Set 6 – due 1 march

"What a pity that I have to die in the age of relativity's development." – H. Minkowski (1909)

1) [20 points] "Why, these monsters had star travel!" In Heinlein's classic novel, "Have Space Suit, Will Travel," the earth is menaced by aliens from Proxima Centauri (distance 4.3 light years), who travel in ships which accelerate for half the trip at $A = 8$ g, and decelerate for the other half. What is the elapsed time of a one-way trip from there to here, as seen by the crew, and as measured by observers on either planet?

I can think of two ways to proceed. The first one is go into the moving frame (of velocity v) and imagine an object with velocity u' and acceleration du'/dt'. In the stationary frame the object has a velocity w and an acceleration dw/dt. Lorentz transformation and velocity addition lead to

$$\frac{dw}{dt} = \frac{1}{\gamma}\frac{du'}{dt'}\frac{1 - v^2/c^2}{(1 + vu'/c^2)^3}. \tag{1}$$

In the limit $u' \to 0$, this reduces to $dv/dt = A/\gamma^3$ for this problem, and you can show $At = \gamma v$, which you can use to find $v(t)$ or $\gamma(t)$, then the total distance and time, integrating using $dt' = dt/\gamma$.

The second way is to realize that the ship's acceleration is a proper acceleration which can be boosted into the earth's frame, and then (in terms of proper time) $A^\mu = dU^\mu(\tau)/d\tau$ where $U^\mu(\tau)$ is the four velocity and τ is the proper time. Then $U^\mu(\tau) = dx^\mu(\tau)/d\tau$. You can integrate everything to get $x^\mu(\tau)$ which gives the same relations between A, the distance traveled, and the elapsed time in both frames, as the first way gives.

2) [10 points] A set of invariants which describe the scattering of unequal mass relativistic particles $(1 + 2 \to 3 + 4)$ are $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$. (a) [5 points] Show $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$. (b) [5 points] Now assume all the particles have equal mass m. From (a) only two of s, t, u are independent. Draw a two dimensional plot of s vs t showing the kinematically allowed region(s)– i.e., ANY scattering experiment will populate only your allowed region. ($c = 1$ here.) The reason why this is interesting: scattering amplitudes are most usefully computed in terms of invariants, because then it is easy to evaluate them in different frames. (For example, $d\sigma/dt$ rather than $d\sigma/d\Omega$.) It is useful to know the ranges of s, t, etc, when you examine the amplitudes to look for "interesting" behavior.

3) [20 points] Two equal mass particles (of mass m) scatter elastically with a scattering angle α in the center of mass frame. Show that the scattering angle

in a frame where one particle is at rest and the other has energy E is given by

$$\cos^2\theta = \frac{\cos^2\alpha/2}{1 - \frac{E - mc^2}{E + mc^2}\sin^2\frac{\alpha}{2}} \tag{2}$$

Comment on the nonrelativistic and extreme relativistic limits.

1) The ship's occupants feel an acceleration $A = 8g$. What is the acceleration in the frame of the Earth? To begin, assume there is an object with velocity u' in the ship's frame, while the ship has a velocity v with respect to the Earth. The object's velocity in the Earth's frame is

$$w = \frac{u'+v}{1+vu'/c^2}$$

so
$$dw = \frac{du'}{1+\frac{vu'}{c^2}} - \frac{v}{c^2}\frac{(u'+v)}{\left[1+\frac{vu'}{c^2}\right]^2}du'$$

$$= \frac{du'}{\left[1+\frac{vu'}{c^2}\right]^2}\left[1+\frac{vu'}{c^2}-\frac{vu'}{c^2}-\frac{v^2}{c^2}\right]$$

$$= \frac{du'\,(1-v^2/c^2)}{\left[1+\frac{vu'}{c^2}\right]}. \qquad (1)$$

Earth time t is related to ship coordinates (x', t') by a Lorentz transformation, so

$$dt = \gamma\left[dt' + \frac{v\,dx'}{c^2}\right] \quad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

so
$$dt = \gamma\,dt'\left[1+\frac{v}{c^2}\frac{dx'}{dt'}\right]$$

$$= \gamma\,dt'\left[1+\frac{vu'}{c^2}\right] \qquad (2)$$

Combine (1) and (2):

$$\frac{dw}{dt} = \frac{1}{\gamma}\frac{du'}{dt'}\frac{[1-v^2/c^2]}{[1+vu'/c^2]} \xrightarrow[u'=0]{} \frac{1}{\gamma^3}\frac{du'}{dt'}$$

