"It is quite easy to speak about symmetries, on one side. Everybody has a
tonight of symmetry, it is a very deeply rooted and widespread concept, ranging
from art to science. In some way or another symmetry is perceived by everybody.
I think it is worth mentioning that about thirty years ago there was strong
interest in experimenting with apes to see how much they were able to learn. One
objective was to see how apes would learn to paint. In one of these experiments
one dot was made at one side of a piece of paper and the ape would then try to
make a dot on the other side to balance it symmetrically. That's exactly what
we are doing in physics." – J. Wess

The midterm exam will be in Benson Earth Sciences 185, Wednesday 11 March,
7:00 – 8:30 PM. The problem set is due Friday the 13th.

1) [15 points] Jackson 11.13

2) One could think of the Schrödinger equation as the classical equation of
motion for a classical complex field \( \psi \), which extremizes the action
\[
S = \int dt d^3x [i\hbar \psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} (\vec{\nabla} \psi)^* \cdot (\vec{\nabla} \psi) - V(r,t)\psi^* \psi].
\] (1)

(a) [10 points] Varying \( \psi^* \) and \( \psi \) separately, and integrating by parts if you
have to, verify this assertion.

(b) [10 points] What are the conserved current and associated charge density
associated with the invariance of the action (and Lagrange density) under global
phase rotations \( \psi(x,t) \rightarrow e^{i\alpha} \psi(x,t) \), \( \psi(x,t)^* \rightarrow e^{-i\alpha} \psi(x,t)^* \).

(c) [10 points] In quantum mechanics we usually write the desired local gauge
transformation as
\[
\psi(x') = \exp(-i\frac{q}{\hbar c} \theta(x))\psi(x)
\] (2)

\[
A_\mu(x') = A_\mu + \partial_\mu \theta(x).
\] (3)

Show that the appropriate covariant derivative is
\[
D_\mu = \partial_\mu + i\frac{q}{\hbar c} A_\mu.
\] (4)

Hint: show \( D'_\mu \psi' = \exp(-i\frac{q}{\hbar c} \theta(x))D_\mu \psi \)

(d) [10 points] Next, make the replacement of the covariant derivatives in
the action and derive the appropriate Schrödinger equation for a nonrelativistic
quantum mechanical particle in an electromagnetic field.
In frame $K'$ the wire is at rest and has a charge per unit length $\lambda = \frac{q}{2\pi r}$. In this frame $\mathbf{B'} = 0$ and so in CGS, Gauss' law is

$$ \int \mathbf{E'} \cdot d\mathbf{A'} = 4\pi \frac{q}{2\pi r} = \frac{2q}{r} $$

meaning $\mathbf{E'}(r') = \mathbf{E} \cdot \frac{4\pi \frac{q}{2\pi r}}{2\pi r} = \frac{2q}{r} \mathbf{E}$

I've written $\mathbf{E'} = (E'_{x'}, E'_{y'}, E'_{z'})$ in cylindrical coordinates.

In the lab frame $K$, the fields are transformed

$E_2 = E_2' = 0 \quad B_2 = B_2' = 0$

$$ E_x = \gamma (E'_{x'} - \beta B'_{y'}) = \gamma E'_{x'} $$

$$ E_y = \gamma (E'_{y'} - \beta B'_{x'}) = \gamma E'_{y'} $$

$$ B_y = \beta \gamma E_{x'} \quad B_x = -\beta \gamma E_{y'} $$

$C$ is transverse to the boost direction so $\mathbf{E} = \mathbf{E'}$.

This is also true for $\mathbf{B}$, thus

$$ \mathbf{E} = \gamma \mathbf{E'} = \gamma \mathbf{E} = \frac{2q}{r} \mathbf{E} $$

$$ \mathbf{B} = \gamma \beta \mathbf{z} \times \mathbf{E'} = \beta \mathbf{z} \times \mathbf{E} $$

and

$$ \mathbf{z} \times \mathbf{E} = \phi $$

$$ \mathbf{B} = \frac{\phi}{c} \cdot \frac{2q}{r} \frac{v}{c} $$

in CGS.
b) Still in the wire's frame, \( \mathbf{J}^{\prime} = (c \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \),
or \[ \mathbf{J}^{\prime} = [c g_0, \mathbf{e}_z] \mathbf{e}^2(c \mathbf{e}_z) \]

Now we boost this to the lab frame
\[ \mathbf{J}^0 = \gamma [\mathbf{J}^0 + \beta \mathbf{J}^2] = \mathbf{J}^0 = \gamma c g_0 \mathbf{e}^2(c \mathbf{e}_z) \]
\[ \mathbf{J}^2 = \mathbf{x} [\mathbf{J}^2 + \beta \mathbf{J}^0] = \beta \mathbf{x} \mathbf{J}^0 = \gamma \mathbf{x} g_0 \mathbf{e}^2(c \mathbf{e}_z) \]

