Phase diagram of strongly interacting four-fermion theory

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- Extension to a model of Yukawa-Higgs type
- A continuum model with topological defects which recovers lattice model upon discretization (if time permits)

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Massless phase $G < G_c^1$, narrow broken phase $G_c^1 < G < G_c^2$, Strong coupling: *massive* phase $G > G_c^2$ SO(4) symmetry restored

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Momentum space

$$F(p) = rac{i\sqrt{6G^2}\sum_\mu sinp_\mu}{\sum_\mu sin^2 p_\mu + m_F^2}$$
 $B(p) = rac{8(6G^2)}{4\sum_\mu sin^2 p_\mu + m_B^2}$

where $m_F^2 = 4(6G^2) - 2$ and $m_B^2 = 4(6G^2) - 8$

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In the limit $G \to \infty$ $G^2 < \epsilon_{abcd} \psi^a \psi^b \psi^c \psi^d >$ tends to a constant Fermions are massive at strong coupling. This corresponds to pairing of elementary fermion ψ with composite fermion $\Psi_a = \epsilon_{abcd} \psi^b \psi^c \psi^d$ Four fermion condensate can be thought of as a bilinear formed from Ψ_a and ψ^a

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 φ^{ab}₊ is self-dual and transforms under SU₊(2) and is a singlet
- ϕ_{+}^{ab} is self-dual and transforms under $SU_{+}(2)$ and is a singlet under $SU_{-}(2)$.

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The self-dual nature of ϕ_+ ensures fermion eigenvalues appear in quartets $(\lambda, \lambda, -\lambda, -\lambda)$ and hence the Pfaffian arising after integrating over ψ is real and positive definite- no sign problem

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 $\kappa=0$ is our original four-fermion model with symmetric mass generation

Nouman Tariq (Syracuse University)

Coleman-Weinberg effective potential

The question of spontaneous symmetry breaking can be examined by computing the effective potential for ϕ_+

$$S_{eff}(\phi_+) = -\frac{1}{2} \operatorname{Tr} \ln(\eta \Delta + \varepsilon(x) G \phi_+)$$

Assume a constant

$$\phi_{+} = \mu \left(\begin{array}{cc} i\sigma_{2} & \mathbf{0} \\ \mathbf{0} & i\sigma_{2} \end{array} \right)$$

and diagonalize the kinetic operator

$$S_{eff}(\phi_{+}) = -rac{1}{4}tr[\ln|(\Delta_{\mu}^{2} - m^{2})|]$$

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where $m = G^2 \mu^2$ and *tr* denotes a functional trace The $\varepsilon(x)$ factor is important to get an effective potential of symmetry breaking type because $\{\varepsilon(x), \eta_{\mu}\Delta^{\mu}\} = 0$

Computing the functional trace in momentum space the effective potential takes the form

$$V_{eff}=-rac{1}{4}\int d^4 p \ln |rac{ ilde{p}_{\mu}^2+m^2}{ ilde{p}_{\mu}^2}|$$
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where
$$lpha=\int d^4 p rac{1}{ ilde{
ho}_{\mu}^2}$$
 and $eta=\int d^4 p rac{1}{ ilde{
ho}_{\mu}^24}$

This argument doesn't generalize to non-local SO(4) invariant bilinears breaking the shift symmetry. To understand the phase structure fully turn to (RHMC) simulation





Connected susceptibility $\chi_{conn} = \frac{1}{V} \sum_{x} < \chi^{a}(x)\chi^{b}(x)\chi^{a}(0)\chi^{b}(0) > \chi_{conn} \sim L^{4}$ in the transition region G = 1.05



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To look for spontaneous symmetry breaking we augment the action with a symmetry breaking $m \sum_{x} \varepsilon \psi^{a} \psi^{b}$ term and scan in $m \to 0$ limit while $V \to \infty$

Narrow broken phase





Figure: Left : *G* = 1.05, Right : *G* = 1.15, Bottom : *G* = 0.95

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opology and strongly interacting fermions





The susceptibility peak widens as you move towards negative κ and it shrinks towards positive κ .



