Real time gauge theory dynamics: Programming gauge theory on quantum computers

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Real time dynamics

In many body quantum systems real time dynamics suffers from a severe sign problem: this is physics, interference in quantum mechanics.

$$\int \mathcal{D}X \, \exp(iS)$$

Hilbert space usually grows too quickly for other methods to work:

$$\dim(Hilb) \simeq A^{\#dof}$$

Especially true when we want to take continuum limit!

$$\#dof \to \infty$$

Enter quantum computers

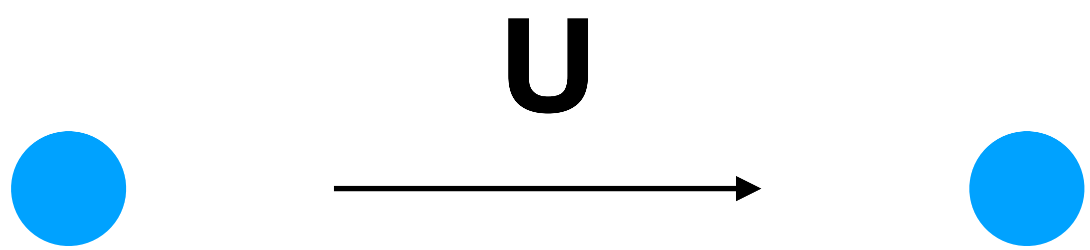
- System is quantum mechanical: interference is built in in the memory (it is a Hilbert space)
- Hilbert space grows exponentially in number of qubits
- Can do Hamiltonian evolution directly: possibility of doing real time dynamics.
- Needs an initial condition.

Outline

- Hamiltonian formulation of Gauge theory
- Truncation and approximation: many choices here.
- Example: U(1) GT in 2+1 D.
- D-theory: building large representations dynamically
- Plaquettes, ferromagnets, gaps

Hamiltonian formulation of lattice gauge theory

Kogut-Susskind formulation



Discrete space, continuous time.

$$A_0 = 0$$

Classical variables: unitaries (group elements) on links
+ canonical conjugates (electric fields)

Canonical conjugates generate gauge transformations on links: on both ends of the link.

$$\mathcal{H} = g_E \sum_{links} E^2 + g_\Box \sum_\Box Plaquette$$

When quantizing, turn everything into operators plus we need wave functions.

Position representation

$$\psi(U) \in L^2(U)$$

Problem: this Hilbert space is infinite dimensional per link.
Need to truncate.

This problem has been mentioned already in conference by Lewis, Stryker, Meurice, Hughes

- Discretizing U (as a group) only works for U(1): the other non-abelian gauge groups do not have discrete subgroups that can become dense in a limit (can’t extrapolate to continuum).

- Can discretize the space of wave functions directly instead and keep gauge invariance. (This approach has a lot of history)

Decompose into representations of gauge transformations

Peter-Weyl theorem

$$L^2(U) \simeq \oplus_R R \otimes \bar{R}$$

This is decomposed into all possible irreps.

The kinetic (electric) term is the quadratic Cassimir of R.

This representation is an electric basis for the wave functions.

Advantage: full gauge transformations preserved.

Higher representations cost more energy (bigger Cassimir)

Can truncate to a finite Hilbert space for low energy effective field theory: need to put cutoffs in Cassimir.

Goal

- Write truncated theory in terms of qubits: encode link space into qubits (many choices here)
- Write Hamiltonian in terms of qubits (fewer many body terms better)
- Can be done for arbitrary gauge groups (D-theory: Brower, Wiesse …). —We have not costed resources for these yet.
- Time evolve: Trotter steps

Trotter steps

$$\exp(iHt) \simeq \left(\prod_i \exp(i(t/n)H_i) \right)^n$$

The individual t H/n pieces can be computed in a finite number of steps, many of them in parallel. We need to assemble product.

Bottleneck: Clock time is number of consecutive gate steps that can not be done in parallel. Loss of coherence limits clock time (depth of circuit). We need to try to have very few gates.

