Set 8 – due 22 March

"It is quite easy to speak about symmetries, on one side. Everybody has a notion of symmetry, it is a very deeply rooted and widespread concept, ranging from art to science. In some way or another symmetry is perceived by everybody. I think it is worth mentioning that about thirty years ago there was strong interest in experimenting with apes to see how much they were able to learn. One objective was to see how apes would learn to paint. In one of these experiments one dot was made at one side of a piece of paper and the ape would then try to make a dot on the other side to balance it symmetrically. That's exactly what we are doing in physics." – J. Wess

1) [15 points] Jackson 11.13

2) [10 points] Jackson 12.14

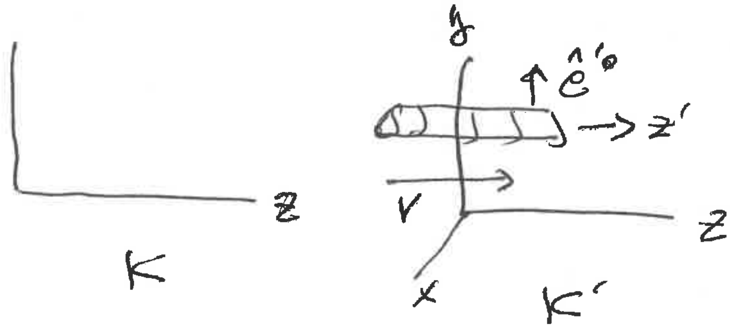
3) [10 points] Jackson 11.14, parts a, b only

4) (30 points) The axion is a candidate dark matter field with potential JILA or NIST search modes. The Lagrangian for electromagnetism coupled to an axion field $a(x,t)$ is (in some units)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu + \frac{1}{4}CaF_{\mu\nu}\tilde{F}^{\mu\nu} \tag{1}$$

where C is a coupling constant and $\tilde{F}^{\mu\nu}$ is the dual field strength tensor defined in Jackson (11.140), $\tilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. To set the stage for thinking about experiments, derive the analog Maxwell's equations from the Lagrangian, Eq. 1, expressing the electromagnetic fields in terms of $\vec{E}$ and $\vec{B}$. You'll get source terms involving the a field appearing along with the charge and current density. These new terms are the experimental hooks which different discovery experiments attempt to exploit.

11.13



In frame K' the wire is at rest and carries a charge/unit length $\equiv \rho_0$. In this frame $B' = 0$ and, using the CGS version of Gauss' law

$$\oint E' \cdot \hat{n}\, dA = 4\pi \rho_{enclosed},$$

$$E'(\rho') = \hat{\rho} \cdot \frac{4\pi \rho_0}{2\pi \rho'} = \frac{2\rho_0}{\rho'}\hat{\rho}.$$

I've written $x' = (\rho', \varphi', z')$ in cylindrical coordinates

In frame K the field transformations are

$$E_z = E'_z = 0, \quad B_z = B'_z = 0$$

$$E_x = \gamma(E'_x - \beta B'_y) = \gamma E'_x$$

$$E_y = \gamma(E_{y'} - \beta B'_x) = \gamma E'_y$$

$$B_y = \beta\gamma E'_x, \quad B_x = -\beta\gamma E'_y$$

ρ is transverse to the boost direction (as are x and y)

so $$\vec{E} = \gamma E' = \gamma \cdot \frac{2q_0}{\rho}\hat{\rho}$$

$$\vec{B} = \gamma\beta\, \hat{z} \times E' = \beta\, \hat{z} \times \vec{E}$$

Finally $\hat{z} \times \hat{\rho} = \hat{\varphi}$ so

$$\vec{B} = \hat{\varphi} \cdot \frac{2\rho_0}{\rho}\frac{v}{c} \quad \text{in CGS.}$$

b) In the wire's frame $J^{\mu'} = (c\varrho', 0)$

or $J^{\mu'} = [c q_0, 0]\, \delta^2(\varrho_\perp)$.

Boosting this to the lab frame

$$J^0 = \gamma\left[J^{0'} + \beta J_z'\right] = \gamma J^{0'} = \gamma c q_0 \delta^2(\varrho_\perp)$$

$$J_z = \gamma\left[J_z' + \beta J_0'\right] = \beta\gamma J_0' = v\gamma q_0 \delta^2(\varrho_\perp)$$

c) And we use J^0 and J_z to compute the fields directly:

$\nabla\cdot E = 4\pi\varrho$ $\quad \oint E\cdot n\, dA = 4\pi Q_{encl}$

Gauss' law: $\vec{E} = 2\dfrac{[\gamma q_0]}{\varrho}\hat{\varrho}$

Ampere's law uses $I = \int d^2x_\perp J_z(x_\perp)$

$\oint B\cdot dl = \frac{4\pi}{c} I$ in CGS

$$\vec{B} = \hat{\varphi}\,\frac{4\pi}{c}\,\frac{v\gamma q_0}{2\pi\varrho} = \hat{\varphi}\cdot 2\frac{v}{c}\gamma\frac{q_0}{\varrho},$$

the answers we found in part (a).

