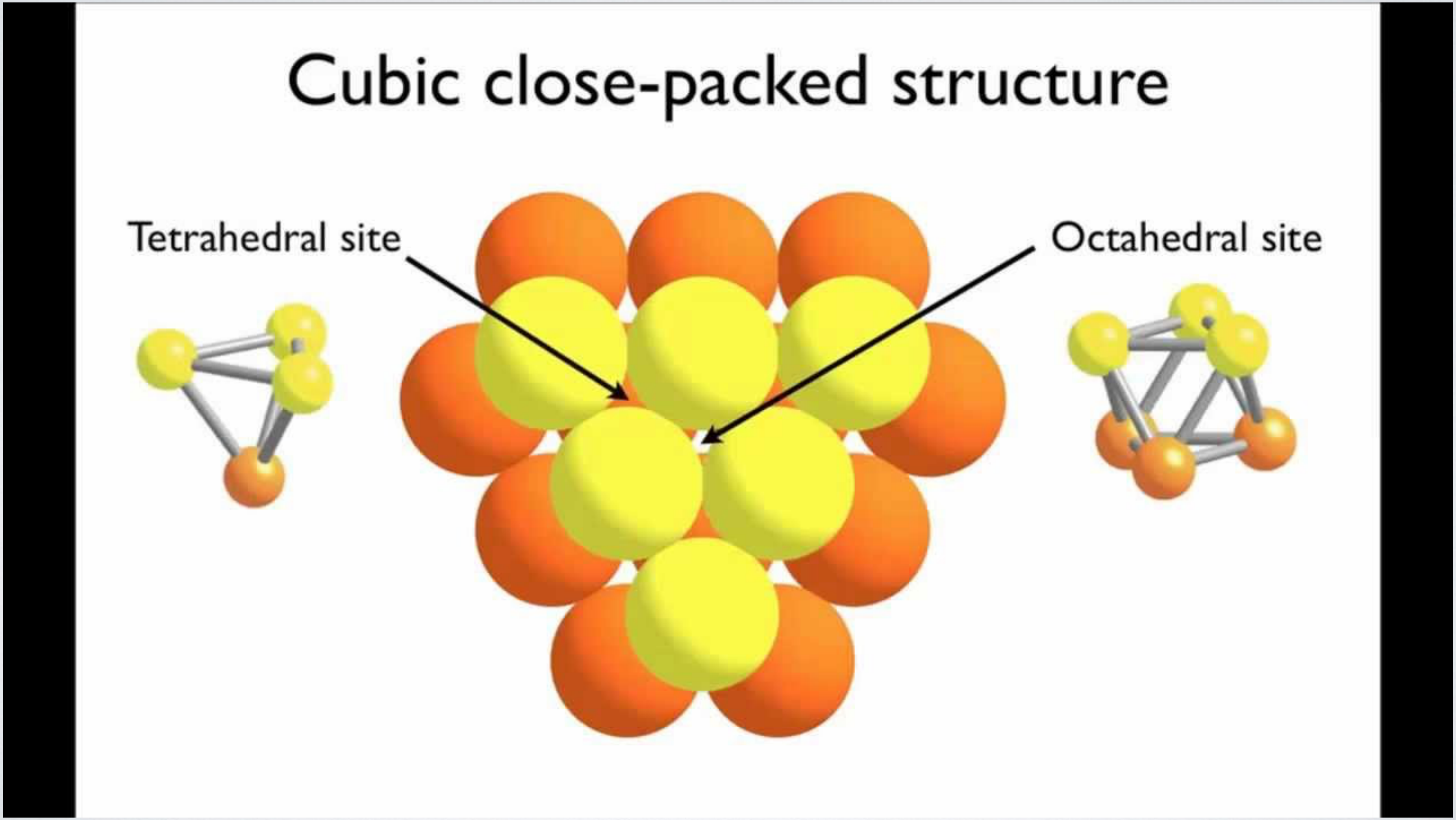




BACKUP SLIDES!

