New Lattice Approach for $\beta$-function in the Chirally Broken Phase

Chik Him (Ricky) Wong

Lattice Higgs Collaboration (LatHC):
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**β-function at strong coupling** is crucial in the studies of nearly conformal theories, but simulation becomes hard as we approach the conformal window.

Here we present an alternate way of obtaining the $\beta$-function making use of knowledge in the $p$-regime for theories outside the conformal window.
Outline

- Review: Lattice Studies of $\beta$-function of nearly conformal gauge theories
- An alternative Lattice approach for $\beta$-function in the $\chi$SB phase
- Application example: $\beta$-function of the Sextet model
- Conclusion
Review: Lattice Studies of $\beta$-function of nearly conformal gauge theories

- Lattice study of $\beta$-function
  - Gradient Flow [Lüscher JHEP 1008:071,2010]
    \[
    \frac{dA_\mu(t)}{dt} = D_\nu F_{\nu\mu}, \ D_\mu = \partial_\mu + [A_\mu, \cdot]
    \]
  - Perturbation Theory $\overline{MS}$, RG scale: $\mu = 1/\sqrt{8t}$
    \[
    E = \frac{3(N_c^2 - 1)g^2}{128\pi^2t^2}(1 + c_1g^2 + O(g^4)), \ E = \frac{1}{4}(F_{\mu\nu}^a)^2
    \]
  - Non-perturbative definition, (overall normalization depends on boundary conditions etc)
    \[
    g^2(t) \propto \left( \frac{128\pi^2}{3(N_c^2 - 1)} \right) t^2 \langle E \rangle_{\text{latt}}
    \]
- $\beta$-function $\propto -\mu^2 \frac{dg^2}{d\mu^2} = t \frac{dg^2}{dt}$
- Defined in continuum and infinite volume
Review: Lattice Studies of $\beta$-function of nearly conformal gauge theories

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  - Compare $g_c^2$ at $L/a$ with $g_c^2$ at $sL/a$ for some finite ratio $s$

$$\beta(g_c^2, a/L) = \frac{g_c^2(sL/a) - g_c^2(L/a)}{\ln(s^2)}$$

$g_c^2$ is obtained by either tuning $6/g_0^2$ or interpolating $g_c^2(6/g_0^2)$

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  (odd powers are absent for staggered fermions)

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- In our previous studies, massless staggered fermions anti-periodic in all directions is used
- More recent examples:
  $N_f = 10, 12$ in Fundamental representation [Fodor et al, Phys. Lett. B779 (2018)]
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  $N_f = 10, 12$ in Fundamental representation [Fodor et al, Phys.Lett. B779 (2018)]
Latest results: $N_f = 12 \ (\beta \equiv 6/g^2_0 \ \text{here})$

\[ g^2 \text{ (tuned)} = 6.9842 \pm 0.0014 \]

$\chi^2/\text{dof} = 0.3 \quad Q = 0.91$

Added $L = 32 \rightarrow L = 64$
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![Graph showing lattice studies of beta function](image)

- More data points planned
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\[ \frac{(g^2(sL) - g^2(L))}{\log(s^2)} = c_0 + c_1 \cdot \frac{a^2}{L^2} \]

\[ c_0 = 0.117 \pm 0.011 \]
\[ c_1 = -70.5 \pm 4.7 \]

$\chi^2$/dof = 0.44
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- Strong disagreement with [Hasenfratz and Schaich, JHEP 1802 132 (2018)]
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Latest results: $N_f = 10$

\[\frac{g^2(sL) - g^2(L)}{\log(s^2)} = c_0 + c_1 \cdot \frac{a^2}{L^2}\]

- $c_0 = 0.727 \pm 0.029$
- $c_1 = -30.1 \pm 9$
- $\chi^2/dof = 1.2$
- Targeted $g^2 = 7$
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Alternative approach
Application example
Conclusion

- **Latest results**: $N_f = 10$

![Graph showing the $\beta$-function with $N_f = 10$](image)

- **Strong disagreement with** [Chiu, PoS LATTICE2016 228 (2017)]
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Two ways to determine boundary of conformal window