Jackson 12.14 : $\mathcal{L}' = -\frac{1}{8\pi} \partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha$

$$= -\frac{1}{8\pi} g_{\alpha\mu} g_{\beta\nu} \partial^\mu A^\nu \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha.$$

Evaluate the equation of motion $\partial^\mu \frac{\partial \mathcal{L}'}{\partial(\partial^\mu A^\nu)} - \frac{\partial \mathcal{L}'}{\partial A^\nu} = 0$

$$0 = -\frac{2}{8\pi} \partial^\mu [\partial_\mu A_\nu] + \frac{1}{c} J_\nu$$

or $\quad \partial^\mu \partial_\mu A_\nu = \frac{4\pi}{c} J_\nu \quad (1)$

This ~~looks~~ looks familiar but strange. If we added a $-\partial^\mu \partial_\nu A_\mu$ term, the LHS would be $\partial^\mu F_{\mu\nu}$. We can do this if $\partial^\mu \partial_\nu A_\mu = 0$, which it is, because we can switch the order of derivatives, $\partial^\mu \partial_\nu A_\mu = \partial_\nu \partial^\mu A_\mu$ and in Lorentz gauge $\partial^\mu A_\mu = 0$. So (1) is just the usual equation of motion in Lorentz gauge.

b) Now start with $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$.

The first term is, $\mathcal{L}_F = -\frac{1}{16\pi} [\partial_\mu A_\nu - \partial_\nu A_\mu][\partial^\mu A^\nu - \partial^\nu A^\mu]$

$$= -\frac{1}{16\pi} \left[\partial_\mu A_\nu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu \right].$$

The indices are just dummy indices which are summed over so just flip them on the 2nd & 4th terms ($\mu \rightleftarrows \nu$)

to get $\mathcal{L}_F = -\frac{1}{8\pi}\left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right]$

$\mathcal{L}_F = \mathcal{L}' + E$ where $\mathcal{L}'$ is the expression at the top of p. 12.14.1 and

$$E = \frac{1}{8\pi}(\partial_\mu A_\nu)(\partial^\nu A^\mu).$$

If it happened that $\partial_\mu A^\mu = 0$, we could write

$$E = \frac{1}{8\pi}\partial_\mu\left[A_\nu \partial^\nu A^\mu\right].$$

This is a total divergence. In the action, $S = \int d^4x\, \mathcal{L}$,

a total divergence does not affect the equations of motion - (as long as A_μ vanishes at infinity). The total divergence just gives a contribution to S from the boundary at spatial infinity.

Jackson 11.14 ab.

To write $F^{\alpha\beta}F_{\alpha\beta}$ and $\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}$ in terms of E + B, just write out the components.

$$F^{\alpha\beta}F_{\alpha\beta} = F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} + \dots$$

$$= -E_x^2 - E_y^2 - E_z^2 - E_x^2 - E_y^2 - E_z^2 + B_x^2 + \dots$$

$$= -2\left[E^2 - B^2\right].$$

$\mathcal{F}^{\alpha\beta}$ just exchanges $\vec{E}$ to $\vec{B}$ and $\vec{B}$ to $-\vec{E}$, compared to $F^{\alpha\beta}$, so this is just $-F^{\alpha\beta}F_{\alpha\beta}$.

$$F^{\alpha\beta}\mathcal{F}_{\alpha\beta} = F^{01}G_{01} + \dots = -E_x B_x - E_y B_y \dots$$

$$= -4\,\vec{E}\cdot\vec{B}.$$

These are the only invariants that are quadratic in E + B that one can write.

b) Can you have $\vec{B}=0$ in frame K and $\vec{E}=0$ in frame K'? No! Because if you can,

$$F^{\alpha\beta}F_{\alpha\beta} = -2E^2 < 0 \text{ in } K$$

$$= 2B'^2 > 0 \text{ in } K'$$

But $F^{\alpha\beta}F_{\alpha\beta}$ is an invariant!

