Supersymmetry breaking and gauge/gravity duality on the lattice

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1709.07025 [PRD, in press], 1800.00012 [PRD 97, 054504] and work in progress with Simon Catterall, Joel Giedt, Anosh Joseph, David Schaich & Toby Wiseman

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Outline

- Motivation and possibilities
- Two dimensional $\mathcal{N} = (2,2)$ SYM – supersymmetry breaking
- Holographic connection - two and three dimensional SYM (16 supercharges)
Why lattice supersymmetry (SUSY) ?

Discretization on the lattice furnishes gauge-invariant regularization of gauge theories and provides non-perturbative insights into

- Gauge/gravity (AdS/CFT) duality - potential non-perturbative definition of string theory
- Finite $N$ regime and large $N$ limit of supersymmetric theories.
- Confinement, phase transitions, symmetry breaking and conformal field theories.
Lattice SUSY: Problem and resolution

Problem
Supersymmetry generalizes Poincaré symmetry by adding spinorial generators $Q$ and $\bar{Q}$ to translations, rotations, boosts.

The algebra includes $Q\bar{Q} + \bar{Q}Q = 2\sigma^\mu P_\mu$, $P_\mu$ generates infinitesimal translations, which don’t exist on the lattice. Supersymmetry explicitly broken at the classical level.

Solution
Preserve a subset of SUSY algebra exactly on the lattice. Possible for theories with $Q \geq 2^D$. For ex: $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM). Methods are based on orbifold construction and topological twisting. I will focus only on the twisted construction in this talk.
<table>
<thead>
<tr>
<th>THEORY</th>
<th>R-SYMMETRY</th>
<th>LATTICE CONSTRUCTION</th>
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<tbody>
<tr>
<td>$d = 2, Q = 4$</td>
<td>$SO(2) \otimes U(1)$</td>
<td>✓</td>
</tr>
<tr>
<td>$d = 2, Q = 8$</td>
<td>$SO(4) \otimes SU(2)$</td>
<td>✓</td>
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<tr>
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<td>$SO(8)$</td>
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<td>$SO(7)$</td>
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<tr>
<td>$d = 4, Q = 16$</td>
<td>$SO(6)$</td>
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SUSY breaking : Witten index

To understand susy breaking non-perturbatively, Witten introduced index, $W$. As it turns out, $W$ can be written as,

$$W = \int_{PBC} \mathcal{D}(\cdots) e^{-S}$$

don’t have a way of evaluating this using simulations

Alternatively : Look for ground state energy as order parameter for breaking
\[ \mathcal{N} = (2,2) \text{ SYM in } d=2 \]

The action of continuum \( \mathcal{N} = (2,2) \text{ SYM} \) takes the following \( Q \)-exact form after topological twisting

\[ S = \frac{N}{2\lambda} \int d^2 x \Lambda, \]

where

\[ \Lambda = \text{Tr} \left( \chi_{\mu\nu} F_{\mu\nu} + \eta [\overline{D}_\mu, D_\mu] - \frac{1}{2} \eta d \right), \]

and \( \lambda = g^2 N \) is the 't Hooft coupling.
The nilpotent supersymmetry transformations associated with the scalar supercharge $Q$ are given by

\[
\begin{align*}
Q A_\mu &= \psi_\mu, \\
Q \psi_\mu &= 0, \\
Q A_\bar{\mu} &= 0, \\
Q \chi_{\mu \nu} &= -\overline{F}_{\mu \nu}, \\
Q \eta &= d, \\
Q d &= 0.
\end{align*}
\]
The four degrees of freedom appearing in this theory are just the twisted fermions \((\eta, \psi_\mu, \chi_{\mu\nu})\) and complexified gauge field \(A_\mu\). The complexified field is constructed from the usual gauge field \(A_\mu\) and the two scalars \(B_\mu\) present in the untwisted theory: \(A_\mu = A_\mu + iB_\mu\). The twisted theory is naturally written in terms of the complexified covariant derivatives

\[
\mathcal{D}_\mu = \partial_\mu + A_\mu, \quad \overline{\mathcal{D}}_\mu = \partial_\mu + \overline{A}_\mu, \quad (1)
\]

and complexified field strengths

\[
\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu], \quad \overline{\mathcal{F}}_{\mu\nu} = [\overline{\mathcal{D}}_\mu, \overline{\mathcal{D}}_\nu]. \quad (2)
\]
The action can be written as, \( S = S_B + S_F \), where the bosonic action is

\[
S_B = \frac{N}{2\lambda} \sum_n \text{Tr} \left( -\overline{F}_{\mu\nu}(n)F_{\mu\nu}(n) + \frac{1}{2}[\overline{D}_\mu, D_\mu]^2 \right),
\]

and the fermionic piece

\[
S_F = \frac{N}{2\lambda} \sum_n \text{Tr} \left( -\chi_{\mu\nu}(n)D_{[\mu}\psi_{\nu]}(n) - \eta(n)\overline{D}_\mu\psi_\nu(n) \right).
\]