Simulation time is number of Trotter steps times the time parameter per step

Problem: variable U changes on truncation, not Unitary any longer

We only need to show that we can recover the theory of interest in a limit.
Hopefully can get away with very drastic truncations.

Example: U(1) in 2+1

$$L^2(S^1) \simeq \oplus_q R_q \otimes R_{-q}$$

This is just Fourier decomposition on the circle.

$$U = \exp(i\theta)$$

$$E = -i\partial_\theta$$

Naive truncation: cutoff in the charge symmetrically

$$L^2(S^1) \sim Span(\exp(iq\theta)), \quad |q| \leq q_{max}$$

$$\begin{aligned} Q|q\rangle &= q|q\rangle \\ U|q\rangle &= \alpha_q|q+1\rangle \\ U^*|q\rangle &= \alpha^*_{q-1}|q-1\rangle \end{aligned}$$

In the continuum

$$\alpha_q = 1 \quad \forall \quad q$$

Can choose other values. To preserve gauge transformations all we need is

$$[Q, U] = U$$

A plaquette on a single link is propotional to

$$\beta U + \beta^* U^\dagger$$

We get the full 1-link Hamiltonian
(Coefficients are usually operator valued on other links)

$$H_{1-link} \simeq \begin{pmatrix} (q+1)^2 & \beta\alpha_q & 0 & \ddots \\ \beta\alpha_q^* & q^2 & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Truncation works as low energy effective theory because diagonal terms grow quickly with q, and off-diagonal terms are bounded.

We need to implement this Hamiltonian on a particular Hilbert space built from qubits.

Our choice

$$Q = L_z$$
$$U = L_+/\ell$$
$$U^* = L_-/\ell$$

For spin l representation

Easy to show that

$$\alpha_q \simeq 1 + O(L_z/\ell)$$

Still, need to put it in qubits

Porting to qubits

Use addition of Angular momentum:

$$L^i_{tot} = \sum \sigma^i/2$$

We have an SU(2) on an individual qubit, put many together (2 l) to get right spin.

For a plaquette with k sides (links)

$$P \simeq O((2\ell)^k) terms$$

“All to all”: too many terms. Run out of time before a single Trotter step on current devices.

Optimize: smaller k, reduce number of terms.

Idea

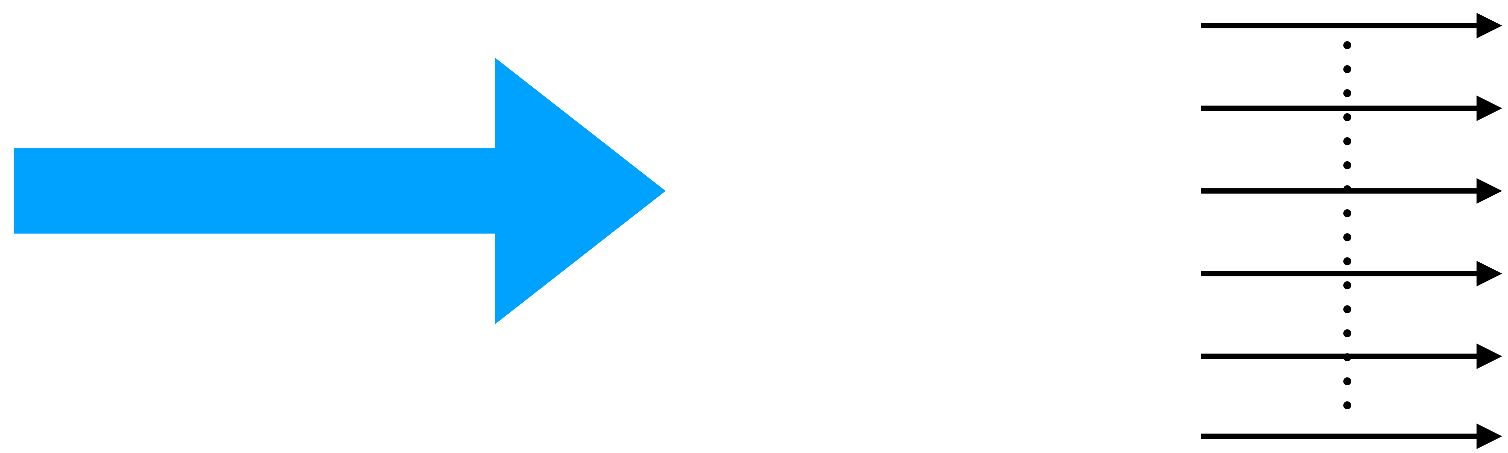
Get to maximal spin dynamically
(Essentially what D-theory does)
as part of full Hamiltonian

