"Some things are so serious that all you can do is joke about them." — N. Bohr

The midterm exam will be in Benson Earth Sciences 185, Wednesday 11 March, 7:00 – 8:30 PM.

1) [15 points] The Lagrange density for a two component self-interacting scalar field is

\[ \mathcal{L} = \frac{1}{2} \left[ (\partial_\mu \phi_1)(\partial^\mu \phi_1) + (\partial_\mu \phi_2)(\partial^\mu \phi_2) \right] + \frac{1}{2} \mu_0^2 \left( \phi_1^2 + \phi_2^2 \right) - \frac{\lambda}{4!} \left( \phi_1^2 + \phi_2^2 \right)^2. \]  

For \( \mu_0^2 > 0 \) with the sign conventions as shown, the vacuum has broken symmetry and there is a Goldstone boson.

(a) [10] A nice way to see this is to consider the change of variables to two new fields \( \rho(x, t) \) and \( \chi(x, t) \) with \( \phi_1(x, t) = \rho(x, t) \cos \chi(x, t) \) and \( \phi_2(x, t) = \rho(x, t) \sin \chi(x, t) \) or \( \phi(x, t) = \phi_1(x, t) + i \phi_2(x, t) = \rho(x, t) e^{i\chi(x, t)}. \) Show that \( \rho \) is the massive field and \( \chi \) is the Goldstone boson.

(b) [5] Next, suppose that the last symmetry of the Lagrangian is broken by a term \( V' = -e\phi_1 \). Show that the Goldstone bosons also get a mass, and that \( m_\chi^2 \sim \epsilon/\rho_0 \) where \( \rho_0 \) is the vacuum expectation value of the \( \rho \) field. This behavior is seen in ferromagnets, in the presence of an external magnetic field, and in QCD, where the pion is the Goldstone boson and \( \epsilon \) represents the quark mass. (Pedantic people call this small mass particle a "pseudo Goldstone boson.")

2) [10 points] Jackson 12.14

3) [10 points] Jackson 11.14, parts a, b only

4) [15 points] A plane wave is represented by a four-potential

\[ A^\mu = a^\mu \exp ik^\alpha x_\alpha \]  

where \( a^\mu \) is a constant four-vector and \( k^\mu k_\mu = 0. \)

(a) [3 points] If the electromagnetic field is in Lorentz gauge, how are the \( a^\mu \)'s related?
(b) [3 points] Find an expression for $F^{\mu \nu}$ and (c) [3 points] show that $F^{\mu \nu} k_\nu = 0$.
(d) [6 points] Show that $\vec{B} \times \vec{k} = \omega \vec{E}/c$ follows from (c).
\[ L = \frac{1}{2} \sum_{\alpha} \left( \partial_\mu \phi_\alpha \right) \left( \partial^\mu \phi_\alpha \right) + \frac{1}{2} \mu_0^2 \phi^2 \left( \partial_\alpha \phi \right)^2 - \frac{\lambda}{4!} \left( \phi^4 \right)^2 \]

\[ \phi^2 = \sum \phi^2. \]

The sign of \( \mu_0^2 \) is chosen so that positive \( \mu_0^2 \) means that spontaneous symmetry breaking occurs. Decomposing \( \phi \) into radial and angular modes,

\[ \phi_1(x, t) = c(x, t) \cos X(x, t) \]
\[ \phi_2(x, t) = c(x, t) \sin X(x, t) \]

gives simpler mathematics. Clearly \( \phi_1^2 = e^2 \).

The kinetic terms are

\[ \partial_\mu \phi_1 = [\alpha n e] \cos X \phi - e \sin X \alpha \nabla \phi \]
\[ \partial_\mu \phi_2 = [\alpha n e] \sin X + e \cos X \alpha \nabla \phi \]

so

\[ (\partial_\mu \phi_1)(\partial^\mu \phi_1) + (\partial_\mu \phi_2)(\partial^\mu \phi_2) = \alpha n e \partial_\mu e + e^2 \partial_\mu X \partial^\mu X \]

and

\[ L = \frac{1}{2} \alpha n e \partial_\mu e + \frac{1}{2} \mu_0^2 e^2 - \frac{\lambda}{4!} e^4 \]

\[ + \frac{1}{2} \alpha \partial_\mu X \partial^\mu X \]