1. Rejecting $\chi_{SB}$: Look for IRFP
   - $\beta$-function diminishes as we approach near conformal window
     $\Rightarrow$ higher accuracy and better systematic controls are needed
     $\Rightarrow$ Costly simulations are needed to hopefully resolve controversies
   - Absence of IRFP in limited search range of $g^2$ is not decisive
     $\Rightarrow$ An IRFP is always possible at stronger $g^2$ out of reach

2. Rejecting IR conformality: Look for signals of $\chi_{SB}$
   - If the theory is $\chi_{SB}$, chiral symmetry is spontaneously broken beyond certain $g_{critical}^2$
   - The value of $g_{critical}^2$ has to be consistent with results from the p-regime simulations
     $\Rightarrow$ p-regime simulations predict how the model should behave beyond $g_{critical}^2$
   - If we can compute $\beta$-function from p-regime calculation and covers the range of $g^2$ values achievable by step scaling (hopefully includes $g_{critical}^2$), it is a consistency test for $\chi_{SB}$ behavior
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     ⇒ An IRFP is always possible at stronger $g^2$ out of reach

2. Rejecting IR conformality: Look for signals of $\chi$SB
   - If the theory is $\chi$SB, chiral symmetry is spontaneously broken beyond certain $g^2_{\text{critical}}$
   - The value of $g^2_{\text{critical}}$ has to be consistent with results from the p-regime simulations
     ⇒ p-regime simulations predict how the model should behave beyond $g^2_{\text{critical}}$
   - If we can compute $\beta$-function from p-regime calculation and covers the range of $g^2$ values achievable by step scaling (hopefully includes $g^2_{\text{critical}}$), it is a consistency test for $\chi$SB behavior
Step scaling is a costly approach:

- Massless fermions with specific boundary conditions
- Lattice ensembles are different from p-regime simulations
- Cost comparable with p-regime simulations, but not as useful
- If $\beta$-function can be studied in p-regime for theories outside the conformal window, we can reuse ensembles from p-regime simulations
Review: Lattice Studies of $\beta$-function of nearly conformal gauge theories

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An alternative Lattice approach for $\beta$-function in the $\chi$SB phase

Chik Him (Ricky) Wong

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[Fodor et al, LATTICE 2017(2017)]

- Along the flow: Obtain $\beta(g^2) = -\mu^2 \frac{dg^2}{d\mu^2} = t \frac{dg^2(t)}{dt}$

Approximated by arbitrarily small $dt = \varepsilon$:

$$
\left. \frac{dg^2}{dt} \right|_t = \frac{1}{12\varepsilon} (-g^2(t + 2\varepsilon) + 8g^2(t + \varepsilon) - 8g^2(t - \varepsilon) + g^2(t - 2\varepsilon)) + O(\varepsilon^4)
$$

- Choose $g^2 = g_{\text{target}}^2$, determine $t_0$ such that

$g^2(t_0/a^2, L/a, am) = g_{\text{target}}^2$ at each $L/a$ and $am$

- Obtain the corresponding $\beta(g^2(t_0/a^2, L/a, am))$

- Extrapolate $t_0$ and $\beta$ at fixed value of $g_{\text{target}}^2$:
  - Infinite volume limit $L/a \to \infty$
  - Chiral limit $am \to 0$
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- Repeat at another $g_{\text{target}}^2$
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[Modor et al, LATTICE 2017(2017)]

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An alternative Lattice approach for \( \beta \)-function in the \( \chi_{SB} \) phase

Chik Him (Ricky) Wong

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Choose \( g^2 = g_{\text{target}}^2 \), determine \( t_0 \) such that
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Application example: \( \beta \)-function of the Sextet model

- **SU(3) Gauge Theory with** \( N_f = 2 \) fermions in Two-index Symmetric Representation

- Our previous studies show \( \chi \)SB behaviors, e.g. Hadron spectrum with pseudoscalars as Goldstone bosons of \( \chi \)SB and non-vanishing Goldstone decay constant in the chiral limit [Fodor et al EPJ Web Conf. 175 08027 (2018)]

- **SSC:**
  - HMC gauge action: Symanzik, 2 steps of \( \rho = 0.15 \) stout
  - Flow gauge action: Symanzik
  - Discretization of E: Clover

- **Target:** \( g^2(t_0/a^2, L/a, am) = g^2_{\text{target}} = 6.7 \)
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Application example: \( \beta \)-function of the Sextet model