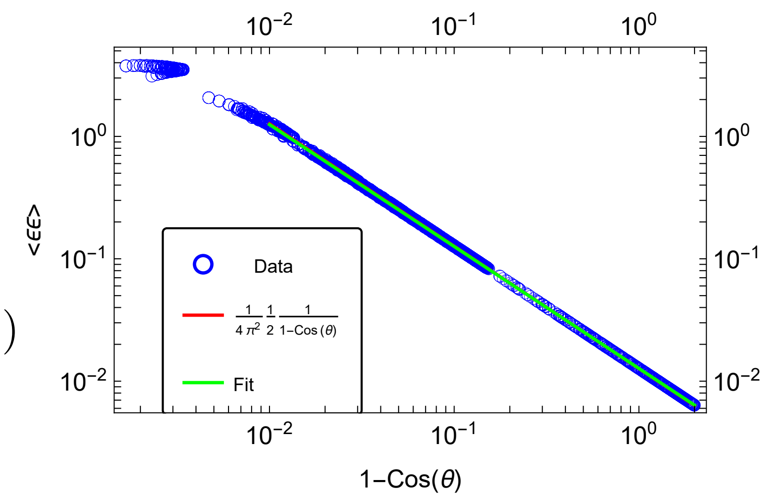
NEED COLLABORATORS & SUPPORT

3D BCC OR A3*

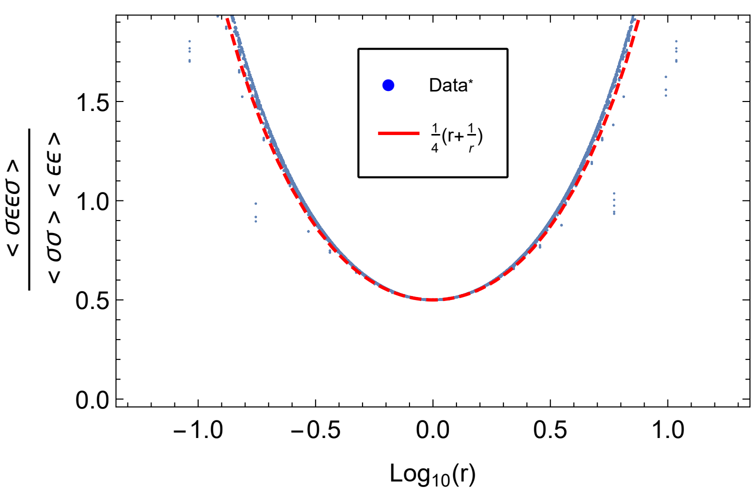

Cubic close-packed structure
Tetrahedral site
Octahedral site

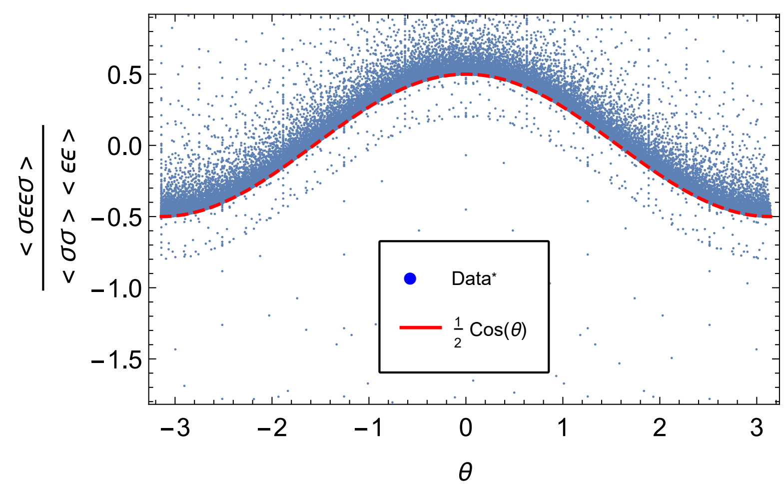
FREE MAJORANA FERMIONS ON S2

$$\langle \psi(z_1)\bar{\psi}(z_1)\bar{\psi}(z_1)\psi(z_2)\rangle = \left[\frac{1}{\partial}\right]_{z_1,z_2}\left[\frac{1}{\bar{\partial}}\right]_{z_1,z_2} = \frac{1}{4\pi^2}\frac{1}{|z_1-z_2|^2}$$



$$\frac{\langle \sigma(0)\epsilon(z_2)\epsilon(z_3)\sigma(\infty)\rangle}{\langle \epsilon(z_2)\epsilon(z_3)\rangle}$$
$$= \frac{1}{4}|\sqrt{z_1/z_2}+\sqrt{z_2/z_1}|^2 = \frac{1}{4}(r+1/r+2\cos\theta)$$





Using Binder Cumulants

$$U_4 = \frac{3}{2}\left(1 - \frac{m_4}{3\ m_2^2}\right) \qquad m_n = \langle \phi^n \rangle$$

$$U_6 = \frac{15}{8}\left(1 + \frac{m_6}{30\ m_2^3} - \frac{m_4}{2\ m_2^2}\right)$$

$$U_8 = \frac{315}{136}\left(1 - \frac{m_8}{630\ m_2^4} + \frac{2\ m_6}{45\ m_2^3} + \frac{m_4^2}{18\ m_2^4} - \frac{2\ m_4}{3\ m_2^2}\right)$$

$$U_{10} = \frac{2835}{992}\left(1 + \frac{m_{10}}{22680\ m_2^5} - \frac{m_8}{504\ m_2^4} - \frac{m_6\ m_4}{108 m_2^5} + \frac{m_6}{18\ m_2^3} + \frac{5\ m_4^2}{36\ m_2^4} - \frac{5\ m_4}{6\ m_2^2}\right)$$

$$U_{12} = \frac{155925}{44224}\left(1 - \frac{m_{12}}{1247400\ m_2^6} + \frac{m_{10}}{18900\ m_2^5} + \frac{m_8\ m_4}{2520\ m_2^6} - \frac{m_8}{420\ m_2^4} + \frac{m_6^2}{2700 m_2^6} - \frac{m_6 m_4}{45\ m_2^5} + \frac{m_6}{15\ m_2^3} - \frac{m_4^3}{108\ m_2^6} + \frac{m_4^2}{4\ m_2^4} - \frac{m_4}{m_2^2}\right)$$

