Set 9 – due 5 April

"Some things are so serious that all you can do is joke about them." – N. Bohr

1) [30 points] The Lagrange density for a two component self-interacting scalar field is

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\phi_1)(\partial^\mu\phi_1) + (\partial_\mu\phi_2)(\partial^\mu\phi_2)] + \frac{1}{2}\mu_0^2[\phi_1^2 + \phi_2^2] - \frac{\lambda}{4!}[\phi_1^2 + \phi_2^2]^2. \qquad (1)$$

For $\mu_0^2 > 0$ with the sign conventions as shown, the vacuum has broken symmetry and there is a Goldstone boson.

(a) [15] A nice way to see this is to consider the change of variables to two new fields $\rho(x,t)$ and $\chi(x,t)$ with $\phi_1(x,t) = \rho(x,t)\cos\chi(x,t)$ and $\phi_2(x,t) = \rho(x,t)\sin\chi(x,t)$ or $\phi(x,t) = \phi_1(x,t) + i\phi_2(x,t) = \rho(x,t)e^{i\chi(x,t)}$. Show that ρ is the massive field and χ is the Goldstone boson. Compute the mass of the Higgs $(rho(x,t) = \rho_o + \eta(x,t))$ field to make sure you get the same answer as in class.

(b) [10 points] Perhaps you recall that Eq. 1 has a conserved current due to the symmetry

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \qquad (2)$$

Re-derive the expression for the associated Noether current in terms of (ϕ_1, ϕ_2). Then make the change of variables into (ρ, χ) and reconstruct J^μ. Notice (and compute) the form of the symmetry transformation and the conserved current in the (ρ, χ) variables. This "shift symmetry" is often used as a diagnostic for the presence of Goldstone bosons.

(c) [5] Next, suppose that the last symmetry of the Lagrangian is broken by a term $V' = -\epsilon\phi_1$. Show that the Goldstone bosons also get a mass, and that $m_\chi^2 \simeq \epsilon/\rho_0$ where ρ_0 is the vacuum expectation value of the ρ field. This behavior is seen in ferromagnets, in the presence of an external magnetic field, and in QCD, where the pion is the Goldstone boson and ϵ represents the quark mass. (Pedantic people call this small mass particle a "pseudo Goldstone boson.")

Oftentimes, the literature will make an assumption that one is working at momentum scales much less than the Higgs mass and the radial degree of freedom is "frozen" to ρ_0.

2) [15 points] A plane wave is represented by a four-potential

$$A^\mu = a^\mu \exp ik^\alpha x_\alpha \qquad (3)$$

where a^μ is a constant four-vector and $k^\mu k_\mu = 0$.

(a) [3 points] If the electromagnetic field is in Lorentz gauge, how are the a^μ's related?
(b) [3 points] Find an expression for $F^{\mu\nu}$ and (c) [3 points] show that $F^{\mu\nu}k_\nu = 0$.
(d) [6 points] Show that $\vec{B} \times \vec{k} = \omega\vec{E}/c$ follows from (c).

1) $\mathcal{L} = \frac{1}{2}\sum_{i=1}^{2}(\partial_\mu \varphi_i)(\partial^\mu \varphi_i) + \frac{1}{2}\mu_0^2 |\phi|^2 - \frac{\lambda}{4!}(|\phi^2|)^2$

and $|\phi|^2 = \sum_i \phi_i^2$ – pick sign of μ_0^2 so we have spontaneous symmetry breaking. Decompose ϕ into radial and angular modes,

$$\phi_1(x,t) = \rho(x,t)\cos\chi(x,t)$$
$$\phi_2(x,t) = \rho(x,t)\sin\chi(x,t)$$

gives simpler mathematics. Clearly $|\phi^2| = \rho^2$.
The kinetic terms are

$$\partial_\mu\varphi_1 = [\partial_\mu\rho]\cos\chi - \rho\sin\chi\,\partial_\mu\chi$$
$$\partial_\mu\varphi_2 = [\partial_\mu\rho]\sin\chi + \rho\cos\chi\,\partial_\mu\chi \text{, so}$$

$$\sum_i \partial_\mu\varphi_i\,\partial^\mu(\varphi_i) = \partial_\mu\rho\,\partial^\mu\rho + \rho^2\partial_\mu\chi\,\partial^\mu\chi,$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2}\mu_0^2\rho^2 - \frac{\lambda}{4!}\rho^4 + \frac{1}{2}\rho^2\partial_\mu\chi\,\partial^\mu\chi.$$