- **Example:**
  \( 56^3 \times 96, \ 6/g_0^2 = 3.20, \ am = 0.001 \Leftrightarrow t_0/a^2 = 5.487 \pm 0.077 \)

- **Approximate derivative**
  \( \varepsilon = 0.05 \Rightarrow \beta(t_0) = t \left( \frac{dg^2}{dt} \right) = 0.753 \pm 0.019 \)
Application example: β-function of the Sextet model

Example:

\[ 56^3 \times 96, \ 6/g_0^2 = 3.20, \ am = 0.001 \leftrightarrow t_0/a^2 = 5.487 \pm 0.077 \]

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\[ \varepsilon = 0.05 \Rightarrow \beta(t_0) = t \left( \frac{dg^2}{dt} \right) = 0.753 \pm 0.019 \]
Application example: \( \beta \)-function of the Sextet model

- Infinite volume limit using \( \chi \)PT [Bar and Golterman, Phys. Rev. D 89, 034505 (2014)]

- Point-like source approximation \( \sqrt{8t_0} M_\pi \ll 1 \)
  \( \Rightarrow \) Finite volume correction by wrap-around Goldstone bosons
  \( g_1(M_\pi L_t, \eta = L_t/L_s) \)

- Infinite volume limit of \( M_\pi \): \( aM_\pi = 0.08118 \pm 0.00018 \)

\[
M_\pi (L) = M_\pi + c_1 \cdot g_1(M_\pi L, \eta)
\]
\[
M_\pi = 0.08118 \pm 0.00018
c_1 = 0.0175 \pm 0.0019
\]
\[
\chi^2/dof= 0.06 \quad Q= 0.81
\]

fitted volumes: \( 40^3 \times 80, 48^3 \times 96, 56^3 \times 96 \)
**Application example:**

**β-function of the Sextet model**

- Infinite volume limit using $\chi$PT [Bar and Golterman, Phys. Rev. D 89, 034505 (2014)]
- Point-like source approximation $\sqrt{8t_0} M_\pi \ll 1$
  $\Rightarrow$ Finite volume correction by wrap-around Goldstone bosons $g_1(M_\pi L, \eta = L_t/L_s)$
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Application example: \(\beta\)-function of the Sextet model

- Infinite volume limit using \(\chi PT\) (ignoring the effects of low lying \(0^{++}\) scalar)
- Point-like source approximation \(\sqrt{8t_0} \, M_\pi << 1\)
  \(\Rightarrow\) Finite volume correction by wrap-around Goldstone bosons \(g_1(M_\pi L, \eta = L_t/L_s)\)

**Graphs**

- **Left graph**: 
  \[ t_0(L) = t_0 + c_1 \cdot g_1(M_\pi L, \eta) \]
  \[ t_0 = 5.36 \pm 0.10 \quad g_1 \text{ fitted} \]
  \[ c_1 = 2.15 \pm 0.61 \]
  \[ M_\pi = 0.08118 \text{ (18)} \quad \text{input} \]
  \[ \chi^2/\text{dof} = 0.50 \quad Q = 0.48 \]

- **Right graph**: 
  \[ \beta(t) = t \, \frac{d\beta(t)}{dt} \]
  \[ \beta(L) = \beta + c_i \cdot g_1(M_\pi L, \eta) \]
  \[ \beta = 0.774 \pm 0.017 \]
  \[ c_i = -0.356 \pm 0.084 \]
  \[ M_\pi = 0.08118 \text{ (18)} \quad \text{input} \]
  \[ \chi^2/\text{dof} = 0.24 \quad Q = 0.63 \]
Application example: \( \beta \)-function of the Sextet model

- Infinite volume limit using \( \chi \)PT (ignoring the effects of low lying \( 0^{++} \) scalar)
- Point-like source approximation \( \sqrt{8t_0} M_\pi \ll 1 \)
  \[ \Rightarrow \] Finite volume correction by wrap-around Goldstone bosons
  \[ g_1(M_\pi L, \eta = L_t / L_s) \]

\[ \begin{align*}
  t_0(L) &= t_0 + c_1 \cdot g_1(M_\pi L, \eta) \\
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  c_1 &= 2.15 \pm 0.61 \\
  M_\pi &= 0.08118 (18) \quad \text{input} \\
  \chi^2 / \text{dof} &= 0.50 \quad Q = 0.48 \\
\end{align*} \]
Application example: \( \beta \)-function of the Sextet model