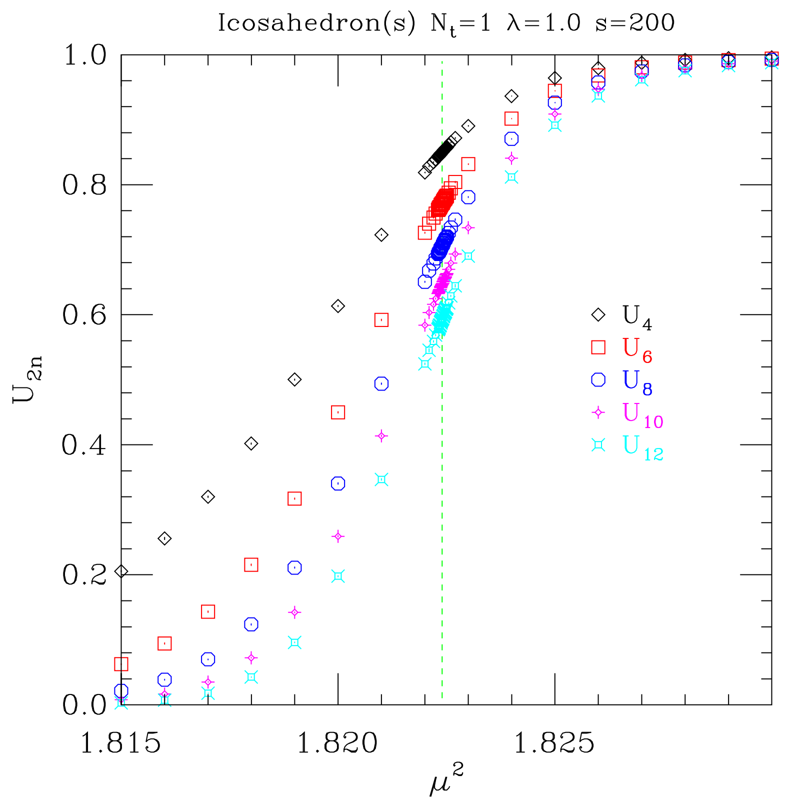
In infinite volume

$U_{2n}=0$ in disordered phase

$U_{2n}=1$ in ordered phase

$0<U_{2n}<1$ on critical surface

- $U_{2n,cr}$ are universal quantities.
- Deng and Blöte (2003): $U_{4,cr}=0.851001$
- Higher critical cumulants computable using conformal $2n$-point functions: Luther and Peschel (1975) Dotsenko and Fateev (1984)

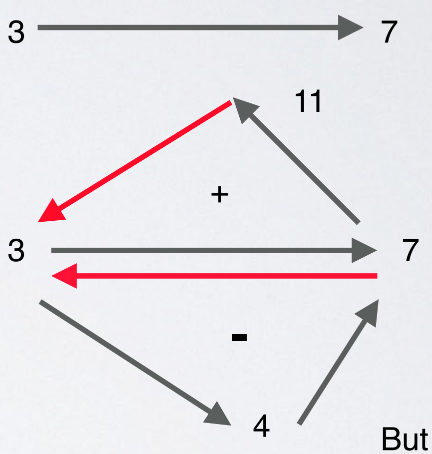


HOW DO WE KEEP TRACK OF SIGNS EVEN IN 2D & 3D?

Pick arbitrary lexical order of vertices. Positive simplex is in this order.

$$e.g. \quad \sigma_1(3,7) = -\sigma_1(7,3) > 0$$

$$e.g. \quad \sigma_2(3,7,11) = -\sigma_2(11,7,3) > 0$$



But 2 o 8 flips it!

In general even/odd permutation of lexical order

Note Adjacent Triangles are not both oriented.

Boundary sign operator sign works for triangle

$$\partial\sigma_n(i_0 i_1 \cdots i_n) = \sum_{k=0}^{n} (-1)^k \sigma_{n-1}(i_0 i_1 \cdots \widehat{i}_k \cdots i_n)$$

$$\partial\sigma_2(11,3,7) = \sigma_1(3,7) - \sigma_1(11,7) + \sigma_1(11,3) = \sigma_1(3,7) + \sigma_1(7,11) + \sigma_1(11.3)$$

How to construct an oriented manifold? Must start with orientation of D-dimensional simplicial complex?

What is * sign? Oriented Dual Lattice?

Form boundary vs co-boundary to d vs *d*

DISCRETE EXTERIOR CALC.

$$U_{ij} = P[\exp[i \int_{x_j}^{x_i} dx^\mu A_\mu(x)] \simeq 1 + i\theta_{ij} = 1 + l^\mu_{ij} A_\mu(x_c)$$

$$(\theta_{ij} + \theta_{jk} + \theta_{ki}) \simeq$$

$$\oint_{\triangle_{ijk}} dx^\mu A_\mu(x) = \iint_{\triangle_{ijk}} F_{\mu\nu}(x) dx^\mu \wedge dx^\mu \simeq A^{\mu\nu}_{ijk} F_{\mu\nu}(x_c)$$

Using Stokes theorem for the DEC: F = dA

How to see F^F on 4 simplex with sites 0,1,2,3,4? Need to prove this!

$$\epsilon^{ijkl} Tr[U_{\triangle_{0ij}} U_{\triangle_{0kl}}] \simeq V_{ijkl} \epsilon^{\mu\nu\rho\sigma} Tr[F_{\mu\nu}(0) F_{\rho\sigma}(0)]$$

$$U_{0ij} = U_{0i} U_{ij} U_{j0} \quad , \quad U^\dagger_{0ij} = U_{0j} U_{ji} U_{i0}$$

Sum over base vertex 5 base vertex 0

$$F_{k:ij} = [U_{ki}U_{ij}U_{jk} - U_{kj}U_{ji}U_{ik}]/(2i) \simeq A^{\mu\nu}_{ijk}F_{\mu\nu}$$

$$A^{\mu\nu}_{ijk} = \frac{1}{2}[r^{\mu}_{ik}r^{\nu}_{jk} - r^{\nu}_{ik}r^{\mu}_{jk}]$$

$$V_{12} = \frac{1}{2}|det[r^{\mu}_{i0}]| = (x_1 y_2 - x_2 y_1)/2$$

Because $$r_{kj} + r_{ki} + r_{ij} = 0$$

This oriented dual area two tensor is independent of the “open vertex”

$$r_{ij} = r_i - r_j \quad , \quad r_{jk} = r_j - r_k \quad , \quad r_{ki} = r_k - r_i$$