In the $u' \to 0$ limit, w reduces to v. ~~Recalling~~

Recalling $A = \frac{du'}{dt'}$, we have the acceleration in our frame $\frac{dv}{dt} = \frac{A}{\gamma^3}$.

This can be ~~intdg~~ integrated:

$$At = \int_0^v dv' \gamma^3(v') = \int_0^v \frac{dv'}{\left[1-\frac{v'^2}{c^2}\right]^{3/2}}$$

$$= \frac{v}{\sqrt{1-v^2/c^2}} = \gamma v$$

$At = \gamma v$ is the hint.

[check: ($c=1$) $\frac{d}{dt}\frac{v}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-v^2}} + \frac{v^2}{(1-v^2)^{3/2}} = \frac{1-v^2+v^2}{(1-v^2)^{3/2}}$]

Then $At = \frac{v}{\sqrt{1-v^2/c^2}} \Longrightarrow A^2t^2 = \frac{v^2}{1-v^2/c^2}$

This is $v^2 = A^2t^2\left(1-\frac{v^2}{c^2}\right)$ or

$$v^2\left[1+\frac{A^2t^2}{c^2}\right] = A^2t^2 \quad \text{or} \quad \frac{v}{c} = \frac{At/c}{\sqrt{1+A^2t^2/c^2}}.$$

Note also $\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} = 1 - \frac{A^2t^2/c^2}{1+\frac{A^2t^2}{c^2}} = \frac{1}{1+A^2t^2/c^2}$.

The ship accelerates halfway and decelerates halfway, so the total distance D and time T (both measured in Earth's frame) are related by

$$D = 2\int_0^{T/2} v\, dt.$$

The ship's differential time is $dt' = \frac{dt}{\gamma}$, which can be integrated to give the elapsed time in the ship's frame.

$$T' = 2\int_0^{T/2} \frac{dt}{\sqrt{1+\frac{A^2t^2}{c^2}}} = \frac{2c}{A} \sinh^{-1} \frac{AT}{2c}$$

or $$\frac{AT}{2c} = \sinh \frac{AT'}{2c}.$$

$$D = 2\int_0^{T/2} \frac{At\,dt}{\sqrt{1+A^2t^2/c^2}} \quad ; \quad \text{set } u = 1+A^2t^2/c^2, \quad du = \frac{2A^2 t\,dt}{c^2}$$

$$D = \frac{c^2}{A}\int_1^{1+A^2T^2/4c^2} \frac{du}{\sqrt{u}} = \frac{2c^2}{A}\left\{\sqrt{1+\left(\frac{AT}{2c}\right)^2} - 1\right\}$$

or $$\left[\frac{AD}{2c^2}+1\right]^2 = 1+\left(\frac{AT}{2c}\right)^2$$

$$T = \frac{2c}{A}\left[\left(\frac{AD}{2c^2}+1\right)^2 - 1\right]^{1/2} = \frac{D}{c}\left[\left(1+\frac{2c^2}{AD}\right)^2 - \left(\frac{2c^2}{AD}\right)^2\right]^{\frac{1}{2}}$$

$$T = \frac{D}{c}\left[1+\frac{4c^2}{AD}\right]^{1/2}.$$

Check the limiting cases. If $\frac{AD}{c^2} \gg 1$ then T is slightly greater than $\frac{D}{c}$. The ship is mostly moving at a velocity just below c. However if $\frac{AD}{c^2} \ll 1$,

$$T \approx \frac{D}{c}\cdot\frac{2c}{\sqrt{AD}} = 2\sqrt{\frac{D}{A}} \quad \text{or} \quad D = \frac{AT^2}{4} \quad \text{or}$$

$\frac{1}{2}D = \frac{1}{2}A\left(\frac{T}{2}\right)^2$ - this is nonrelativistic motion.