Take the qubits for a single link and add a ferromagnetic interaction to align them.

$$H_{align} \simeq -\alpha_{align} \sum (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y) - \beta_{align} \sum \sigma^z \otimes \sigma^z$$

Spins are ordered in extra dimension (same as D-theory)

Advantage: not all to all, few gates per qubit.

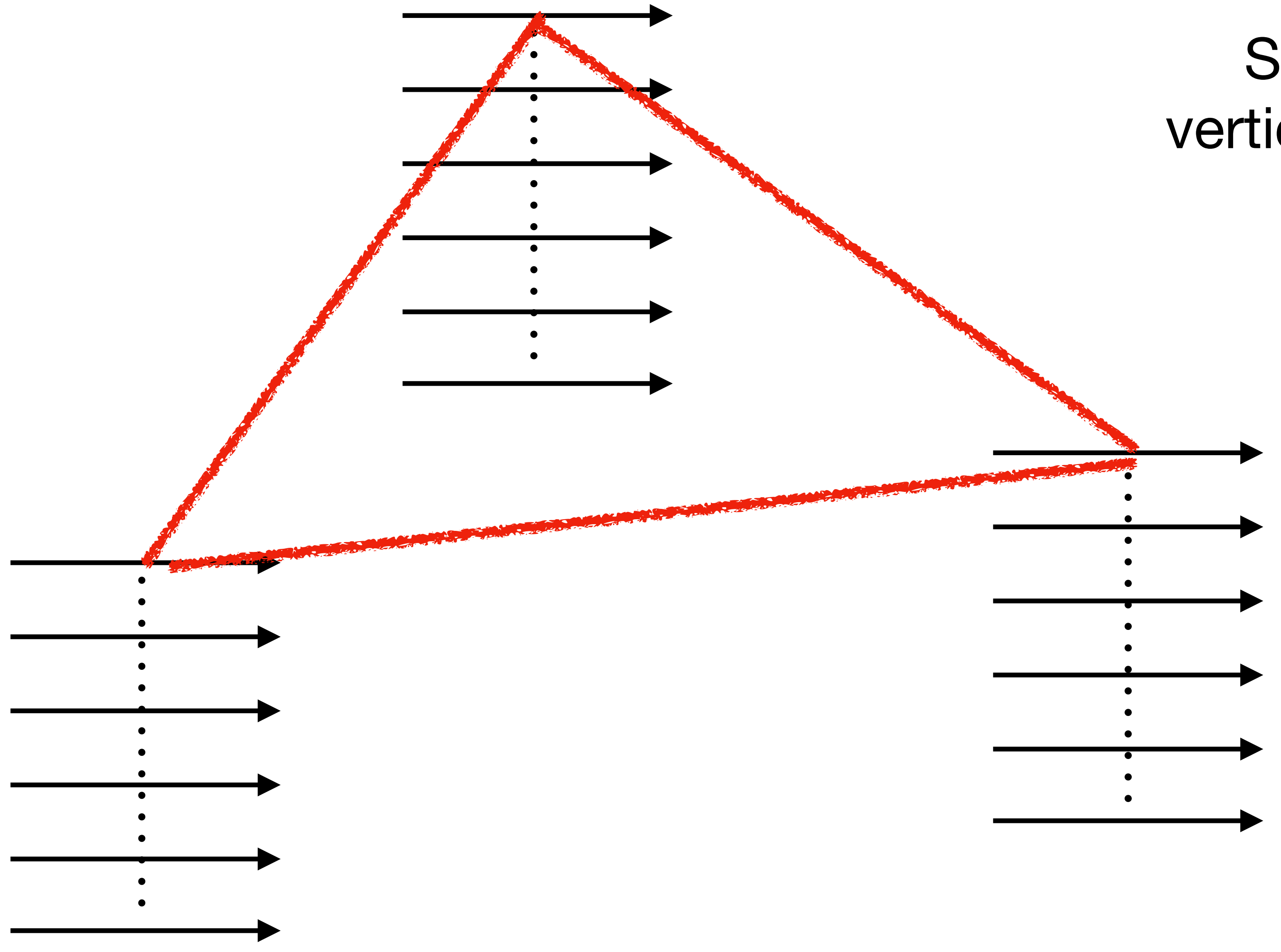


Low energy physics projects on states with maximal spin.

$$\beta_{align} \neq \alpha_{align}$$