Therefore $$\epsilon^{ijkl}Tr[F_{0;ij}F_{0;kl}] \simeq \epsilon^{ijkl}A^{\mu\nu}_{0ij}A^{\lambda\rho}_{0kl}Tr[F_{\mu\nu}F_{\lambda\rho}]$$

$$\epsilon^{ijkl}A^{\mu\nu}_{0ij}A^{\lambda\rho}_{0kl} = \epsilon^{ijkl}r^{\mu}_{0i}r^{\nu}_{0j}r^{\lambda}_{0k}r^{\rho}_{0l} = 4!V_{ijkl}\epsilon^{\mu\nu\rho\lambda}$$

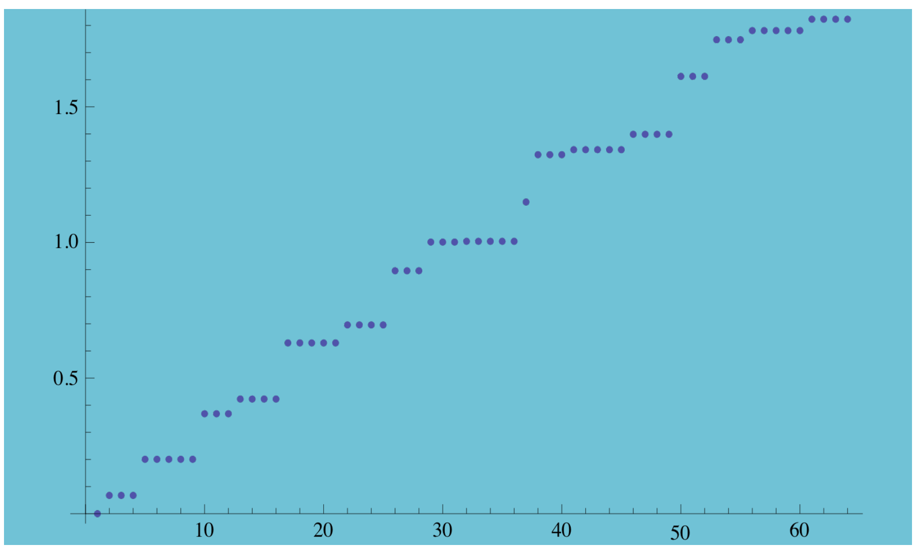
CAN WE PROVE IT USING VOLUME IDENTITY — Check Combinatorics

https://en.wikipedia.org/wiki/Simplex#Volume

FEM FIXES SPECTRAL DEFECTS OF LAPLACIAN ON SPHERE

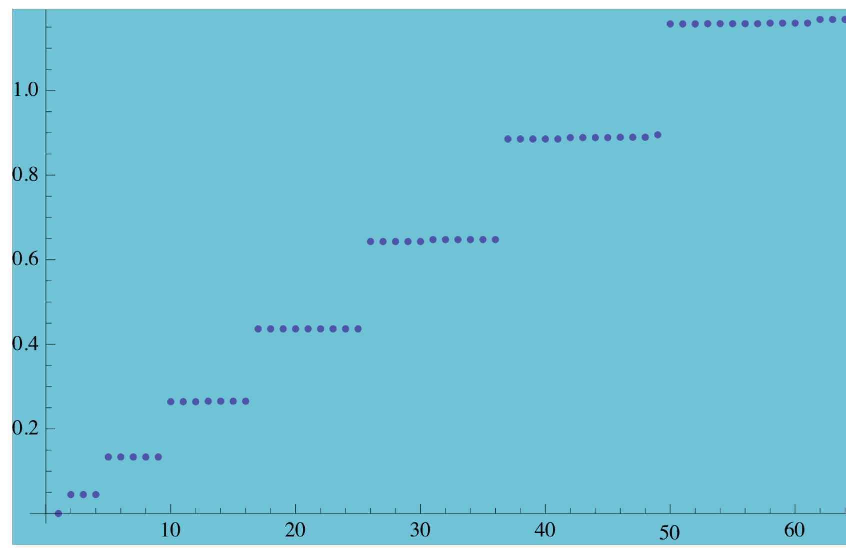
For s = 8 first (l+1)*(l +1) = 64 eigenvalues