- Chiral limit using \( \chi \)PT [Bar and Golterman, Phys. Rev. D 89, 034505 (2014)] (ignoring the effects of low lying \( 0^{++} \) scalar)
- Point-like source approximation \( \sqrt{8t_0} M_\pi << 1 \)
  \[ t_0 = t_0^{(M_\pi=0)} (1 + k_1 \frac{M_\pi^2}{(4\pi f)^2} + \ldots) \]
  \[ \beta(t_0) = \beta(t_0^{(M_\pi=0)}) (1 + l_1 \frac{M_\pi^2}{(4\pi f)^2} + \ldots) \]

- In this analysis, only leading order \( (M_\pi^2) \) is considered
- \( M_\pi = 2Bm \Rightarrow \) Linear in \( m \) for leading order
- Ansatz:
  \[ t_0 = t_0^{(m=0)} (1 + c_1 m), \]
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Application example: 
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- **Chiral limit using χPT** [Bar and Golterman, Phys. Rev. D 89, 034505 (2014)] (ignoring the effects of low lying $0^{++}$ scalar)

  - **Point-like source approximation** $\sqrt{8t_0} M_\pi << 1$
    
    $$t_0 = t_0^{(M_\pi=0)} (1 + k_1 \frac{M_\pi^2}{(4\pi f)^2} + \ldots)$$
    
    $$\beta(t_0) = \beta(t_0^{(M_\pi=0)}) (1 + l_1 \frac{M_\pi^2}{(4\pi f)^2} + \ldots)$$

- In this analysis, only leading order ($M_\pi^2$) is considered

- $M_\pi = 2Bm \Rightarrow$ Linear in $m$ for leading order

- **Ansatz:**
  
  $$t_0 = t_0^{(m=0)} (1 + c_1 m),$$
  
  $$\beta(t_0) = \beta(t_0^{(m=0)}) (1 + d_1 m)$$
Application example: \(\beta\)-function of the Sextet model

- Chiral limit using \(\chiPT\) [Bar and Golterman, Phys. Rev. D 89, 034505 (2014)] (ignoring the effects of low lying \(0^{++}\) scalar)

- Point-like source approximation \(\sqrt{8t_0}M_\pi << 1\)

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t_0 = t_0^{(M_\pi=0)} \left( 1 + k_1 \frac{M_\pi^2}{(4\pi f)^2} + \ldots \right)
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Application example:  
**β-function of the Sextet model**

- **Chiral limit using χPT** [Bar and Golterman, Phys. Rev. D 89, 034505 (2014)] (ignoring the effects of low lying $0^{++}$ scalar)
- **Point-like source approximation** $\sqrt{8}t_0 M_\pi << 1$
  
  $$t_0 = t_0^{(M_\pi=0)} \left(1 + k_1 \frac{M_\pi^2}{(4\pi f)^2} + \ldots \right)$$

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Application example: \( \beta \)-function of the Sextet model

- Chiral limit using \( \chi PT \) [Bar and Golterman, Phys. Rev. D 89, 034505 (2014)] (ignoring the effects of low lying \( 0^{++} \) scalar)
- Point-like source approximation \( \sqrt{8t_0} M_\pi << 1 \)

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t_0 = t_0^{(M_\pi=0)} \left( 1 + k_1 \frac{M_\pi^2}{(4\pi f)^2} + \ldots \right)
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Application example:
β-function of the Sextet model

Chiral limit using χPT

\[ t_0(m) = t_0(0) + c_1 \cdot m \]
\[ t_0(0) = 6.20 \pm 0.14 \]
\[ c_1 = -857 \pm 87 \]

\[ \chi^2/\text{dof} = 0.06 \quad Q = 0.81 \]

\[ \beta(m) = \beta(0) + c_1 \cdot m \]
\[ \beta(0) = 0.614 \pm 0.032 \]
\[ c_1 = 164 \pm 21 \]

\[ \chi^2/\text{dof} = 0.075 \quad Q = 0.78 \]
Application example: \( \beta \)-function of the Sextet model