Now we can compute numbers.

$1 \text{ year} = \pi \times 10^7 \text{ sec}. \quad A = 80 \frac{m}{s^2}$

$D = 4.3$ light years so $\frac{D}{c} = 4.3$ years.

$$\frac{AD}{2c^2} = \frac{A}{2c}\,\frac{D}{c} = \frac{80\ m/s^2}{6 \times 10^8\ m/s} \times 4.3 \times \pi \times 10^7 \text{ sec}$$

$$= \frac{8}{6} \times \pi \times 4.3 = 18$$

$T = 4.3 \text{ years} \times \sqrt{1 + \frac{1}{9}} = 4.5$ years – the length of the journey as seen by an Earth observer.

$$T' = \left[\frac{6 \times 10^8\ m/s}{80 \frac{m}{s^2} \times \pi \times 10^7 \text{ sec}}\right] \text{years} \times \sinh^{-1} \underbrace{\left[\frac{80 \frac{m}{s} \times 4.5 \times \pi \times 10^7}{6 \cdot 10^8\ m/s}\right]}_{18.8}$$

$y = \sinh x = \frac{e^x - e^{-x}}{2} \simeq \frac{e^x}{2}$ if x is large

$2y = e^x$ so $x = \ln 2y = \ln[2 \times 18.8] = 3.6$

The travelers age $T' = \frac{6}{8\pi} \times 3.6$ years

$= 0.86$ years

An alternative derivation uses 4-acceleration and proper time. The ship's acceleration $A = 8g$ in its rest frame is a proper acceleration – The four vector is (time, space) = $(0, A)$.

We can boost it into the Earth's frame by

$$\begin{pmatrix} A^0 \\ A^1 \end{pmatrix} = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} \begin{pmatrix} 0 \\ A \end{pmatrix} = \begin{pmatrix} \beta\gamma A \\ \gamma A \end{pmatrix} \qquad (1)$$

u, the 4-velocity, is $(c\gamma, \beta\gamma)$ and $A^\mu = \dfrac{du^\mu}{d\tau}$ so

eq (1) is $\dfrac{du^0}{d\tau} = \dfrac{A}{c} u^1$ and $\dfrac{du^1}{d\tau} = \dfrac{A}{c} u^0$.

A second derivative gives $\dfrac{d^2 u^\mu}{d\tau^2} = \left(\dfrac{A}{c}\right)^2 u^\mu$; $\mu = 0, 1$

There are more cryptic ways to write the solutions but we can simply say

$$u^0(\tau) = c_1 \cosh \frac{A\tau}{c} + c_2 \sinh \frac{A\tau}{c}$$

$$u^1(\tau) = d_1 \cosh \frac{A\tau}{c} + d_2 \sinh \frac{A\tau}{c} \qquad (1)$$

We start at $\tau = 0$ with $u^1(0) = 0$ and $u^0(0) = c$.

This says $c_1 = c$ and $d_1 = 0$.

$$\frac{A}{c} u^1(\tau) = \frac{du^0}{d\tau} = \frac{A}{c}\left[c_1 \sinh \frac{A\tau}{c} + c_2 \cosh \frac{A\tau}{c}\right] \qquad (2)$$

and $u^1(0) = 0$ says $c_2 = 0$: so far $c_1 = c$, $c_2 = 0$

In general $\dfrac{A}{c} u^1(\tau) = \dfrac{A}{c} c_1 \sinh \dfrac{A\tau}{c} = \dfrac{A}{c} d_2 \sinh \dfrac{A\tau}{c}$

so $d_2 = c_1 = c$.

So the real solution satisfying the $\tau = 0$ boundary conditions is

$$u^0(\tau) = c \cosh \frac{A\tau}{c} \;,\quad u^1(\tau) = c \sinh \frac{A\tau}{c}.$$

Next, $u^0(\tau) = \dfrac{dx^0}{d\tau}$ where $x^0 = ct(\tau)$

$u^1(\tau) = \dfrac{dx^1}{d\tau}$ where $x^1 = x(\tau)$

Integrate using u^μ to get $x^\mu = \int d\tau\, u^\mu(\tau)$

$$x^1(\tau) = x(\tau) = \frac{c^2}{A}\left[\cosh\frac{A\tau}{c} - 1\right] \qquad (a)$$

(the -1 so $x(0) = 0$)

$$\frac{x^0(\tau)}{c} = t(\tau) = \frac{c}{A}\sinh\frac{A\tau}{c} \qquad (b)$$

note $\tau(0) = 0$.