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In both cases the four-fermion condensate exists

Spontaneous symmetry breaking

We repeat the spontaneous symmetry breaking analysis with $\kappa=-0.05$ and $\kappa=0.05$

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Spontaneous symmetry breaking

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No spontaneous symmetry breaking with small positive κ This gives a preliminary phase diagram with four different phases

Preliminary Phase Diagram



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Questions

- Can the four-fermion phase be induced by strong gauge coupling? (work in progress)
- Can the four-fermion phase be explained through a continuum model?

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A model based on Kähler-Dirac fermions discretizes to the staggered fermion model studied **arXiv:1708.06715v2**

Starting Point

Consider 4 Majorana spinors with action

$$S = \int d^4x \sum_{A=1}^4 \overline{\psi}^A \gamma . \partial \psi^A$$

The global symmetry is $G = SO_F(4) \times SO_{Lorentz}(4)$ under which

$$\psi \to L_{\alpha\beta} \psi^B_{\beta} F^T_{BA}$$

Focus on diagonal subgroup *D* where L = F under which $\psi \rightarrow \Psi$ with Ψ a 4 × 4 matrix

$$S = \int d^4 x \operatorname{Tr} (\Psi \gamma . \partial \Psi) \quad \text{with } \overline{\Psi} = C \Psi^T C^{-1} = \Psi$$

take 4 copies and add SO(4) invariant interaction

$$\delta \boldsymbol{S} = -\boldsymbol{G}^{2} \int \boldsymbol{d}^{4} \boldsymbol{x} \operatorname{Tr} \left(\boldsymbol{\Psi}^{a} \boldsymbol{\Psi}^{b} \right) \operatorname{Tr} \left(\boldsymbol{\Psi}^{c} \boldsymbol{\Psi}^{d} \right) \boldsymbol{\epsilon}^{abcd}$$

Notice ..

Interaction breaks global symmetry *G* down to diagonal subgroup *D* Additional *SO*(4) symmetry prevents bilinear mass terms For small *G* expect four fermi term irrelevant and get 16 massless Majorana fermions.

Introduce lattice

$$\Psi^{a} = \sum_{b}^{16} \gamma^{x+b} \chi^{a}(x+b) \quad \text{with } b_{\mu} = 0, 1 \quad \gamma^{x+b} = \prod_{\mu=1}^{4} \gamma^{x_{\mu}+b_{\mu}}_{\mu}$$

If $\partial_{\mu}
ightarrow \Delta^{S}_{\mu}$ and do traces

Staggered action !

Auxiliary field

$$S = S_0 + \int d^4x \, \left[G\phi_+^{ab} \text{Tr} \left(\Psi^a \Psi^b \right) + \frac{1}{2} \left(\phi_+^{ab} \right)^2 \right]$$

where ϕ_+ transforms in adj rep of $SU_+(2)$ subgroup of $SO(4) = SU_+(2) \times SU_-(2)$.

Integrate over fermions Pf (γ . ∂ + $G\phi_+$). Additional SU_- (2) shows real, positive. Effective action:

$$S_{\rm eff}(\phi_+) = \frac{1}{2}\phi_+^2 - \frac{1}{4}{\rm tr}\ln\left(-\partial^2 + G^2\phi_+^2 + G\gamma.\partial\phi_+\right)$$

If ϕ_+ constant find symmetry breaking potential

$$\phi_{+}^{a} = \mu n^{a}$$
 with $n^{a} n^{a} = 1$ vacuum manifold S^{2}

Effective Action

Assume broken phase and expand in powers of $\frac{1}{m}$ with $m = \mu G$

$$S_{\rm eff} = \frac{1}{m^2} \int d^4x \, \left(\partial_\mu n^a\right)^2 + \frac{1}{m^4} \int d^4x \, \left(\epsilon^{abc} \partial_\mu n^a \times \partial_\nu n^b\right)^2 + \dots$$