BEFORE FEM



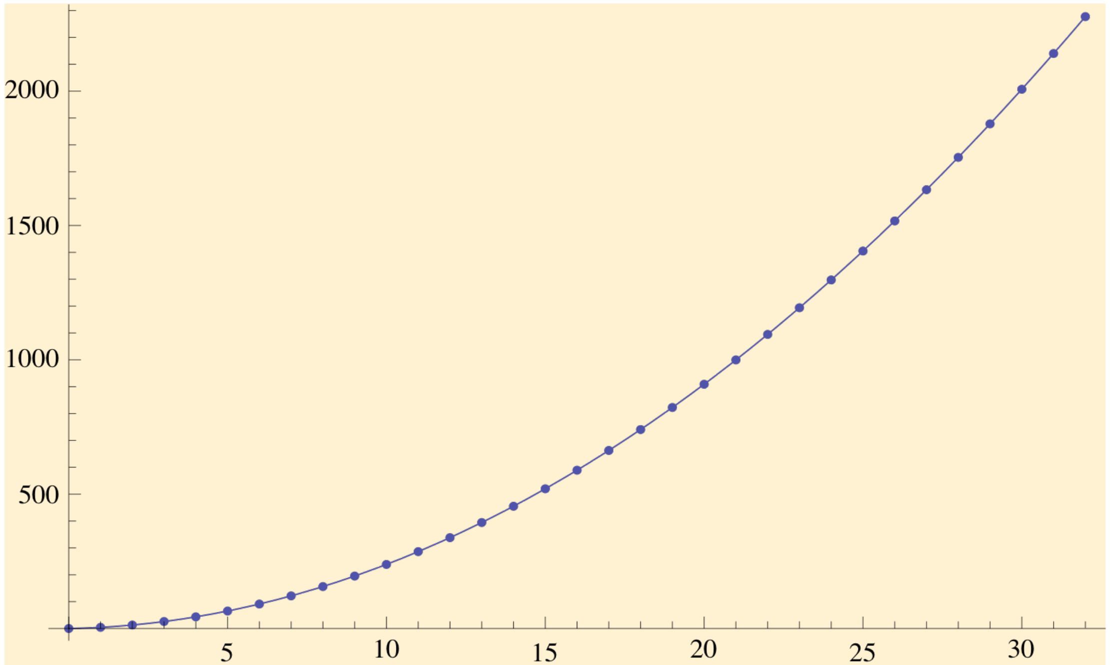
l, m

AFTER FEM



l, m

SPECTRAL FIDELITY ON $\mathbb{S}^2$



s = 128

Fit

$$l + 1.00012\, l^2 - 13.428110^{-6}\, l^3 - 5.5724410^{-6}\, l^4$$

MAXIMALLY SYMMETRIC TRIANGULATIONS.

- *Symmetry: Preserve maximal subgroup of isometries — very useful for testing and build correlators.*
- *Classical Convergence: Shape Regular refinement to maximize "spectral fidelity" accelerated convergence and simplify quantum counter terms.*
- *Efficient Data Parallel Code: To refine with graphs that with regular geometries to enable fast data parallel code*

RG Proof Of UNIVERSAL UV Logs

$$G_{xx}(m) \simeq c_x log(1/m^2 a_x^2) + O(a^2 m^2)$$

$$\implies \gamma(m_0^2) = -m_0 \frac{\partial}{\partial m_0} G_{xx}(m_0^2) \simeq 2c_x + O(m_0^2)$$

$$m_0 = am \to 0$$

FEM Spectral Fidelity

IR: Region. UV

$$G_{xy}(m^2) = \sum_n \frac{\phi_n^*(x)\phi_n(x)}{E_n^{(0)} + m^2}$$

$$\simeq \frac{\sqrt{3}}{8\pi} \sum_{l=0}^{L_0} \frac{(2l+1)P_l(r_x \cdot r_y)}{l(l+1) + \mu_0^2} + \sum_{n=(L_0+1)^2}^{N} \frac{\phi_n^*(x)\phi_n(y)}{E_n^{(0)} + m^2}$$

Insensitive to UV defects

$$\gamma(m_0^2) = -m_0 \frac{\partial}{\partial m_0} G_{xx}(m_0^2)$$

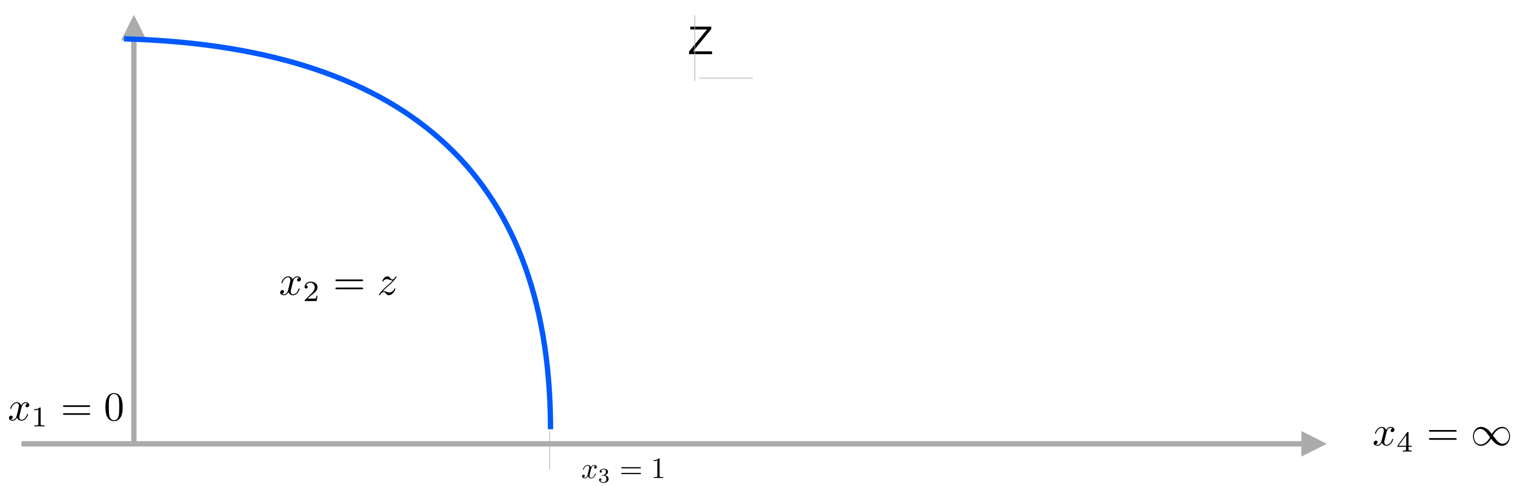
$$\simeq \frac{\sqrt{3}}{8\pi} \int_0^{\Lambda_0^2} dE^{(0)} \frac{m_0^2}{(E^{(0)} + m_0^2)^2} = \frac{\sqrt{3}}{8\pi} \frac{1}{1 + m_0^2/\Lambda_0^2}$$