Also an additional mass term (breaks \( Q \) supersymmetry)

\[
S_{\text{soft}} = \frac{N}{2\lambda} \mu^2 \sum_{n,\mu} \text{Tr} \left( \overline{U}_\mu(n)U_\mu(n) - I_N \right)^2,
\]
Fields on the lattice

\[ U_2(n) \quad \psi_2(n) \]

\[ F_{12}(n) \quad \chi_{12}(n) \]

\[ \bar{U}_2(n) \quad \eta(n) \]

\[ \bar{U}_1(n) \quad U_1(n) \quad \psi_1(n) \]
Extrapolations [PRD 97, 054504]

Figure: Left: $\lim_{a \to 0}$, Right: $\lim_{\mu^2 \to 0}$, Bottom: $\lim_{\beta \to \infty}$
Supersymmetry breaking

- Calculate the ground state energy density in the limit $\beta \to \infty$? (why not just do $T=0$ calculation)
- Need to use small mass term $\mu$ to control flat directions, which we extrapolate to zero after doing continuum extrapolation ($a \to 0$).
- Upper bound on energy density $\frac{\varepsilon_{\text{VAC}}}{N^2 \lambda} = 0.05(2)$, statistically consistent with zero.

[Similar study done earlier by Kanamori, Sugino and Suzuki based on A-twist Sugino’s action]
Applications to holography - gauge/gravity

Original AdS/CFT correspondence

4D $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory associated with $N$ D3-branes, is dual to Type IIB string theory on $AdS_5 \times S_5$ in the large $N$ limit.

More general holographic dualities in lower dimensions

Maximally supersymmetric YM in $p + 1$ dimensions dual to D$p$-branes

At low temperatures, and in the decoupling limit: dual description in terms of black holes in Type II A/B supergravity

Decoupling limit: $N \to \infty$ and $t = T/\lambda^{\frac{1}{3-p}} \ll 1$
Maximal SYM for $p < 3$

- Dimensionally reduce lattice $\mathcal{N} = 4$ SYM along $(3-p)$ spatial directions.
- Dimensional reduction: $A_4^* \to A_{p+1}^*$ giving a skewed torus with $\gamma = -1/(p + 1)$ ($\gamma = \cos \theta$).
- 't Hooft coupling ($\lambda$) is dimensionful in $p < 3$ dimensions and we construct a dimensionless coupling given by $r_{\text{eff}} = \lambda_p \beta^{3-p}$, where $\beta = 1/T$.
- No phase transition (single de-confined phase) in 1-d QM case, richer structure for $p = 1,2$. 
To have a valid SUGRA description, we need:

- Radius of curvature should be large in units of $\alpha'$. This implies $r_{\text{eff}} \gg 1$.
- String coupling should be small.

We can combine both requirements to get a constraint on the effective dimensionless coupling we can probe for a well-defined SUGRA description ($p < 3$)

$$1 \ll \frac{\lambda_p \beta^{3-p}}{N^{\frac{10-2p}{7-p}}}$$
Various dimensions - Existing works

- $p=0$: [Hanada, Nishimura and Takeuchi in 0706.1647 + Catterall & Wiseman, 0706.3518]
- $p=1$: This talk [Our recent work arXiv: 1709.07025 (PRD, in press), also work done using different action by D. Kadoh.]
- $p=2$: This talk [Preliminary work]

Eventual goal, $p=3$: Thermodynamics of $\mathcal{N} = 4$ SYM. Statement: Can we understand $f(\lambda) \ni, f(0) = 1$ and $f(\infty) = 3/4$?
p=1 : Maximal SYM in (1+1)-dimensions

- Interesting phase structure at finite temperature with a deconfinement transition dual to a gravity transition (between uniform D1 and localized D0 phase with spatial Wilson loop being the order parameter) at strong coupling and large $N$.
- Different temperature dependence in both phases for free energy (D0 & D1 thermodynamics)
- Can see the transition but can’t determine the order!
Some results (arXiv: 1709.07025)

The transition strengthens as $N$ increases, while showing little sensitivity to the lattice size.