Implements electric field splitting in the multiplet of interest.

Plaquette



Still gauge invariant if broken vertically: few operations per qubit. (Not all to all)

Why should it work?

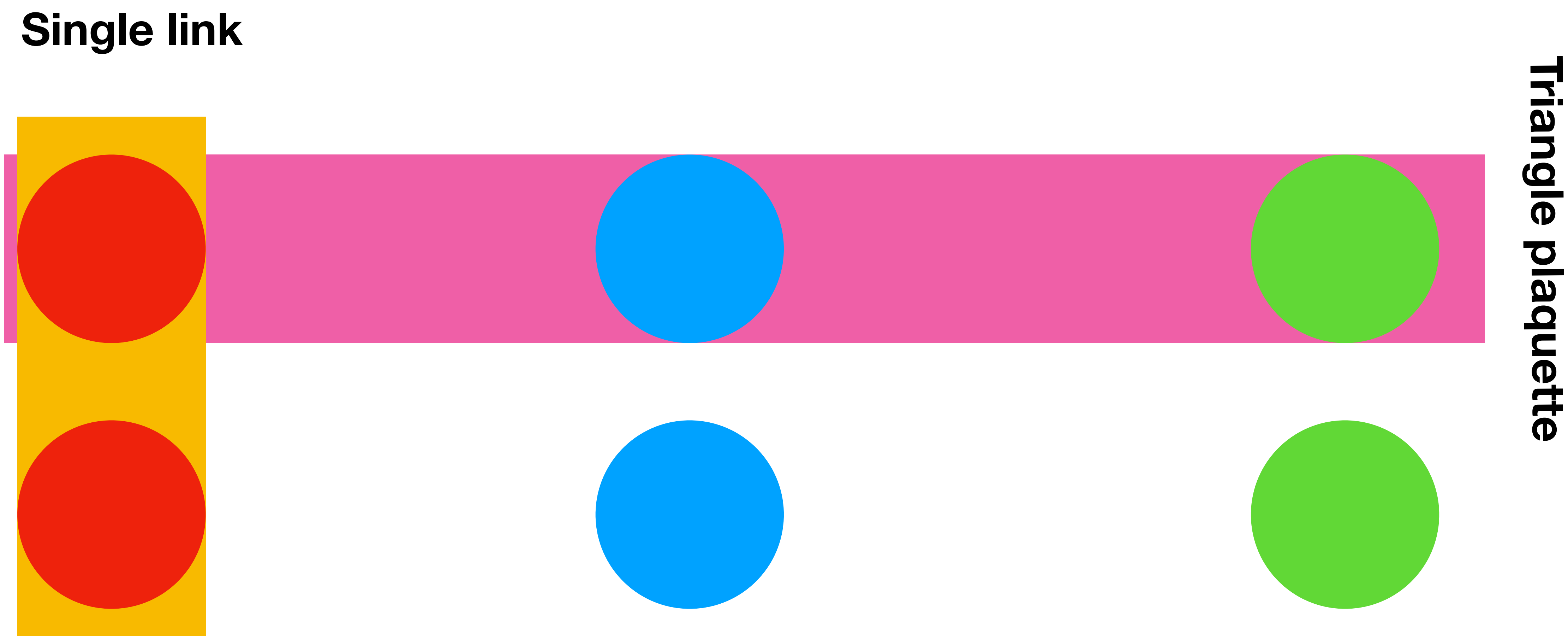
If plaquette is small perturbation, the Wigner Eckart
theorem applied to low energy physics
requires that the result of applying SU(2) generators
on individual qubits must be
compatible with SU(2) structure of
low energy states (maximal spin) in effective Hamiltonian