The potential only depends on \( e \):

\[ V(e) = -\frac{1}{2} \mu_0^2 e^2 + \frac{\lambda}{4!} e^4 \]

The minimum is at \( \frac{\partial V}{\partial e} \mid e = e_0 = -\mu_0^2 e + \frac{\lambda e^3}{3!} \nabla \) or at \( e^2_0 = \frac{6\mu_0^2}{\lambda} \).
Next, introduce the Higgs field $\eta(x,t)$ by writing

$$ L = \frac{1}{2} \partial_{\mu} \eta \partial^{\mu} \eta + \frac{1}{2} \mu_0^2 \left( \eta_0 + \eta \right)^2 - \frac{\lambda}{4!} \left( \eta_0 + \eta \right)^4 $$

$$ + \frac{1}{2} \left( \eta_0 + \eta \right)^2 \partial_{\mu} \eta \partial^{\mu} \eta $$

$$ = \frac{1}{2} \partial_{\mu} \eta \partial^{\mu} \eta - V(\eta) + \frac{1}{2} \left( \eta_0 + \eta \right)^2 \partial_{\mu} \eta \partial^{\mu} \eta $$

where

$$ V(\eta) = - \frac{\mu_0^2}{2} \left( \eta_0^2 + \eta \right) $$

$$ + \frac{\lambda}{4!} \left( \eta_0^4 + \eta_3 \eta_0 + 6 \eta^2 \eta^2 + 4 \eta \eta^2 \eta^2 \right) $$

$$ = \text{constant} + \eta \left[ - \frac{\mu_0^2}{2} \eta_0 + \frac{\lambda}{6} \eta^4 \right] $$

$$ + \eta^2 \left[ - \frac{\mu_0^2}{2} + \frac{\lambda}{3} \eta^2 \right] + \text{higher order} $$

The linear term vanishes (easiest to see from $\partial V = 0$ at $\eta = \eta_0$) leaving a quadratic term

$$ \eta^2 \left[ - \frac{\mu_0^2}{2} + \frac{\lambda}{3} \cdot \frac{6 \mu_0^2}{\lambda} \right] = \eta^2 \mu_0^2 \left( \frac{3}{2} - \frac{1}{2} \right) = \eta^2 \mu_0^2 $$

We can read off the mass from $V(\eta) = \frac{1}{2} m^2 \eta^2$

or

$$ m^2_\eta = 2 \mu_0^2 $$

or from

$$ m^2_\eta = \frac{3 \lambda}{\lambda} \left( \frac{\eta}{\partial \eta} \right) \eta \eta_0 $$

$$ m^2_\eta = - \mu_0^2 + \frac{\lambda}{2} \eta^2 $$

or

$$ m^2_\eta = - \mu_0^2 + \frac{\lambda}{2} \eta^2 = 2 \mu_0^2 $$

$\eta$ has no term proportional to $X^i$, so the $X$ field is massless. There are lots of interaction terms from the non-quadratic part of $L$:

$$ \frac{1}{2} \left[ \partial_{\mu} \eta \partial^{\mu} \eta \right] \partial_{\mu} X \partial^{\mu} X = \frac{1}{2} \left( \eta^2 + \eta \right) \partial_{\mu} X \partial^{\mu} X \ldots $
b) Now add a term \( V' = -\epsilon \phi = -\epsilon \cos x \). Let
\[
\alpha = -\epsilon e (1 - \frac{1}{2} X^2 + \ldots )
\]
Work to quadratic order. When \( \epsilon \) is fixed to \( \epsilon_0 \), this is a \( \frac{1}{2} \epsilon \epsilon_0 X^2 \) term, so the \( X \) part of \( \alpha \) is
\[
\alpha X = \frac{1}{2} \epsilon \epsilon_0 \partial_x \phi \partial_x \phi X - \frac{\epsilon}{2} \epsilon_0 X^2.
\]
The usual definition of a mass comes when the kinetic term has a coefficient of \( \frac{1}{2} \) so define \( \overline{\partial} = \epsilon \epsilon_0 X \) \( \partial_x \phi \partial_x \phi \phi = \frac{1}{2} \epsilon \epsilon_0 \overline{\partial} \phi \)
and
\[
\alpha = \frac{1}{2} \epsilon \epsilon_0 \partial_x \phi \partial_x \phi - \frac{\epsilon}{2} \frac{\epsilon_0}{\epsilon} \overline{\partial}^2
\]
\[
\alpha - \frac{1}{2} M^2 = -\frac{\epsilon}{2} \frac{1}{\epsilon_0} + \boxed{\frac{M^2}{\epsilon_0}}
\]
Alternatively, just look at the equation of motion
\[
\epsilon_0 \partial_x \phi \partial_x \phi \phi + \epsilon \epsilon_0 X = 0
\]
\[
\epsilon_0 \left( \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{\epsilon}{\epsilon_0} \right) X = 0
\]
\( X \sim e^{i(k \cdot r - \omega t)} \) \( \Rightarrow \frac{\omega^2 + k^2 + \frac{\epsilon}{\epsilon_0} = 0}{\omega^2 + k^2 + \frac{\epsilon}{\epsilon_0}} \)
\( \Rightarrow M^2 = \frac{\epsilon}{\epsilon_0} \) again
2) Jackson 12.14

\[\mathcal{L}' = -\frac{1}{8\pi} \partial _{\mu} A^\mu \partial ^{\nu} A_{\nu} - \frac{1}{c} J^\mu A_{\mu}\]

\[= -\frac{1}{8\pi} \frac{\partial}{\partial x^\mu} \left( \epsilon_{\mu\nu} \partial ^{\nu} A^\rho - \frac{1}{c} J^\rho A^\mu \right)\]

The equation of motion is \(\partial ^{\mu} \partial _{\mu} A^\nu - \frac{\partial A^\nu}{\partial x^\mu} = 0\)

\[= -\frac{2}{8\pi} \delta ^{\mu}_{\nu} \left[ \partial \mu A_{\nu} \right] + \frac{1}{c} J^\nu = 0\]

\[\Rightarrow \partial ^{\mu} \partial _{\mu} A_{\nu} = \frac{4\pi}{c} J^\nu.\]

This looks familiar. If we added in \(\partial ^{\nu} A_{\mu}\)

the LHS would be \(\partial ^{\nu} F_{\mu\nu}\). We could do this if the extra term were zero. But it is, in

Lorentz gauge - exchange the order of derivatives,

\[\partial ^{\mu} A_{\nu} = \partial _{\nu} A^\mu = \partial _{\nu} \left[ \partial ^{\mu} A^\nu \right] = 0.\]

So eq. (1) is just the usual equation of motion written in

Lorentz gauge.

b) We know \(\mathcal{L}^2 = -\frac{1}{16\pi} \epsilon_{\mu\nu} F_{\mu\nu} \epsilon^{\rho\sigma} F_{\rho\sigma} - \frac{1}{c} J_{\mu} A^\mu.\)

The first term is \(-\frac{1}{16\pi} \left[ \partial V_{\mu} - \partial V_{\nu} \right] \left[ A^\mu A^\nu - A^\nu A^\mu \right]\)

\[= -\frac{1}{16\pi} \left[ A^\nu \partial _{\nu} A^\mu + A^\mu \partial _{\nu} A^\nu\right.\]

\[-\partial _{\nu} A^\mu \partial _{\nu} A^\nu - A^\nu \partial _{\nu} A^\mu A^\nu - A^\nu \partial _{\nu} A^\mu A^\nu\]

All the indices are dummy indices (they are summed over) so just flip them on the 2nd & 4th term -
\[ \mathcal{L} = -\frac{1}{8\pi} \left[ \partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu} - \partial_{\mu} A_{\nu} \partial^{\nu} A^{\mu} \right] \]

\[ \mathcal{L} = \mathcal{L}^\prime + E \quad \text{where} \quad E = \frac{1}{8\pi} (\partial_{\mu} A_{\nu})(\partial^{\nu} A^{\mu}) \]

If \( \partial_{\mu} A^{\mu} = 0 \), we can write \( E \) as

\[ E = \frac{1}{8\pi} \partial_{\mu} \left[ A_{\nu} \partial^{\nu} A^{\mu} \right] \]

This is a total divergence. In the action

\[ S = \int d^4x \, \mathcal{L} \]

a total divergence does not affect the equations of motion. (It only gives a contribution to \( S \) from the boundary.) If you want to be very careful, you can add the phrase “as long as \( A_{\mu} \) vanishes at infinity.”
3) Jackson 11.14 ab.

