Set 7 – due 8 March

The midterm exam will be in Duane G2B41 on Thursday, March 14, 7:00 – 8:30 PM

"The nonmathematician is siezed by a mysterious shuddering when he hears of 'four-dimensional' things, by a feeling not unlike that awakened by thoughts of the occult. And yet there is no more commonplace statement, that the world in which we live is a four-dimensional space-time continuum."–A. Einstein

1) [10 points] Compton scattering. You might recall that when a photon scatters off a free electron, its wavelength is shifted. Derive the Compton formula. The invariants s, t and u from last week are quite useful. You are working in a frame where the photon comes in along the z direction, so $p_1 = (E, 0, 0, E/c)$, the electron has $p_2 = (mc^2, 0, 0, 0)$ and the outgoing photon is $p_3 = (E', 0, (E'/c)\sin\theta, (E'/c)\cos\theta)$. Recall the relation between photon energy and wavelength, $E = hc/\lambda$. I think I have the c's right in the momenta, but I put them in by hand at the end.

2) [15 points] A particle originally at rest and with initial rest energy $m_1 = m + \Delta E$ emits a photon and decays to a particle of rest energy m. What is the energy of the photon? Check explicitly the limiting cases $m >> \Delta E$, and $m = 0$. ($c = 1$ here). Analogs of these cases are the decay of the 2P state of hydrogen to the 1S state, and the decay Higgs $\to \gamma\gamma$.

3) [20 points] A beam of light of frequency ω travelling in the $x - y$ plane (at an angle θ_0 with respect to the x axis) is reflected from a mirror moving with velocity v in the x direction. Calculate (a) [10 points] the frequency of the reflected beam and b) [10 points] the angle of reflection (in the frame where the incident angle is θ_0).

1) Compton scattering: use $s+t+u = \sum m_i^2 = 2m^2$

where $c=1$, m = electron mass, remember photon is massless. In the lab frame, kinematics labelled as $p = (E, p_x, p_z)$ are

$$p_1 = E(1, 0, 1)$$

$$p_3 = E'(1, \sin\theta, \cos\theta)$$

$$p_2 = (m, 0, 0)$$

so

$$s = (p_1+p_2)^2 = 2p_1 \cdot p_2 + m^2 = 2mE + m^2$$

$$t = (p_1 - p_3)^2 = -2p_1 \cdot p_3 = -2EE'(1-\cos\theta)$$

$$u = (p_2 - p_3)^2 = m^2 - 2p_2 \cdot p_3 = -2mE' + m^2$$

$$2m^2 = [2mE + m^2] - 2EE'(1-\cos\theta) + [m^2 - 2mE']$$

$$0 = 2m(E-E') - 2EE'(1-\cos\theta)$$

$$2mE = 2E'[E(1-\cos\theta) + m]$$

so $$E' = \frac{mE}{E(1-\cos\theta)+m}$$

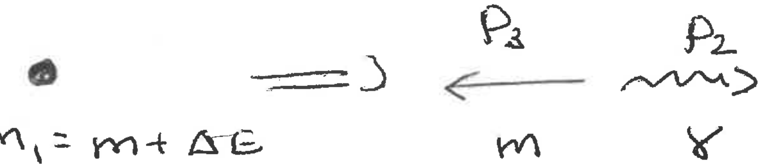
and $$\frac{1}{E'} = \frac{E(1-\cos\theta)+m}{mE} = \frac{1}{E} + \frac{(1-\cos\theta)}{m}$$

Re-insert c's: $$\frac{1}{E'} = \frac{1}{E} + \frac{(1-\cos\theta)}{mc^2}$$

Then $E = \frac{hc}{\lambda}$, $E' = \frac{hc}{\lambda'}$ so

$$\frac{\lambda'}{hc} = \frac{\lambda}{hc} + \frac{(1-\cos\theta)}{m}$$ or $$\boxed{\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)}$$

2)