Figure: Spatial Wilson loop magnitude (left) and susceptibility (right) vs. inverse dimensionless temperature $r_\beta = 1/t$ for SU($N$) gauge groups with $N = 6, 9$ and 12 on $16 \times 4$ and $24 \times 6$ lattices (aspect ratio $\alpha = N_x/N_t = 4$).
Figure: Wilson line phases. Uniform distribution (left) and localized distribution (right) corresponding to different black hole phases in the dual theory.
Figure: The critical temperature for different $\alpha$-lattices (left). D1 phase thermodynamics for $\alpha = 2$ (right). Dashed curve (right) is gravity prediction.
't Hooft coupling has dimensions of energy. Construct $r_{\text{eff}} = \lambda \beta = 1/t$ as dimensionless coupling. Type IIA SUGRA description is valid when the energy scale, $u = r/\alpha'$ (defined as fixed expectation value of a scalar) is in the range shown below:

This translates to the condition (for our dimensionless coupling) as,

$$1 \ll r_{\text{eff}} \ll N^6_5$$
First discussed by [Kabat, Lifshitz and Lowe, hep-th/9910001, hep-th/0105171], the thermal SYM partition function has divergence.

\[ I \sim kN \log(f(\zeta)) + N^2 I_{\text{finite}} \]

So technically, one can avoid the issue of divergence if \( N \to \infty \) (another need for large \( N \)) because the finite contribution dominates. For the \( N \) we can access in our numerical simulations, we need to do more!

Use a mass term for the scalar fields in our lattice action to restrict the moduli space and then extract the finite piece carefully and compare to the thermodynamics of Dp-branes.
An example of divergence showing up at $N=4$. 
Thermodynamics of D2-branes

For a uniform Dp-brane \( p < 3 \), we have a prediction for free energy density which is [Itzhaki et al., hep-th/9802042, Harmark and Obers, hep-th/0407094],

\[
\mathcal{F} = -k_p N^2 \lambda^{\frac{1+p}{3-p}} t^{\frac{14-2p}{5-p}}
\]

where, \( k \) can be read off the table in the above reference.

<table>
<thead>
<tr>
<th>( p )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_p )</td>
<td>((2^{21} 3^2 5^7 7^{-19} \pi^{14})^{1/5})</td>
<td>(2^4 3^{-4} \pi^{5/2})</td>
<td>((2^{13} 3^5 5^{-13} \pi^8)^{1/3})</td>
<td>(2^{-3} \pi^2)</td>
<td>(2^5 3^{-7} \pi^2)</td>
</tr>
</tbody>
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For our case of \( i.e \ p = 2 \), we get:

\[
\mathcal{F} = -2.492 \ N^2 \lambda^3 t^{\frac{10}{3}}
\]
We focus on calculating the free energy density for the SYM theory on the lattice restricting to uniform D2 phase.

Choose temperatures $t \ll 1$ and large $N$ for multiple lattices.

Computational cost scales as $\sim N^{7/2}$, so we restrict to $N_{\text{maximum}} = 8$ on $8^3$, $10^3$ and $12^3$ lattices.

We need to use small mass regulator $\zeta$ (discussed before), which we extrapolate to zero as $\zeta^2 \to 0$.

Publicly available lattice code for arbitrary $N$ (we have explored up to $N=20$ with fermions in 1d, unpublished) : [github.com/daschaich/susy](https://github.com/daschaich/susy)
Preliminary numerical results
Preliminary numerical results

\[ - \frac{s_{\text{Bos}}}{N^2 \lambda} \]

- \( N=4, 8^3 \)
- \( N=6, 8^3 \)
- \( N=6, 12^3 \)

SUGRA: \(-2.492 \, t^{10/3}\)

\( t = T/\lambda \)
Thank you!
Thank you!

Funding and computing resources
A naïve truncation of $U(N)$ supersymmetric theory to $SU(N)$ does not work at finite-$N$.

- Breaks the lattice supersymmetry that relates $U_a$ to $\psi_a$ in the $U(N)$ construction.
- Solution: Represent the truncated gauge links as $U_b = e^{i g a A_b}$ to argue that the continuum supersymmetry relating $A_a$ and $\psi_a$ is approximately realized in the large-$N$ limit even at non-zero lattice spacing since $g \rightarrow 0$ in the decoupling limit.
Continuum vs. lattice coupling

The non-orthogonal basis vectors of the $A_d^*$ lattice leads to mismatch in ’t Hooft coupling between lattice and continuum. The target continuum $(p+1)$-SYM coupling ($r_{\tau,\text{cont.}}$) differs from the lattice coupling as,

$$r_{\tau,\text{lattice}} = \frac{(d + 1)^{\frac{4-p}{6-2p}}}{\sqrt{d}} r_{\tau,\text{cont}}$$
\[ D_{1L_2(\beta)} \]
\[ \text{Tr} \bar{U} = 0 \]
\[ \text{Tr} \bar{V} \neq 0 \]
\[ \text{Tr} \bar{W} \neq 0 \]

\[ D_{2L_2(\beta, L_1)} \]
\[ \text{Tr} U = 0 \]
\[ \text{Tr} V = 0 \]
\[ \text{Tr} W \neq 0 \]

\[ \text{Tr} W \neq 0 \] (gravity)
Lower-dimensional sixteen supercharge SYM with $\text{apbc}$ has no sign problem.