Also if $\vec{E}=0$ in any frame, then $F^{\mu\nu}\mathcal{F}_{\mu\nu}=0$ (it is equal to $-4E\cdot B$) and it must be that $F^{\mu\nu}F_{\mu\nu} > 0$ in all frames.

Axion electrodynamics

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu + \frac{1}{4} C a F_{\mu\nu} \tilde{F}^{\mu\nu}$$

a = axion field, $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$.

Find Maxwell's eqns with the axion source term.

$$\partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} - \frac{\partial \mathcal{L}}{\partial A^\alpha} = 0$$

Start as in Jackson 12.86, writing

$$F_{\mu\nu} F^{\mu\nu} = g_{\lambda\mu} g_{\nu\sigma} (\partial^\mu A^\sigma - \partial^\sigma A^\mu)(\partial^\lambda A^\nu - \partial^\nu A^\lambda)$$

$$\frac{\partial [F_{\mu\nu} F^{\mu\nu}]}{\partial(\partial^\beta A^\alpha)} = g_{\lambda\mu} g_{\nu\sigma} \Big[(\delta^\mu_\beta \delta^\sigma_\alpha - \delta^\sigma_\beta \delta^\mu_\alpha) F^{\lambda\nu}$$

$$+ F^{\mu\sigma} (\delta^\lambda_\beta \delta^\nu_\alpha - \delta^\lambda_\alpha \delta^\nu_\beta) \Big]$$

$$= 4 F_{\beta\alpha}$$ — 4 terms, F is ~~anti symmetric~~ anti symmetric.

Also $F_{\mu\nu} \tilde{F}^{\mu\nu} = \tilde{F}_{\mu\nu} F^{\mu\nu}$ so

$$\frac{\partial F_{\mu\nu} \tilde{F}^{\mu\nu}}{\partial(\partial^\beta A^\alpha)} = 4 \tilde{F}_{\beta\alpha}.$$

Also ~~since~~ $\partial_\alpha \tilde{F}^{\alpha\beta} = 0$ (see Jackson ..). Maxwell's eqns are then

$$\partial^\beta [F_{\beta\alpha} - C a \tilde{F}_{\beta\alpha}] = J_\alpha$$

or $$\partial^\beta F_{\beta\alpha} = J_\alpha + C \partial^\beta [a \tilde{F}_{\beta\alpha}]$$

The homogeneous equations are unchanged

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} - \frac{\partial\vec{B}}{\partial t} = 0 .$$

For the inhomogeneous terms, we need

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & & 0 \end{bmatrix}$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix}$$

$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \quad J_\alpha = (\rho, -\vec{J})$$

Now for the ~~homog~~ inhomogeneous equations

$\alpha = 0,\ \beta = i$: $\quad -\partial_i\left[-E_i - c\,a(-B_i)\right] = \rho$

Dont forget: $a = a(x,t)$. $\partial_i B_i = \nabla\cdot B = 0$, so

$$\underline{\vec{\nabla}\cdot\vec{E} = \rho + c\,\vec{B}\cdot\vec{\nabla}a}$$

Then $\alpha = i$;

$$\partial^0 \left[F_{0i} - Ca \tilde{F}_{0i} \right] + \partial^j \left[F_{ji} - Ca \tilde{F}_{ji} \right] = -J_i$$

$$\frac{\partial}{\partial t}\left(E_i - ca B_i \right) - \partial_j \left[-\epsilon_{jik} B_k + ca\, \epsilon_{jik} E_k \right] = -J_i$$

$$\frac{\partial E_i}{\partial t} - ca \left[\frac{\partial B_i}{\partial t} - \epsilon_{jik} \partial_j E_k \right] + \epsilon_{jik} \partial_j B_k - C B_i \frac{\partial a}{\partial t} - C (\partial_j a) \epsilon_{jik} E_k = -J_i$$

or

$$\frac{\partial \vec{E}}{\partial t} - ca \left[\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} \right] - \vec{\nabla} \times \vec{B} - C \vec{B} \frac{\partial a}{\partial t} - C (\vec{\nabla} a) \times \vec{E} = -\vec{J}$$

$\frac{\partial B}{\partial t} + \nabla \times E = 0$ so we have

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} - C \underbrace{\left[\vec{B} \frac{da}{dt} + \vec{\nabla} a \times \vec{E} \right]}$$

The new source terms involve
E + B multiplied by derivatives
of the axion field.

See P. Sikivie Phys Rev Lett 51 1415 (1983)
F. Wilczek Phys Rev Lett 58, 1799 (1987)
Kahn, Safdi, Thaler Phys Rev Lett 117, 141801 (1986)
[1602.01086?]