Brute force truncations

Two qubits per link: physical Hilbert space is dimension 3 (SU(2) triplet)+
1 (singlet) that should be UV physics

Total: we need 6 qubits per plaquette.

At least one Plaquette can be simulated!
Maybe 4….



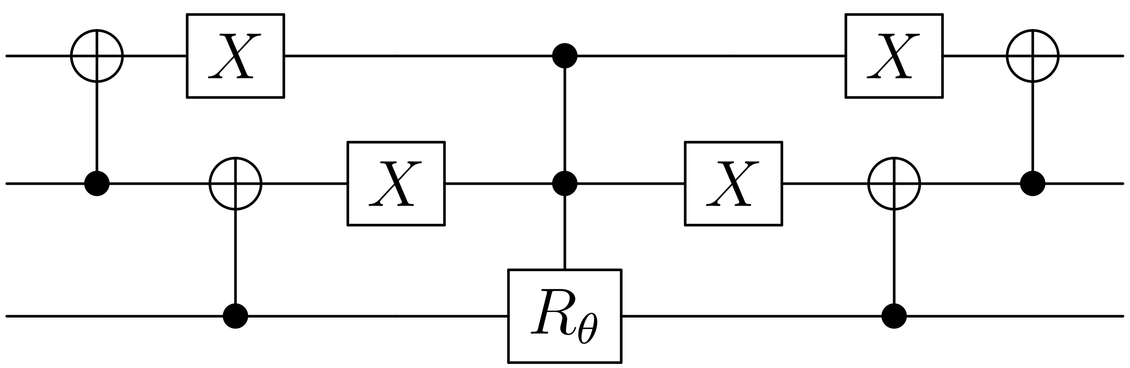
Assumption about architecture: can perform arbitrary 2 qubit operations between nearest neighbors.

Vertical Trotter step is easy.
Horizontal Trotter step is 3 qubit gate: need to implement.

$$\mathcal{H}_\triangle \propto \sigma^+ \otimes \sigma^+ \otimes \sigma^+ + \sigma^- \otimes \sigma^- \otimes \sigma^-$$

$$U = \exp(i\alpha\mathcal{H})$$

This is a rotation on a 2-plane of 8 dimensional Hilbert space (+++) rotating into (- - -).
Can be written in terms of a double control gate after some bit flips which need to be undone



Depending on details of architecture: it can take anywhere between 5 computing cycles and 20 (depth).
For experts: May be done efficiently with ancillas if Toffoli gates available