Now we can fill things in: the halfway point is $x = \frac{D}{2}$, $t = \frac{T}{2}$, which happens at proper time $\tau = \frac{T'}{2}$.

Eq (b) is $\sinh\dfrac{AT'}{2c} = \dfrac{AT}{2c}$ - see p. 1.3

Eq (a) is $\dfrac{D}{2} = \dfrac{c^2}{A}\left[\cosh\dfrac{AT'}{2c} - 1\right]$

or $\cosh\dfrac{AT'}{2c} = \dfrac{AD}{2c^2} + 1$

To find T in terms of D

$$\cosh^2\frac{AT'}{2c} - \sinh^2\frac{AT'}{2c} = 1 = \left(\frac{AD}{2c^2} + 1\right)^2 - \left(\frac{AT}{2c}\right)^2$$

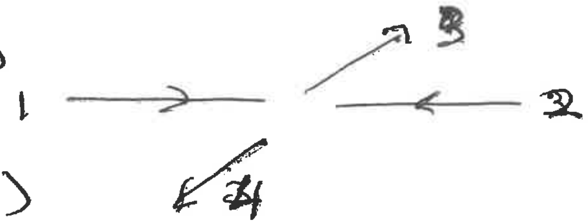
or $$\frac{AT}{2c} = \left[\left(\frac{AD}{2c^2} + 1 \right)^2 - 1 \right]^{1/2}$$

$$= \frac{AD}{2c^2} \left[\left(1 + \frac{2c^2}{AD} \right)^2 - \left(\frac{2c^2}{AD} \right)^2 \right]^{1/2}$$

$$T = \frac{D}{c} \left[1 + \frac{4c^2}{AD} \right]^{1/2}$$

as we previously found on p.1.3.

2)



a)

$$s = (P_1+P_2)^2$$
$$t = (P_1-P_3)^2$$
$$u = (P_1-P_4)^2$$

$$P_1+P_2-P_3-P_4=0$$

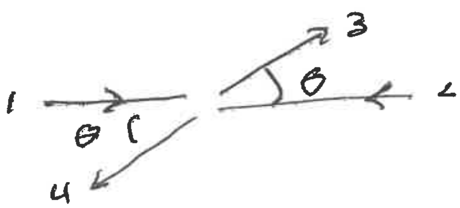
$$\begin{aligned} s+t+u = \; & m_1^2+m_2^2+2P_1\cdot P_2 \\ & + m_1^2+m_3^2-2P_1\cdot P_3 \\ & + m_1^2+m_4^2-2P_1\cdot P_4 \end{aligned}$$

$$0 = P_1\cdot(P_1+P_2-P_3-P_4) = m^2+P_1\cdot P_2-P_1\cdot P_3-P_1\cdot P_4$$

so $\quad s+t+u = 3m_1^2+m_2^2+m_3^2+m_4^2-2m_1^2$

$$= \sum_{i=1}^{4} m_i^2 .$$

b) We now set all masses to be equal, for convenience. In the CM frame, and hence in all frames, $s \geq 4m^2$. To see this, write the kinematics explicitly:



$$P_i = (E_i, P_{zi}, P_{xi}, P_{yi})$$

$$P_1 = (E, P, 0, 0) \quad P_2 = (E, -P, 0, 0)$$

note $E^2-p^2=m^2$ for each

$$P_3 = [E, p\cos\theta, p\sin\theta, 0]$$
$$P_4 = (E, -p\cos\theta, -p\sin\theta, 0)$$

so $\quad s = (P_1+P_2) = 4E^2$

$$p^2 = E^2-m^2 = \frac{s-4m^2}{4} \geq 0 \Rightarrow \boxed{s \geq 4m^2}$$

$$t = (P_1-P_3)^2 = [\,(0, p(1-\cos\theta), -p\sin\theta, 0]^2$$
$$= -p^2(1-\cos\theta)^2 - p^2\sin^2\theta$$
$$= -p^2[1-2\cos\theta+\cos^2\theta-\sin^2\theta] = -2p^2(1-\cos\theta)$$
$$t = -\left[\frac{s-4m^2}{2}\right](1-\cos\theta)$$