- Chiral limit using \( \chi \)PT

\[
\begin{align*}
\beta_0(m) &= \beta_0(0) + c_1 \cdot m \\
\beta_0(0) &= 10.482 \pm 0.23 \\
c_1 &= -1.76e+03 \pm 146
\end{align*}
\]

\[
\chi^2/\text{dof} = 0.31 \quad Q = 0.58
\]

- SSC \( \beta(m) \) linear chiral fit \( g^2 = 6.7 \) \( \beta = 3.25 \)

\[
\begin{align*}
\beta(m) &= \beta(0) + c_1 \cdot m \\
\beta(0) &= 0.563 \pm 0.019 \\
c_1 &= 156 \pm 11
\end{align*}
\]

\[
\chi^2/\text{dof} = 0.12 \quad Q = 0.72
\]
Application example: \( \beta \)-function of the Sextet model

Chiral limit using \( \chiPT \)

- Analysis of new data is ongoing in order to improve all the results, which will remain consistent with the above.
**Application example:**

$\beta$-function of the Sextet model

- **Chiral limit using $\chi$PT**

  ![Graph 1](image1.png)
  ![Graph 2](image2.png)

- **Analysis of new data is ongoing in order to improve all the results, which will remain consistent with the above.**
Application example: \( \beta \)-function of the Sextet model

- Continuum limit
  \[
  \beta(g_{\text{target}}, a^2/t_0) = \beta(g_{\text{target}}) + k \frac{a^2}{t_0}
  \]

- Error of \( t_0 \) is taken into account by
  \[
  \chi^2 = \sum_k \left[ \frac{(X_k - \langle x \rangle)^2}{\sigma^2_{x,k}} + \frac{(Y_k - \langle y \rangle)^2}{\sigma^2_{y,k}} \right]
  \]
  \[
  x = a^2/t_0, X = \langle x \rangle; \ y = \beta, Y = \langle y \rangle; \ \sigma^2 \text{'s: variances} \quad \text{[Krystek and Anton, Measurement Science and Technology 18, 3438 (2007)]}
  \]
Application example: $\beta$-function of the Sextet model

- **Continuum limit**
  \[ \beta(g_{\text{target}}^2,\frac{a^2}{t_0}) = \beta(g_{\text{target}}^2) + k\frac{a^2}{t_0} \]

\[
\begin{align*}
\beta(g^2,\frac{a^2}{t_0}) &= \beta(g^2) + c \cdot \frac{a^2}{t_0} \\
\beta(g^2) &= 0.548 \pm 0.047 \\
c &= 0.321 \pm 0.44 \\
\chi^2/dof &= 1.61 \quad Q = 0.2
\end{align*}
\]

- **Error of $t_0$ is taken into account by**
  \[
  \chi^2 = \sum_k \left[ \frac{(X_k - x_k)^2}{\sigma_{x,k}^2} + \frac{(Y_k - y_k)^2}{\sigma_{y,k}^2} \right]
  \]
  
  \[x = \frac{a^2}{t_0}, X = \langle x \rangle; \quad y = \beta, Y = \langle y \rangle; \quad \sigma's : \text{variances} \quad [\text{Krystek and Anton, Measurement Science and Technology 18, 3438 (2007)}] \]
Application example: \( \beta \)-function of the Sextet model

- The new p-regime \( \beta \)-function is like \( c \rightarrow 0, s \rightarrow 1 \) in step scaling, but they are different schemes
  \( \Rightarrow \) a bridging between the two has to be done, despite that they seem to be close to each other, possibly due to the insensitivity of \( c \) and \( s \) values of step scaling when \( c \) and \( s \) are small enough
Conclusion

An alternative approach of computing non-perturbative $\beta$-function of models in $\chi$SB phase is presented

- It provides a new tool to probe and test $\chi$SB behaviors by bridging p-regime simulations and step scaling $\beta$-function
- We can now recycle p-regime simulations on $\beta$-function calculation

Possible improvements:

- The method was based on $\chi$PT ignoring the existence of light $0^{++}$ scalars near the Conformal Window. The effect of taking this into account is under investigation
- A simultaneous chiral and continuum limit would eliminate the ambiguity of the order of the limits
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