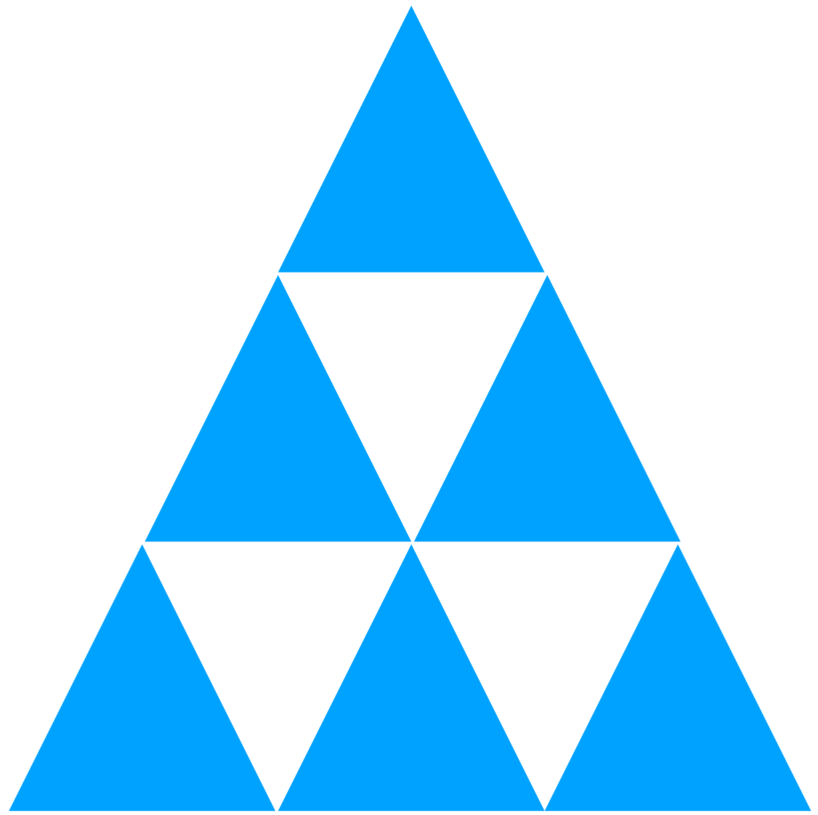
Another realization

$$U_\triangle = \exp\left(i\alpha(\sigma^1 \otimes \sigma^1 \otimes \sigma^1 - \sigma^1 \otimes \sigma^2 \otimes \sigma^2 - \sigma^2 \otimes \sigma^1 \otimes \sigma^2 - \sigma^2 \otimes \sigma^2 \otimes \sigma^1)\right)$$

$$\exp\left(i\tfrac{\pi}{4} 1 \otimes \sigma^3 \otimes \sigma^2\right) \exp\left(i\alpha(\sigma^1 \otimes \sigma^1 \otimes 1 - \sigma^2 \otimes \sigma^2 \otimes 1)\right) \exp\left(-i\tfrac{\pi}{4} 1 \otimes \sigma^3 \otimes \sigma^2\right)$$
$$= \exp\left[i\alpha(-\sigma^1 \otimes \sigma^2 \otimes \sigma^2 - \sigma^2 \otimes \sigma^1 \otimes \sigma^2)\right]$$

$$\exp\left(i\tfrac{\pi}{4} 1 \otimes \sigma^3 \otimes \sigma^1\right) \exp\left(i\alpha(\sigma^1 \otimes \sigma^2 \otimes 1 + \sigma^2 \otimes \sigma^1 \otimes)\right) \exp\left(-i\tfrac{\pi}{4} 1 \otimes \sigma^3 \otimes \sigma^1\right)$$
$$= \exp\left[i\alpha(\sigma^1 \otimes \sigma^1 \otimes \sigma^1 - \sigma^2 \otimes \sigma^2 \otimes \sigma^1)\right]$$

These two pieces commute: no violation of gauge invariance in the Trotterization for any value of parameters. (Does not include quantum errors)