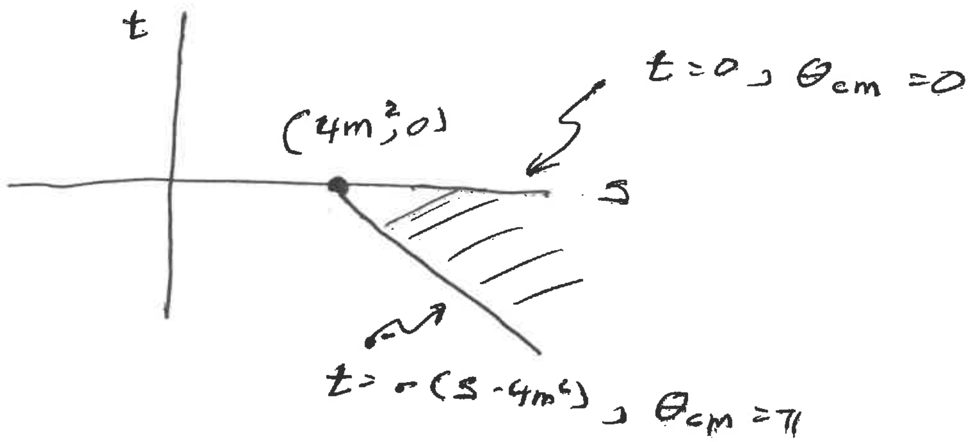
Of course, $-1 \le \cos\theta \le 1$.

This says that in the equal mass case

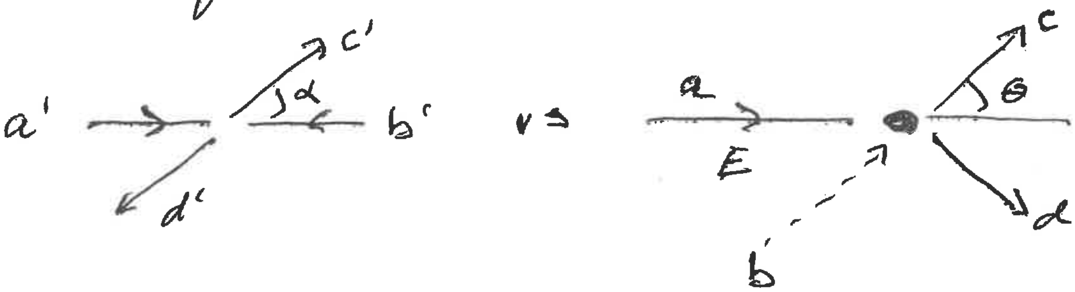
$$0 > t > -\left[\frac{s-4m^2}{2}\right]\cdot 2$$

$$0 > t > -(s-4m^2)$$

The physical region is shaded:



3) Equal mass elastic scattering in the lab and CM frames.



We are asked to show $(c=1)$

$$\cos^2\theta = \frac{\cos^2\frac{\alpha}{2}}{1-\sin^2\alpha\cdot\left[\frac{E-m}{E+m}\right]}.$$

Scalar quantities are invariant so $P_a\cdot P_d = P_a'\cdot P_d'$.
We need to write the kinematics explicitly.

$P_a' = (E', p', 0, 0)$

$P_b' = (E', -p', 0, 0)$

$P_c' = (E', p'\cos\alpha, p'\sin\alpha, 0)$

equal masses means all $|\vec{p}|$'s and all E's are equal in the CM

$P_a = (E, P, 0, 0)$

$P_b = (m, 0, 0, 0)$

$P_c = (E_c, P_c\cos\theta, P_c\sin\theta, 0)$

There are at least 2 ways to get the answer.