EXACT FOUR POINT FUNCTION

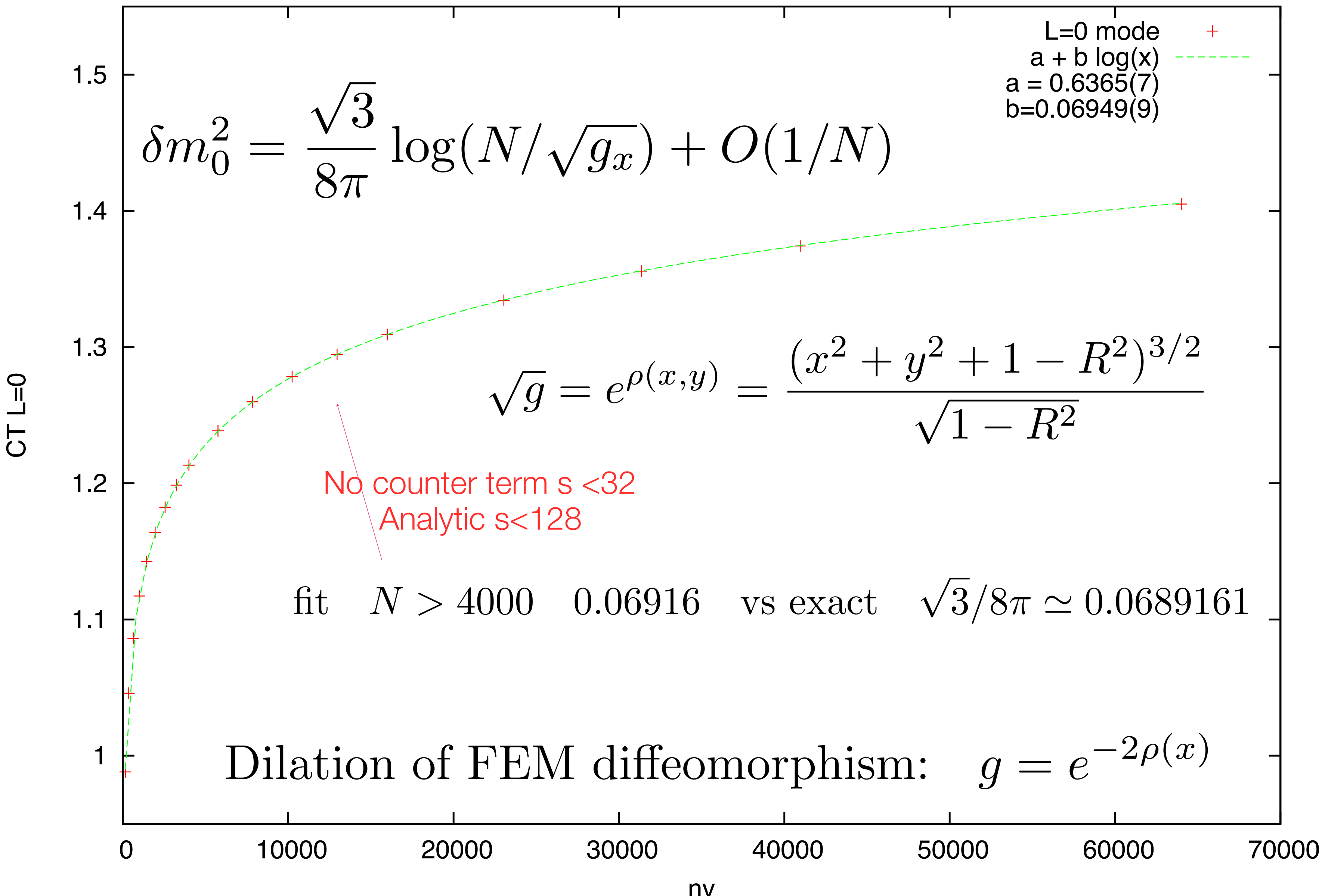
$$g(u,v) = \frac{\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle}{\langle \phi_1 \phi_3 \rangle \, \langle \phi_2 \phi_4 \rangle}$$

