Set 5 – due 23 February

"The ideas of space and time which I wish to develop before you grew from the soil of experimental physics. Therein lies their strength. Their tendency is radical. From now on, space by itself and time by itself must sink into the shadow, while only a union of the two preserves independence." – H. Minkowski (1908)

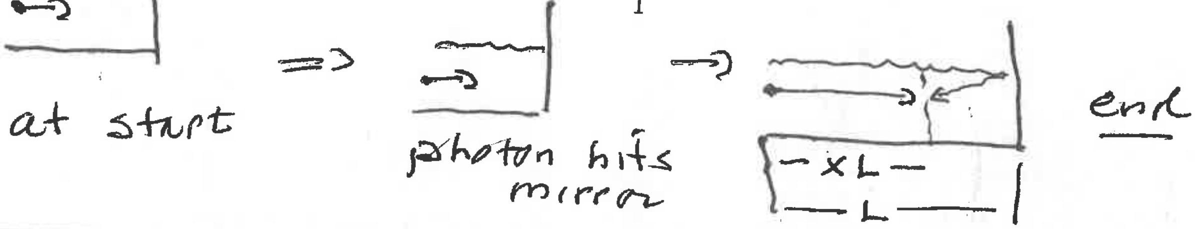
1) [5 points] Jackson 11.4 part (a). (b) [5 points] Instead of Jackson's part (b), do the following: "Length contraction" occurs when an observer infers a length from an elapsed time. Suppose a rod moves parallel to its length at velocity v, past an observer. The (unprimed) observer sees one end of the rod pass the origin at time $t_1 = 0$. The other end passes the origin at time t_2. The observer infers a length $l = vt_2$. A (primed) observer on the rod also has $(x'_1, t'_1) = (0, 0)$. The other end of the rod is at l_0 and the moving first observer passes this end at time $t'_2 = l_0/v$. Show $l = l_0/\gamma$.

2) [10 points] Consider a stick with a mechanism which, in its proper rest frame, can simultaneously release a drop of ink from each end. The stick moves parallel to its length with a velocity v along the floor. The stick has a length (measured in its rest frame) of l. The floor has lines spaced a distance l apart, again as measured in the floor's frame, aligned perpendicular to the direction of motion of the stick. When the mechanism is set off, how far apart (in the rest frame of the floor) are the marks on the floor? Justify your result from the point of view of an observer in the rest frame of the stick and an observer in the rest frame of the floor.

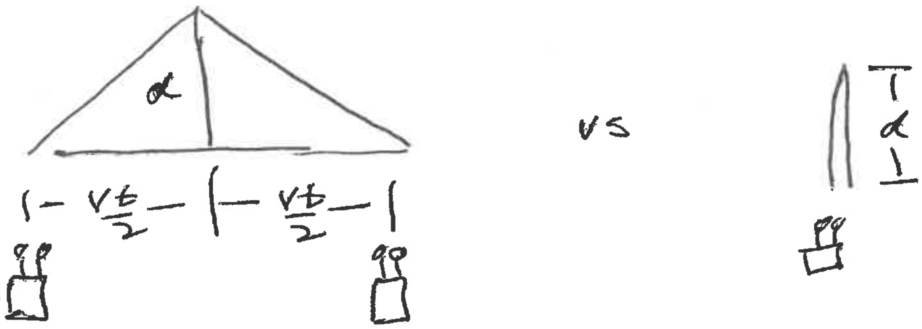
3) [15 points] Consider a stick with a mirror on the right end. At a given moment a photon and a particle moving with velocity $u < c$ leave the left end, moving to the right along the stick. The photon reaches the mirror first, is reflected, and, moving back to the left, encounters the particle still moving to the right, a fraction x of the way along the stick. The situation at three typical moments is shown below.

Using the fact that x is an invariant quantity (think about it!) show using only the constancy of the speed of light (i. e. you don't have to assume anything about the shrinking factor for the moving stick) that for an observer moving to the left with velocity v with respect to the stick, the velocity of the particle must be

$$w = \frac{u+v}{1+\frac{uv}{c^2}} \tag{1}$$



1) a - 11-4 a - time dilation



One clock tick takes a time t' in the clock's frame and a time t in the observer's frame. In the clock's frame, light goes a distance $2d$ with velocity c, so

$$t' = \frac{2d}{c}.$$

In the observer's frame, light goes a distance

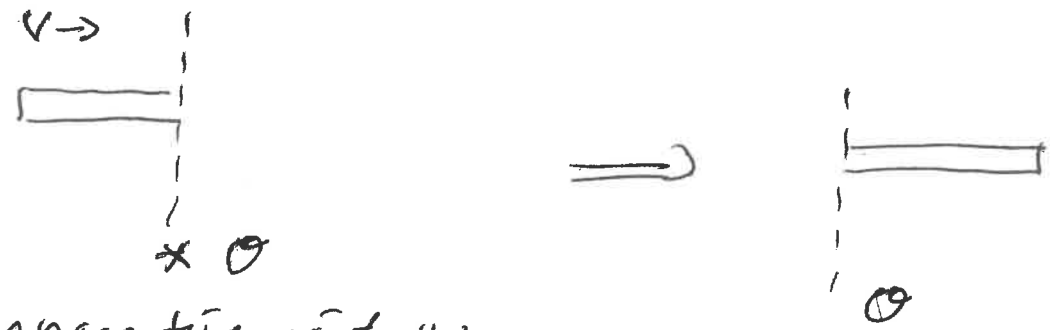
$$s = 2\left[d^2 + \left(\frac{vt}{2}\right)^2\right]^{1/2}$$

in a time $t = \frac{s}{c}$, so

$$\left(\frac{ct}{2}\right)^2 = d^2 + \left(\frac{vt}{2}\right)^2$$

$$\left(\frac{t}{2}\right)^2 = \frac{d^2}{c^2\left[1 - \frac{v^2}{c^2}\right]}$$

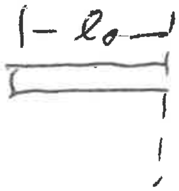
Therefore $t = \frac{2d}{c} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma t'.$



space time point #1 - unprimed observer at $(t_1=0,\ x_1=0)$

observer at time t_2, $x_2=0$ infers $l = vt_2$ so space-time point #2 is $\left(t_2 = \frac{l}{v},\ x_2=0\right)$