The first way:

1) $(p_a' + p_b')^2 = 4E'^2 = (p_a + p_b)^2 = 2m^2 + 2Em$

or $E = \dfrac{2E'^2 - m^2}{m}$ or $\cancel{E'^2 = \frac{m^2 + mE}{2}}$

2) $p_a' \cdot p_c' = p_a \cdot p_c$: $E'^2 - p'^2 \cos\alpha = E E_c - p p_c \cos\theta$

3) $p_b' \cdot p_c' = p_b \cdot p_c$: $E'^2 + p'^2 \cos\alpha = m E_c$

Pause to recall useful things:

$\cos^2\frac{\alpha}{2} = \dfrac{1+\cos\alpha}{2}$, $\sin^2\frac{\alpha}{2} = \dfrac{1-\cos\alpha}{2}$, $1 - 2\sin^2\frac{\alpha}{2} = \cos\alpha$

$p'^2 = E'^2 - m^2$ and $E'^2 = \dfrac{mE + m^2}{2}$ says $p'^2 = \dfrac{mE - m^2}{2}$

We also need $p_d^2 = m^2 = (p_a + p_b - p_c)^2$

$$m^2 = 3m^2 + 2p_a \cdot p_b - 2p_a \cdot p_c - 2p_b \cdot p_c$$

$$0 = 2m^2 + 2mE - 2(EE_c - pp_c\cos\theta) - 2mE_c$$

$$= 2pp_c\cos\theta + 2(m+E)(m - E_c)$$

This says $\cos\theta = \left(\dfrac{E+m}{p}\right)\left(\dfrac{E_c - m}{p_c}\right)$

Now become devious: $p = \sqrt{(E+m)(E-m)}$

$p_c = \sqrt{(E_c+m)(E_c-m)}$

so

4) $\cos^2\theta = \left(\dfrac{E+m}{E-m}\right)\left(\dfrac{E_c-m}{E_c+m}\right)$

3) is $E_c = \dfrac{E'^2 + p'^2\cos\alpha}{m} = \dfrac{1}{m}\left[\dfrac{m^2 + mE}{2} + \left(\dfrac{mE - m^2}{2}\right)\cos\alpha\right]$

or $E_c = \dfrac{E+m}{2} + \left(\dfrac{E-m}{2}\right)\cos\alpha$.

Fill in (4):

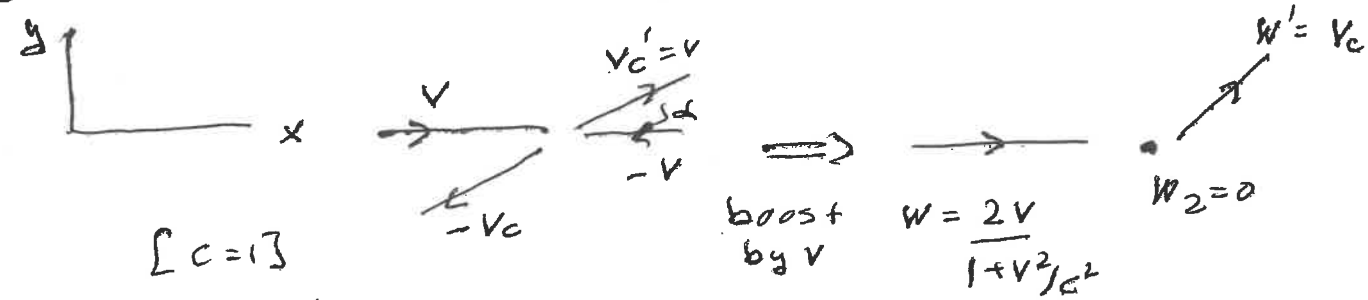
$$\cos^2\theta = \left(\frac{E+m}{E-m}\right)\frac{\left(\frac{E+m}{2} + \left(\frac{E-m}{2}\right)\cos\alpha - \frac{2m}{2}\right)}{\left(\frac{E+m}{2} + \left(\frac{E-m}{2}\right)\cos\alpha + \frac{2m}{2}\right)}$$

$$= \left(\frac{E+m}{E-m}\right)\frac{(E-m)(1+\cos\alpha)}{E+m+(E-m)(1-2\sin^2\frac{\alpha}{2})+2m}$$

$$= \frac{(E+m)\ (2\cos^2\frac{\alpha}{2})}{2(E+m)-2(E-m)\sin^2\frac{\alpha}{2}}$$

$$\cos^2\theta = \frac{\cos^2\frac{\alpha}{2}}{1-\left(\frac{E-m}{E+m}\right)\sin^2\frac{\alpha}{2}} \quad (!)$$