To write $F^\alpha_\beta F^\gamma_\delta$ and $\Phi^\alpha_\beta \Phi^\gamma_\delta$ in terms of $E-B$, just look at the components of $F^\alpha_\beta$:

$$F^\alpha_\beta = F^0_1 F^0_1 + F^0_2 F^0_2 + F^0_3 F^0_3 + \ldots + F^0_4 F^0_4 + \ldots$$

$$= -E_x^2 - E_y^2 - E_z^2 - E_x E_y - E_z + \ldots + B_x^2 + \ldots$$

$$= -2 \left[ E^2 - B^2 \right]$$

$F^\alpha_\beta$ just exchanges $\vec{E}$ to $B$ and $\vec{B}$ to $-\vec{E}$, compared to $F^0_\beta$, so this is just $-F^\alpha_\beta F^\gamma_\delta$.

$$F^\alpha_\beta F^\gamma_\delta = F^0_1 F^0_1 + \ldots = -E_x B_x - E_y B_y \ldots$$

$$= -4 E \cdot B.$$ 

These are the only invariants quadratic in $E \cdot B$ that you can write.

b) Suppose $\vec{B} = 0$ in frame $K$ and $\vec{E}' = 0$ in frame $K'$.

Then $F^\alpha_\beta F^\gamma_\delta = -2 E^2 < 0$ in $K$.

$$= 2 B^2 > 0 \text{ in } K'.$$

This can't happen! $F^\alpha_\beta F^\gamma_\delta$ is an invariant.

Also, if $E = 0$ in any frame, then $F^\mu_\nu \Phi^\mu_\nu = 0$ (since it is equal to $-4E \cdot B$), and

$$F^\mu_\nu F^\mu_\nu > 0 \text{ in all frames.}$$
4) \[ A^\mu = e^{i k \cdot x} \]

\[ a^k = \text{constant}, \quad k \cdot k = 0 \]

a) In Lorentz gauge \( \partial \mu A^\mu = 0 \) or \( k_\mu a^\mu = 0 \)

b) \[ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \]

\[ \partial^\mu = \frac{\partial}{\partial x^\mu} \quad \text{so} \quad F^{\mu\nu} = i [ k^\mu A^\nu - k^\nu A^\mu ] e^{i k \cdot x} \]

c) \[ F^{\mu\nu} k_\nu = i [ k^\mu (\partial^\nu A^\nu) - i (k^\nu k_\nu) a^\mu ] \]

\[ = 0 \quad \text{from part (a)} \quad \text{in the first term} \]

\[ \text{plus} \quad k^\nu k_\nu = 0 \]

d) \[ E_i = F^i_0 \quad \text{and} \quad F^{i\nu} = -\epsilon_{ijk} B^j_k \]

\[ k_0 = \frac{v}{c} \quad \text{and} \quad k_\nu = -k_j, \quad \text{for} \; \nu = j, \quad \text{because} \]

\[ k_\nu \text{ is a dual vector} \quad \text{so} \quad k_\nu = (\frac{v}{c} - \vec{k}) \]

\[ k_\nu k_\nu = 0 \]

\[ F^{i\nu} \text{ with } \mu = \nu \text{ is } F^i_0 k_0 + F^{i\nu} k_\nu = 0 \]

\[ E_i \frac{w}{c} + (\vec{k} \times \vec{B})_i = 0 \]

\[ \frac{w}{c} E_i + (\vec{k} \times \vec{B})_i = 0 \]

\[ \Rightarrow \quad \frac{w}{c} \vec{E} = \vec{B} \times \vec{k} \]