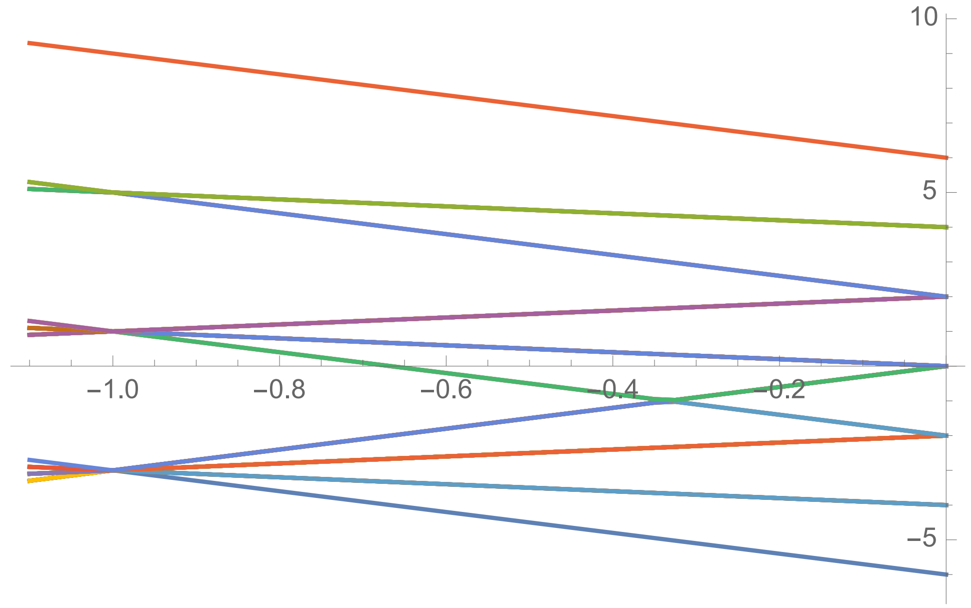


To evaluate lattice: alternate between
two coloring of triangles.

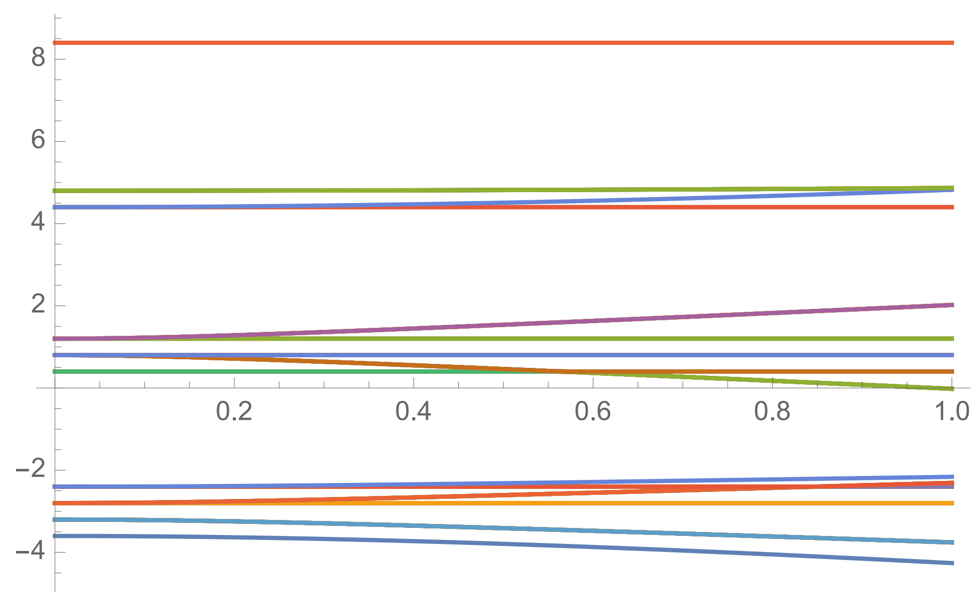
Total: about 15-20 consecutive gate operations (coherence time) per qubit per Trotter step

Estimate of current machines: 3 Trotter steps on Lattice

Parameter fitting Hamiltonian: eigenvalues of 64x64 matrices



Just XXZ piece: need to avoid level crossings, close to XXX is better



Together with plaquette operator: Gap persists: suggests simulation will not be too polluted by UV

Conclusion

- The ingredients for a gauge theory calculation are there: finite resources: not too many qubits per link, not too many gates per qubit.
- Architecture assumptions for setup: success will depend on details of architecture.
- Can in principle initialize to zero electric field + random assortment of plaquette actions on reference state.
- Can evolve for some time, in principle avoid UV pollution
- This is a Physics approach rather than “Computer science digitalization approach” (no judgement)

- Goal is now to deploy and check error rates etc.
- How long can the Trotter steps be?
- Running out of time is a big issue (pun intended).