Another way to get the answer is to directly boost the CM variables by some $-v$ to go to the target's rest frame. 3.4



$[c=1]$

$$V_{cx} = \frac{v + v_c' \cos\alpha}{1 + v v_c' \cos\alpha} \quad \text{and} \quad V_{cy} = \frac{1}{\gamma_v} \frac{v_c' \sin\alpha}{1 + v v_c' \cos\alpha} \qquad (v_c' = v)$$

$$\cos^2\theta = \frac{V_{cx}^2}{V_{cx}^2 + V_{cy}^2} = \frac{v^2(1+\cos\alpha)^2}{v^2(1+\cos\alpha)^2 + \frac{v^2}{\gamma_v^2}\sin^2\alpha} \qquad \text{'(1')}$$

Now it's just thrashing

$$\gamma_w^2 = \frac{E^2}{m^2} = \frac{1}{1-w^2} = \frac{1}{1-\left(\frac{2v}{1+v^2}\right)^2} = \frac{(1+v^2)^2}{1+2v^2+v^4-4v^2}$$

so $\frac{E^2}{m^2} = \left(\frac{1+v^2}{1-v^2}\right)^2$ or $\frac{E}{m} = \frac{1+v^2}{1-v^2}$ or $\boxed{v^2 = \frac{E-m}{E+m}}$

$$\frac{1}{\gamma_v^2} = 1 - v^2 = 1 - \left(\frac{E-m}{E+m}\right) = \frac{2m}{E+m}$$

so (1') is

$$\cos^2\theta = \frac{1}{1 + \left(\frac{\sin\alpha}{1+\cos\alpha}\right)^2 \cdot \frac{2m}{E+m}}$$

$$= \frac{(1+\cos\alpha)^2}{1 + 2\cos\alpha + \cos^2\alpha + \sin^2\alpha\left[\underbrace{\frac{2m}{E+m} - 1}_{-\left(\frac{E-m}{E+m}\right)} + 1\right]}$$

$$\cos^2\theta = \frac{(1+\cos\alpha)^2}{2(1+\cos\alpha) - \sin^2\alpha \cdot \left(\frac{E-m}{E+m}\right)}$$

And $\quad 1+\cos\alpha = 2\cos^2\frac{\alpha}{2}$

$\quad \sin^2\alpha = 4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}$

so

$$\cos^2\theta = \frac{\left[\cos^2\frac{\alpha}{2}\right]^2}{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\left(\frac{E-m}{E+m}\right)}$$

$$= \frac{\cos^2\frac{\alpha}{2}}{1-\left(\frac{E-m}{E+m}\right)\sin^2\frac{\alpha}{2}}$$ as on p.3.3

In the NR limit, $E=m$ plus a small correction, so $\theta = \frac{\alpha}{2}$ which is the familiar (?) result for equal mass NR scattering. The extreme relativistic limit is $E \gg m$ where $\frac{E-m}{E+m} \simeq \frac{E-m}{E}\left(1-\frac{m}{E}\right) \sim 1-\frac{2m}{E}$

and $\cos^2\theta = \frac{\cos^2\frac{\alpha}{2}}{1-\sin^2\frac{\alpha}{2}\left(1-\frac{2m}{E}\right)} = \frac{\cos^2\alpha/2}{\cos^2\frac{\alpha}{2}+\frac{2m}{E}\sin^2\frac{\alpha}{2}}$

$$= \frac{1}{1+\frac{2m}{E}\tan^2\frac{\alpha}{2}}$$

$$= 1 - O\left(\frac{m}{E}\right)$$

Regardless of α, everything goes forward: $\cos\theta = 1$ or $\theta = 0°$.