Fadeev-Skyrme term - supports solitons

Analysis easier if change variables:

$$n^{a}(x)\sigma^{a} = U^{\dagger}(x)\sigma_{3}U(x)$$

Mapping invariant under

Local U(1)

$$U(x) \rightarrow e^{i\sigma_3\beta(x)}U(x)$$

Also global SU(2) symmetry: $U(x) \rightarrow U(x)G$

Effective Action II

$$S_{\text{eff}} = \frac{1}{m^2} \int d^4 x \operatorname{tr} \left[(D_{\mu}U)^{\dagger} (D_{\mu}U) \right] + \frac{1}{m^4} \int d^4 x F_{\mu\nu}^2$$

where $D_{\mu} = \partial_{\mu} + iA_{\mu}\sigma_3$.

Topological defects

As
$$m \to \infty$$
 set $D_{\mu}U = 0$ for large r . Find

$$A_{\mu} = \frac{i}{2} \operatorname{tr} \left(\partial_{\mu} U U^{\dagger} \sigma_{3} \right) \quad \text{and}$$

$$S = b \int d^{4}x \, \frac{1}{4} \operatorname{tr} \left(\partial_{\mu} U \partial_{\nu} U^{\dagger} \sigma_{3} \right)^{2} \quad \text{with}$$

$$U = \left(\begin{array}{c} \alpha_{1} + i\alpha_{2} & -\alpha_{3} + i\alpha_{4} \\ \alpha_{3} + i\alpha_{4} & \alpha_{1} - i\alpha_{2} \end{array} \right) \quad \alpha_{i} = \frac{x_{i}}{r}$$

Hopf map: $\Pi_3(S^2) = Z$

Role of defects ?

Topological charge gotten from $\int \epsilon_{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}$.

Action $\int F_{\mu\nu}^2$ of single defect $S_{
m Hopf} \sim \ln V$

Recall 2D XY model

Single vortex has log divergent action. Pair of vortices has finite action and logarithmic interaction

Conjecture something similar here - pairs of Hopf-antiHopf defects bind in pairs. Play no role for small *G* but unbind to populate vacuum as $G \rightarrow \infty$.

Fermion mass

$$G_{F}(x,y) = \langle \Psi^{a}(x)\Psi^{a}(y)\rangle$$

= tr $\left[\frac{-\gamma_{\mu}\partial_{\mu} + m n^{a}\sigma^{a}}{(-\partial_{\mu}^{2} + m^{2} + mP)}\right]$

where

$$m{P} = \gamma_{\mu} \left(\partial_{\mu} U^{\dagger}(x) \sigma_{3} U(x) + U^{\dagger}(x) \sigma_{3} \partial_{\mu} U(x)
ight)$$

Far from core P = 0 and

$$G_F(x,y) = rac{-\gamma_\mu \partial_\mu}{-\Box + m^2}$$

Fermions acquire mass without breaking symmetries !

(1)

Phase structure

As G increased generically expect 3 phases

- *G* < *G*¹_{*c*} free massless fermions. Trivial IR fixed point. Lorentz and flavor symmetries restored.
- G¹_c < G < G²_c Phase with broken SO(4) symmetry. Fermion mass determined by bilinear condensate. Conventional NJL scenario.
- $G > G_c^2$ Four fermion condensate. Auxiliary field picture proliferation of topological defects. Fermions acquire masses propagating in this background. IR behavior depends on whether phase transition at G_c^2 continuous ...

Summary/Prospects

- Introduced a model based of Kähler-Dirac fermions that discretizes to staggered fermion models studied recently.
- In broken phase topological defects are possible. Give an understanding of how fermions acquire mass at strong coupling without breaking symmetries.

Critical exponents ?

Nature of continuum symmetries ? Is theory Lorentz invariant ? Can we use this as mechanism for gapping mirrror states in Eichten-Preskill approaches to constructing chiral lattice gauge theories ?

Thank you!

Collaborators Simon Catterall, David Schaich

Funding and computing resources



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