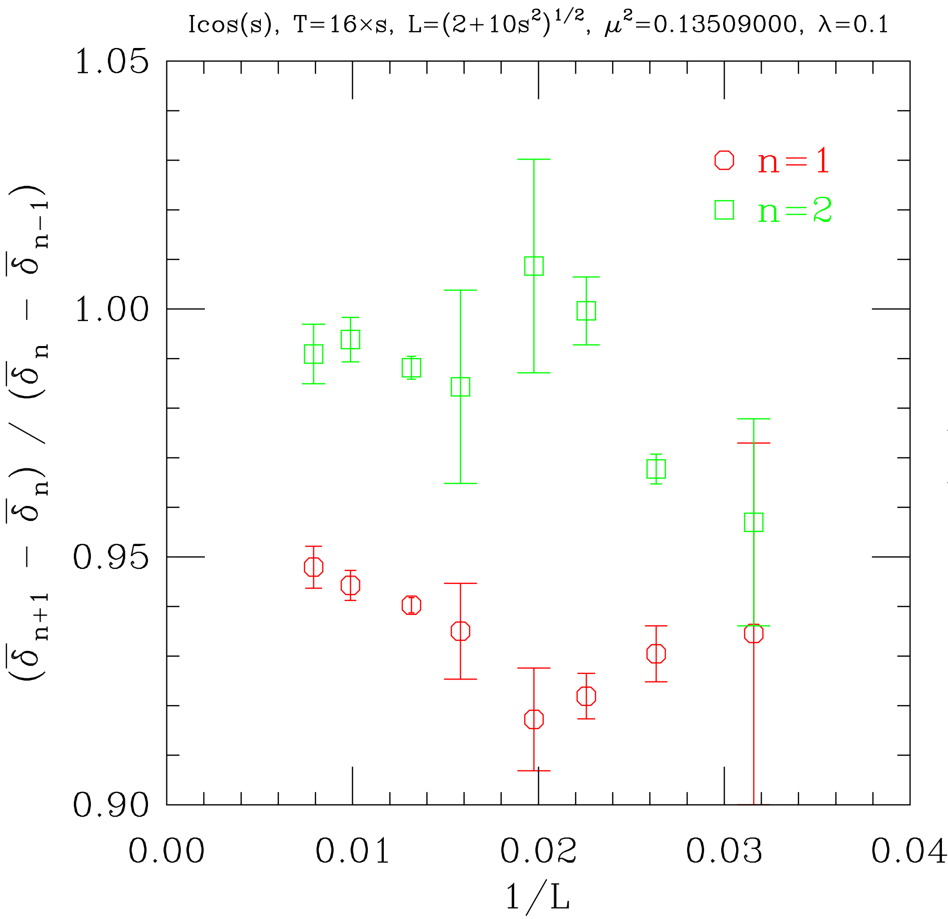
$$= \frac{1}{\sqrt{2}|z|^{1/4}|1-z|^{1/4}} \left[|1+\sqrt{1-z}| + |1-\sqrt{1-z}| \right]$$

Crossing Sym: $|1+\sqrt{1-z}| + |1-\sqrt{1-z}| = \sqrt{2+2\sqrt{(1-z)(1-\bar{z})}+2\sqrt{z\bar{z}}}$