(c) and we compute the fields directly from the charge and current:

Gauss' law \( \Rightarrow \mathbf{E} = 2 [c g_0] \mathbf{e} \)

Amperes law with \( \mathbf{I} = \int d^2 x_1 \mathbf{J}_2(x_1) \) is

\[ \mathbf{I} = \gamma v g_0 \]

\[ \oint \mathbf{B} \cdot d\mathbf{e} = \frac{4 \pi}{c} \mathbf{I} \text{ in CS} \]

\[ \mathbf{B} = \frac{\mathbf{e}^2}{2 \pi c} \left[ 2 \mathbf{x} g_0 \right] = \frac{\mathbf{e}^2}{2 \pi c} \left[ 2 \mathbf{x} g_0 \right] / \left[ 2 \pi c \right] \]

just as we found in part (a).
The Lagrange density is
\[ L = \text{im} \, \psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \left( \nabla \psi^* \right)^2 - V(x) \psi^* \psi. \]

We can treat \( \psi \) and \( \psi^* \) as independent fields. This is just like treating a complex number \( z \) and its conjugate \( z^* \) as independent, which is completely equivalent to treating the real and imaginary parts as independent. The Lagrange equation of motion for either \( \phi = \psi \) or \( \phi = \psi^* \) is
\[ \frac{\partial \phi}{\partial t} - \sum \frac{\partial}{\partial x_j} \left[ \frac{\partial \phi}{\partial (\psi \psi^*)} \right] - \frac{\hbar}{\psi} \frac{\partial \psi}{\partial x} = 0. \]

Setting \( \phi = \psi^* \) gives the Schrödinger equation
\[ -V(x) \psi(x,t) + \frac{\hbar^2}{2m} \nabla^2 \psi + \hbar \frac{\partial \psi}{\partial t} = 0. \]

If you wanted to set \( \phi(x) = \psi(x) \), it might be better to integrate the action by parts first:
\[ \int dt \, \psi^* \frac{\partial \psi}{\partial t} = -\int dt \, \frac{\partial \psi^*}{\partial t} \psi. \]

Then, variation with respect to \( \psi \) would give the Schrödinger equation for \( \psi^* \).

For currents, adopt the convention of same \( c \)
\[ \psi \rightarrow \exp \left[ -\frac{ie \phi}{\hbar c} \right] \psi \propto \left[ 1 - \frac{ie \phi}{\hbar c} \right] \psi \rightarrow \delta \psi = \frac{ie}{\hbar c} \psi \]
and \( \psi^* \rightarrow \exp \left[ \frac{i \phi}{\hbar c} \right] \psi^* \propto \left[ 1 + \frac{i \phi}{\hbar c} \right] \psi^* \rightarrow \delta \psi^* = \frac{i \phi}{\hbar c} \psi^* \).
The conserved Noether current is
\[ J^μ = \frac{\partial \hat{\Psi}^*}{\partial (\partial_\mu \hat{\Psi})} \hat{\Psi} + \frac{\partial \hat{\Psi}^*}{\partial (\partial_\mu \hat{\Psi})} \hat{\Psi} \]

In components
\[ J^0 = \left( i \hbar \hat{\Psi}^* \right) \left( -i \frac{\hbar \Theta}{\hbar c} \right) \hat{\Psi} + 0 \; ; \; \text{note & mono } \frac{\partial \hat{\Psi}^*}{\partial z} \]
\[ = \frac{\hbar \Theta}{\hbar c} \hat{\Psi}^* \hat{\Psi} \; ; \; \text{this is } \frac{\hbar \Theta}{\hbar c} \text{ times the usual probability density} \]
\[ J^z = -i \frac{\hbar \Theta}{\hbar c} \left[ -\frac{\hbar^2}{2m} \left( \nabla \hat{\Psi} \right)^* \hat{\Psi} + \frac{\hbar^2}{2m} \hat{\Psi}^* \nabla \hat{\Psi} \right] \]
\[ = \frac{\hbar \Theta}{\hbar c} \left[ -\frac{i \hbar}{2m} \left( \hat{\Psi}^* \nabla \hat{\Psi} - (\nabla \hat{\Psi})^* \hat{\Psi} \right) \right] \]
\[ = \frac{\hbar \Theta}{\hbar c} \text{ times the usual probability current.} \]

Recall \[ \frac{\partial \hat{\Psi}^*}{\partial z} + \nabla \cdot \hat{J} = 0 \] : the conservation of probability follows from the invariance of \( \hat{\Psi} \) under a global phase rotation of \( \hat{\Psi} \).