$m_1 = m + \Delta E$ $\qquad (c=1)$

$P_1 = (m_1, \vec{0})$

$P_2 = (E_\gamma, -\vec{p})$ and $|p| = E_\gamma$ since photon is massless

$P_3 = (E_3, \vec{p})$; $E_3 = m_1 - E_\gamma = m + \Delta E - E_\gamma$

$$P_1^2 = m_1^2 = (P_2 + P_3)^2 = P_2^2 + P_3^2 + 2 P_2 \cdot P_3$$

$$= 0 + m^2 + 2(E_\gamma E_3 + |\vec{p}|^2)$$

so $$(m + \Delta E)^2 = 0 + m^2 + 2E_\gamma(m + \Delta E - E_\gamma) - 2E_\gamma^2$$

$$m^2 + 2m\Delta E + (\Delta E)^2 = m^2 + 2E_\gamma(m + \Delta E)$$

$$E_\gamma = \frac{\Delta E(2m + \Delta E)}{2(m + \Delta E)}$$

or $$E_\gamma = \Delta E\left[\frac{2(m+\Delta E) - \Delta E}{2(m+\Delta E)}\right] = \Delta E\left[1 - \frac{\Delta E}{2(m+\Delta E)}\right]$$

The limiting cases are

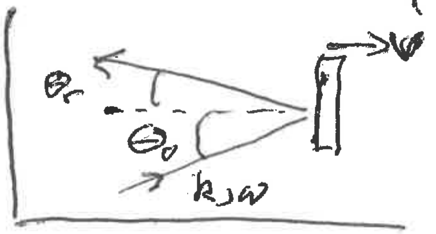
i) $\Delta E \ll m$ (for example, emission of a photon by an atom in an electronic level transition)

$$E_\gamma = \Delta E$$

ii) $m = 0$ (example, $H \to \gamma\gamma$): $E_\gamma = \frac{\Delta E}{2} = \frac{m_1}{2}$

This is also easy to visualize - the massive particle decays into two equal mass particles, and their momenta are equal and opposite, so they share the initial energy equally.

3) Label the lab rest frame as K, the mirror's frame K'. Label $\beta = v/c$, $k = \frac{\omega}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$



$$k_\perp = k \sin\theta_0$$
$$k_\parallel = k \cos\theta_0$$

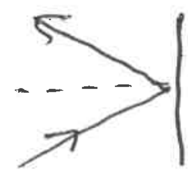
In frame K the photon's 4-momentum is

$$k = \left(\frac{\omega}{c}, k_\parallel, k_\perp\right)$$

To proceed, transform to the mirror's frame K'

$$\omega' = \gamma(\omega - vk\cos\theta_0)$$
$$k'_\parallel = \gamma(k_\parallel - v\omega/c^2) \qquad (1)$$
$$k'_\perp = k_\perp$$

The mirror is at rest in K'. "Reflection" means the reflected wave has $\tilde{k}' = (\omega', -k'_\parallel, k'_\perp)$ - that is, $k'_\parallel$ flips sign



Now return to the lab frame. In the Lorentz transformation inverting (1), β goes to $-\beta$, but

$$\omega'' = \gamma(\omega' - vk_\parallel)$$

because there are <u>two</u> minus signs, $v \to -v$ and $-k'_\parallel$.

In $k''_\parallel = \gamma\left(-k'_\parallel + \frac{v\omega'}{c^2}\right)$ β changes sign

$$k''_\perp = k'_\perp.$$

All together

$$\omega'' = \gamma\left[\gamma(\omega - vk\cos\theta_0) - v\gamma\left(k\cos\theta_0 - \frac{v\omega}{c^2}\right)\right]$$

$$= \gamma^2\omega\left[1 - \beta\cos\theta_0 - \beta\cos\theta_0 + \beta^2\right]$$

$$= \gamma^2\omega\left[1 - 2\beta\cos\theta_0 + \beta^2\right]$$

$$k''_\parallel = \gamma^2\left[-\left(k\cos\theta_0 - \frac{v\omega}{c^2}\right) + \frac{v}{c^2}(\omega - vk\cos\theta_0)\right]$$

$$= \gamma^2\frac{\omega}{c}\left[-\cos\theta_0 + \beta + \beta - \beta^2\cos\theta_0\right]$$

$$= -\gamma^2\frac{\omega}{c}\left[(1+\beta^2)\cos\theta_0 - 2\beta\right]$$

The reflected angle is

$$\tan\theta_r = -\frac{k''_\perp}{k''_\parallel} = -\frac{k_\perp}{k''_\parallel} = \frac{\sin\theta_0}{\gamma^2\left[(1+\beta^2)\cos\theta_0 - 2\beta\right]}$$

If $\beta = 0$, $\theta_r = \theta_0$. Otherwise, as γ becomes large, θ_r falls to zero as $\frac{1}{\gamma^2}$.