The potential only depends on ρ so the minimum is at

$$\left.\frac{\partial V}{\partial\rho}\right)_{\rho=\rho_0} = 0 = -\mu_0^2\rho + \frac{\lambda}{3!}\rho^3 \quad\text{or}\quad \rho_0^2 = \frac{6\mu_0^2}{\lambda}.$$

For the Higgs mass, write $\rho(x,t) = \rho_0 + \eta(x,t)$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta - V(\eta) + \frac{1}{2}(\rho_0+\eta)^2\partial_\mu\chi\,\partial^\mu\chi$$

$$V(\eta) = -\frac{1}{2}\mu_0^2[\rho_0+\eta]^2 + \frac{\lambda}{4!}[\rho_0+\eta]^4.$$

explicitly

$$V(\eta) = -\frac{\mu_0^2}{2}\left(e_0^2 + 2e_0\eta + \eta^2 \right) + \frac{\lambda}{4!}\left(e_0^4 + 4e_0^3\eta + 6e_0^2\eta^2 + 4\eta^3 e_0 + \eta^4 \right)$$

$$= \text{constant} + \eta\left[-\mu_0^2 e_0 + \frac{\lambda}{6} e_0^3 \right] + \eta^2\left[-\frac{\mu_0^2}{2} + \frac{\lambda}{4} e_0^2 \right] + \text{higher order}.$$

The linear term vanishes, the ~~coefficient~~ quadratic term is

$$\eta^2\left[-\frac{\mu_0^2}{2} + \frac{\lambda}{4}\cdot\frac{6\mu_0^2}{\lambda} \right] = \eta^2\mu_0^2.$$

From $V(\eta) = \frac{1}{2} m_\eta^2 \cdot \eta^2$, $m_\eta^2 = 2\mu_0^2$.

Much easier, of course, to say

$$m_\eta^2 = \left.\frac{\partial^2 V}{\partial e^2}\right)_{e=e_0} = -\mu_0^2 + \frac{\lambda}{2} e_0^2 = -\mu_0^2 + \frac{\lambda}{2}\frac{6\mu_0^2}{\lambda}$$

$$= 2\mu_0^2 \text{ again}.$$

$\mathcal{L}$ has no terms proportional to χ^2 so the χ field is massless. There are lots of interaction terms from

$$\frac{1}{2}\left[e_0 + \eta \right]^2 \partial_\mu\chi\partial^\mu\chi = \frac{1}{2} e_0^2 \partial_\mu\chi\partial^\mu\chi \quad \text{(kinetic)}$$

$$+ \partial_\mu\chi\partial^\mu\chi\left[e_0\eta + \eta^2 \right] \quad \text{(interaction)}$$

Conserved current in terms of ρ, χ:

Begin with $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2 - V(|\phi|^2)$

global symmetry is

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

or $\delta\varphi_1 = \alpha\varphi_2$

$\delta\varphi_2 = -\alpha\varphi_1$

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_1)}\delta\varphi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_2)}\delta\varphi_2$$

$$= (\partial^\mu\varphi_1)\alpha\varphi_2 - (\partial^\mu\varphi_2)\cdot\alpha\varphi_1$$

$$\propto [\partial^\mu\varphi_1 \cdot \varphi_2 - (\partial_\mu\varphi_2)\cdot\varphi_1]$$