c) Now consider a \textit{local} gauge transformation
\[ A_\mu' = A_\mu + \partial_\mu \Theta(x,t) \]
\[ \hat{\Psi}'(x,t) = \exp \left[ -i \frac{\hbar \Theta}{\hbar c} \right] \hat{\Psi}(x,t) \]

To show that the covariant derivative is
\[ D_\mu = \partial_\mu + i \frac{\hbar}{\hbar c} A_\mu \] we can insist that
\( D_\mu' \Psi'(x, t) = \exp \left[ -i \frac{\Phi}{\hbar c} \right] D_\mu \Psi(x, t) \)

Because then

\[ i \hbar \left( \frac{-\hbar^2}{2m} D_x^2 + V \right) \Psi' \]

is

\( \left( \exp -i \frac{\Phi}{\hbar c} \right) i \hbar \left( D_x^2 \Psi \right) = \exp -i \frac{\Phi}{\hbar c} \left\{ \frac{-\hbar^2}{2m} D_x^2 \Psi \right\} \)

i.e. a solution of the Schrödinger eqn remains a solution after a gauge transformation.

So \( D_\mu \Psi = \frac{\partial \Psi}{\partial x} + i \frac{\hbar}{\hbar c} A_\mu \Psi \).

\[ D_\mu' \Psi' = \left[ \partial_\mu + i \frac{\hbar}{\hbar c} (A_\mu + \partial_\mu \Theta) \right] e^{-i \frac{\Phi}{\hbar c}} \Psi \]

\[ = e^{-i \frac{\Phi}{\hbar c}} \left[ \partial_\mu \Psi - i \frac{\hbar}{\hbar c} \partial_\mu \Theta + i \frac{\hbar}{\hbar c} A_\mu + i \frac{\hbar}{\hbar c} \partial_\mu A_\mu \right] \Psi \]

\[ = e^{-i \frac{\Phi}{\hbar c}} D_\mu \Psi' \quad \text{as we desired.} \]

d) \( \partial_\mu = \left[ \frac{1}{c} \frac{\partial}{\partial t}, \nabla \right] \Rightarrow A_\mu = \left[ \Phi, -\vec{A} \right] \)

So \( \frac{1}{c} \frac{\partial}{\partial t} \rightarrow \frac{1}{c} \frac{\partial}{\partial t} + \frac{\hbar}{\hbar c} \Phi \equiv D_0 \)

\( i \hbar \frac{\partial}{\partial t} \rightarrow i \hbar \frac{\partial}{\partial t} - \Phi \)

\( \nabla \rightarrow \nabla - i \frac{\hbar}{\hbar c} \vec{A} \quad \text{or} \quad \frac{\hbar}{i} \nabla \rightarrow \frac{\hbar}{i} \nabla - i \frac{\hbar}{\hbar c} \vec{A} \)
\[ L = \psi^* \left( i \frac{\partial}{\partial t} - \mathbf{A} \cdot \mathbf{\nabla} - q \mathbf{A} \right) \psi \]

\[-\frac{1}{2m} \left[ \left( \frac{\hbar}{i} \mathbf{\nabla} - q \mathbf{A} \right) \psi^* \right] \left[ \left( \frac{\hbar}{i} \mathbf{\nabla} - q \mathbf{A} \right) \psi \right] \]

Integrate the space term by parts to get the derivative away from \( \psi^* \).

\[-\frac{1}{2m} \left[ -\frac{\hbar}{i} \mathbf{\nabla} \psi^* - q \mathbf{A} \cdot \mathbf{\nabla} \psi^* \right] \mathbf{\hat{D}} \psi \]

\[ = -\frac{1}{2m} \left[ \psi^* \left( \frac{\hbar}{i} \mathbf{\nabla} - q \mathbf{A} \right) \mathbf{\hat{D}} \psi \right] \]

Then we can use the Lagrange equation of motion for \( \psi^* \) without any \( \partial \psi^* / \partial \mathbf{\hat{r}} \) term.

\[ 0 = i \hbar \frac{\partial \psi^*}{\partial t} - q \mathbf{A} \cdot \mathbf{\nabla} \psi^* - \frac{1}{2m} \left( \mathbf{\hat{p}} - q \mathbf{A} \right)^2 \psi^* \]

where \( \mathbf{\hat{p}} = \frac{\hbar}{i} \mathbf{\nabla} \). Rearrange this to

\[ i \hbar \frac{\partial \psi^*}{\partial t} = q \mathbf{A} \cdot \mathbf{\nabla} \psi^* + \frac{1}{2m} \left( \mathbf{\hat{p}} - q \mathbf{A} \right)^2 \psi^* \equiv \mathbf{H} \psi^* \]

i.e. the Hamiltonian for a charged particle in an electromagnetic field is \( \mathbf{H} \).