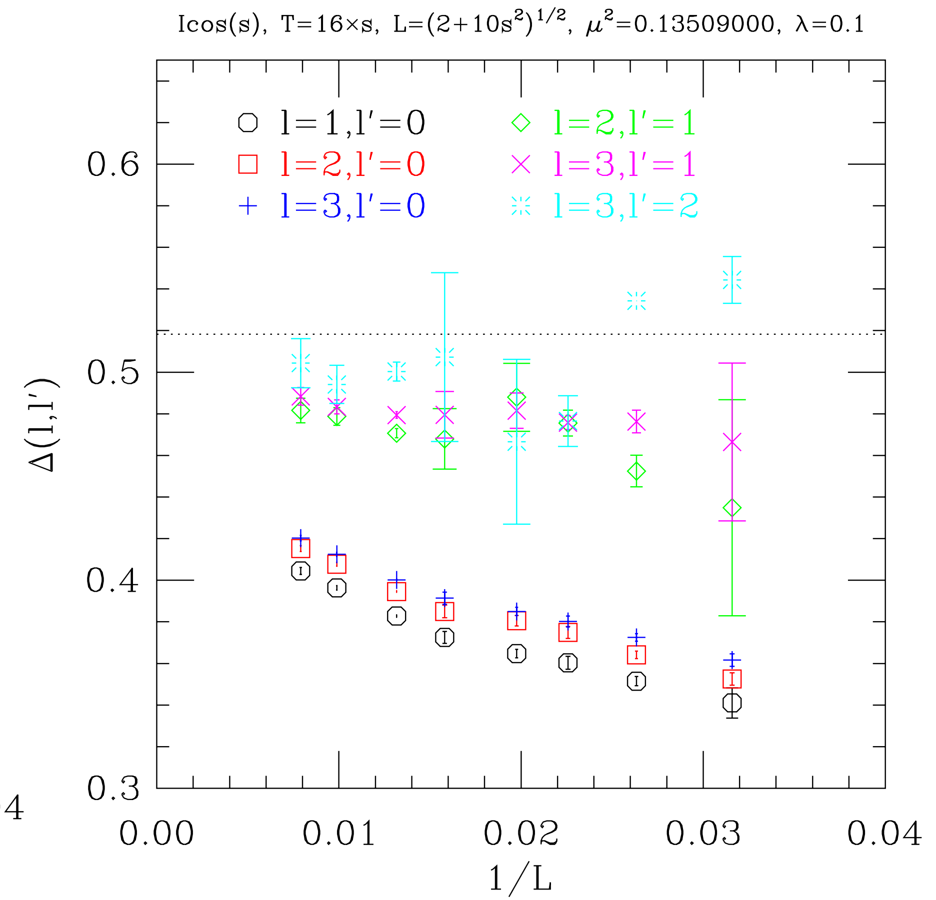


MODEL OF COUNTER TERM

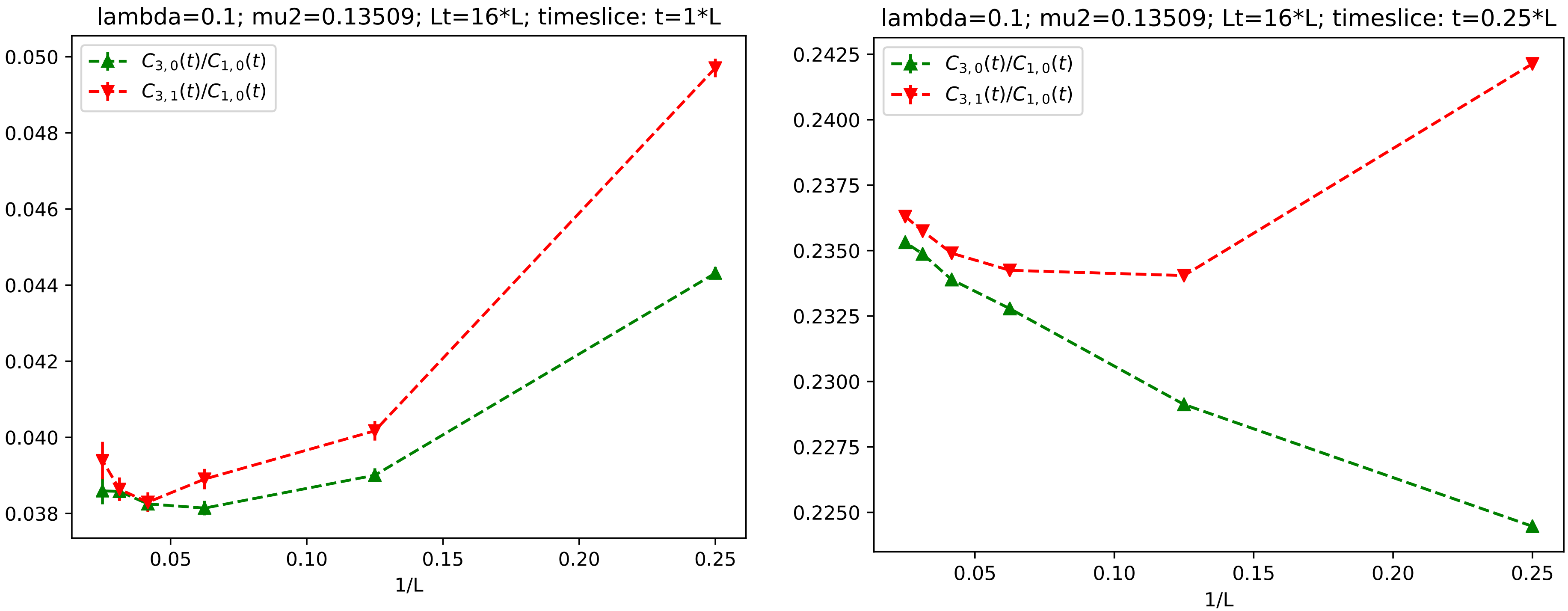




Icos(s), T=16×s, L=(2+10s²)^{1/2}, μ²=0.13509000, λ=0.1
n=1
n=2
1/L



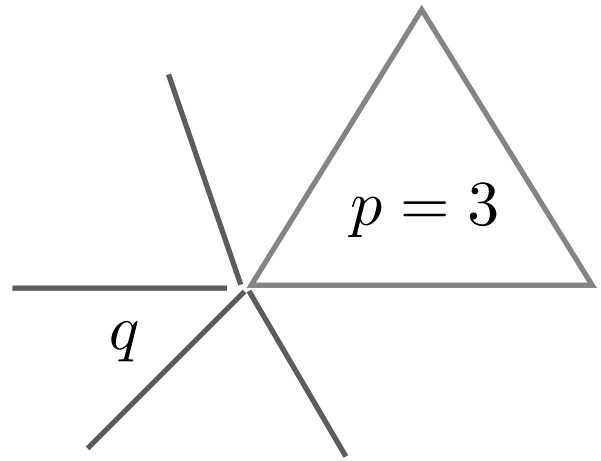
Icos(s), T=16×s, L=(2+10s²)^{1/2}, μ²=0.13509000, λ=0.1
l=1,l′=0
l=2,l′=0
l=3,l′=0
l=2,l′=1
l=3,l′=1
l=3,l′=2
Δ(l,l′)
1/L



lambda=0.1; mu2=0.13509; Lt=16*L; timeslice: t=1*L
$C_{3,0}(t)/C_{1,0}(t)$
$C_{3,1}(t)/C_{1,0}(t)$
0.050
0.048
0.046
0.044
0.042
0.040
0.038
0.05
0.10
0.15
0.20
0.25
1/L
lambda=0.1; mu2=0.13509; Lt=16*L; timeslice: t=0.25*L
$C_{3,0}(t)/C_{1,0}(t)$
$C_{3,1}(t)/C_{1,0}(t)$
0.2425
0.2400
0.2375
0.2350
0.2325
0.2300
0.2275
0.2250
0.05
0.10
0.15
0.20
0.25
1/L

TRIANLE GROUP BASES SIMPLEX

Triangle case



Preserves Discrete
Subgroup of Isometries

$$\frac{1}{p} + \frac{1}{q} > 1/2 \quad \text{de Sitter} \quad \mathbb{S}^2 \qquad \text{vertex} \quad q = 3, 4, 5$$

$$\frac{1}{p} + \frac{1}{q} = 1/2 \quad \text{flat} \quad \mathbb{T}^2 \qquad \text{vertex} \quad q = 6$$

$$\frac{1}{p} + \frac{1}{q} < 1/2 \quad \text{Hyperbolic} \quad \mathbb{A}dS^2 \qquad \text{vertex} \quad q = 7, 8, 9, \cdots$$