Being very pedantic, plug in

$$J^\mu = ([\partial^\mu\rho]\cos\chi - (\rho\sin\chi)\partial^\mu\chi)) \times [\rho\sin\chi]$$

$$- ([\partial^\mu\rho]\sin\chi + (\rho\cos\chi)\partial^\mu\chi)\cdot[\rho\cos\chi]$$

$$= (\rho\partial^\mu\rho)[\cos\chi\sin\chi - \sin\chi\cos\chi]$$

$$- \rho^2(\sin^2\chi + \cos^2\chi)\partial^\mu\chi$$

$$J^\mu = -\rho^2\partial^\mu\chi$$

This is $(\varphi_1 + i\varphi_2) \to e^{i\alpha}(\varphi_1 + i\varphi_2)$

and with $(\varphi_1 + i\varphi_2) \equiv \rho e^{i\chi}$

the shift is $\chi \to \chi + \alpha$, $\delta\chi = \alpha$; $\delta\rho = 0$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} \delta\chi$$

$$\mathcal{L} = \frac{1}{2} \rho^2 \partial_\mu \chi \partial^\mu \chi + \cdots$$

so $J^\mu \propto \rho^2 \partial^\mu \chi$

Notice that the shift symmetry eliminates all non-derivative χ terms from $\mathcal{L}$!

c) Now add a term $V' = -\epsilon\phi_1 = -\epsilon e \cos\chi$

$$\simeq -\epsilon e\left(1 - \frac{1}{2}\chi^2\right).$$

Work to quadratic order. When e is fixed to e_0, there is a $\frac{1}{2}\epsilon e_0 \chi^2$ term, the χ part of $\mathcal{L}$ is

$$\mathcal{L}_\chi = \frac{1}{2} e_0^2 \partial_\mu\chi\partial^\mu\chi - \frac{\epsilon}{2} e_0 \chi^2.$$

The usual definition of mass comes when the kinetic term has a coefficient $\frac{1}{2}$ - so - define $\Phi = e_0\chi$

$$\partial_\mu\chi\partial^\mu\chi = \frac{1}{e_0^2}\partial_\mu\Phi\partial^\mu\Phi,$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{\epsilon}{2}\frac{e_0}{e_0^2}\Phi^2 ; \quad -\frac{1}{2}M^2 = -\frac{\epsilon}{2e_0}$$

$$\boxed{M^2 = \frac{\epsilon}{e_0}}$$

Alternatively, the equation of motion is

$$e_0^2\partial_\mu\chi\partial^\mu\chi + \epsilon e_0\chi = 0$$

$$e_0^2\left[\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{\epsilon}{e_0}\right]\chi = 0$$

$$\chi \sim e^{i(k\cdot r - \omega t)}$$

$$\Rightarrow -\omega^2 + k^2 + \frac{\epsilon}{e_0} = 0$$

$$M^2 = \frac{\epsilon}{e_0}$$

The η^* also gets a shift - but I didn't ask you to find it!

2) $A^\mu = a^\mu \exp i k^\alpha x_\alpha$, a^μ = constant, $k^\mu k_\mu = 0$

a) In Lorentz gauge $\partial_\mu A^\mu = 0$ or $k_\mu a^\mu = 0$

b) $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$\partial^\mu = \dfrac{\partial}{\partial x_\mu}$ so $F^{\mu\nu} = i\left[k^\mu a^\nu - k^\nu a^\mu\right] e^{ik_\alpha x^\alpha}$

c) $F^{\mu\nu} k_\nu = i\left[k^\mu (a^\nu k_\nu) - i(k^\nu k_\nu) a^\mu\right]$

$\times \exp i(k_\alpha x^\alpha)$

$= 0$ from part (a) for the first term,

and from $k^\nu k_\nu = 0$

d) $E_i = F^{0i}$, $F^{ij} = -\epsilon_{ijk} B_k$

$k_0 = \frac{\omega}{c}$ and $k_\nu = -k_j$ for $\nu = j$ since

k_ν is a dual vector, $k_\nu = (\frac{\omega}{c}, -\vec{k})$

$F^{\mu\nu} k_\nu$ with $\mu = i$ is $F^{i0} k_0 + F^{ij} k_j = 0$

which is $\frac{\omega}{c} E_i + (-)(-)\epsilon_{ijk} B_k k_j = 0$

$\frac{\omega}{c} E_i + (\vec{k} \times \vec{B})_i = 0$

or $\frac{\omega}{c} \vec{E} = \vec{B} \times \vec{k}$