The primed observer riding on the rod knows its length is l_0. The two space-time points are



$(t_1'=0,\ x_1'=0)$ and

$t_2' = \frac{l_0}{v}$

$x_2' = l_0$

$(0,0)$ and $(t_2',\ x_2') = \left(\frac{l_0}{v},\ l_0\right)$

Lorentz transform to check $x_2 = \gamma\left[x_2' - vt_2'\right]$

$$= \gamma\left[l_0 - v\frac{l_0}{v}\right] = 0$$

This just checks our sign conventions are consistent

Then Lorentz transform the time -

$$t_2 = \gamma\left[t_2' - \frac{v}{c}x_2'\right] = \gamma\left[\frac{l_0}{v} - \frac{v\,l_0}{c^2}\right]$$

$$= \frac{\gamma}{v}\left[1 - \frac{v^2}{c^2}\right]l_0 = \frac{1}{v}\frac{l_0}{\gamma}$$

And $t_2 = \frac{l}{v}$ hence $l = \frac{l_0}{\gamma}$.

2) A stick drops ink drops from both ends, simultaneously (in its own frame). It moves parallel to its length, perpendicular to lines drawn on the floor. The stick's length is l (in its frame), the spacing of the lines is l in the floor's frame. How far apart are the dots on the floor?

In the stick's (primed) frame $\Delta x' = l$, $\Delta t' = 0$. The observer in the floor's frame sees the dots placed $\Delta x = \gamma[\Delta x' + v\Delta t'] = \gamma l$ apart:

$$\frac{\text{dot spacing}}{\text{line spacing}} = \frac{\gamma l}{l} = \gamma.$$

The dots are not emitted simultaneously,

$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) = \gamma v \frac{l}{c^2}.$$

If you want to think about length contraction, the dots are spaced $\Delta x = \frac{l}{\gamma} + v\Delta t$ apart,

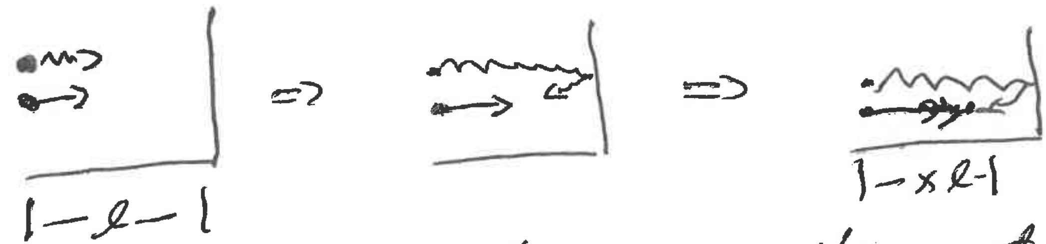
$$\Delta x = \frac{l}{\gamma} + \gamma\frac{v^2 l}{c^2} = \gamma l\left[\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right] = \gamma l \text{ apart}$$

(again).

The observer on the rod says the spacing of the dots is l but the spacing of the lines on the floor is length contracted to l/γ. Either way, both observers agree.

$$\frac{\text{dot spacing}}{\text{line spacing}} = \frac{\gamma l}{l} = \frac{l}{[l/\gamma]} = \gamma.$$

3) Velocity addition with stick and mirror



they meet at time T in the rod's rest frame at location $x\ell$.

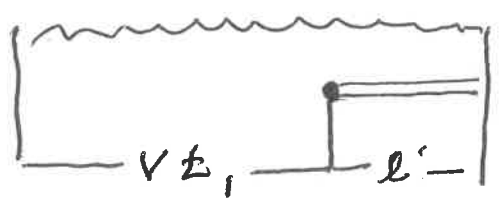
In the rest frame of the rod, the particle has velocity u and goes a distance $x\ell = uT$.
In the same time, the photon, with velocity c, goes a distance $\ell + (1-x)\ell = cT$

so $\quad T = \dfrac{x\ell}{u} = (2-x)\dfrac{\ell}{c}$

or $\quad 2-x = x\dfrac{c}{u}, \quad 2 = x\left(\dfrac{c+u}{u}\right)$ or $\boxed{x = \dfrac{2u}{c+u}}$

Now go to a frame where the rod moves at velocity v and the particle has velocity w. Analyze the three parts of the motion separately.

a) In time t_1, the mirror moves to the right, and the photon hits the mirror

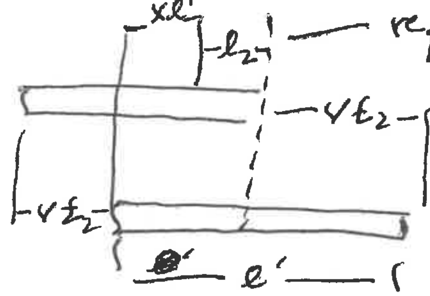


$ct_1 = vt_1 + \ell'$

so $\quad t_1 = \dfrac{\ell'}{c-v}$

(ℓ' = length of rod in this frame)

b) In a time t_2 the stick moves right a distance vt_2. The photon moves left a distance $l_2 = ct_2$. It ends up a distance xl' from the left end of the stick



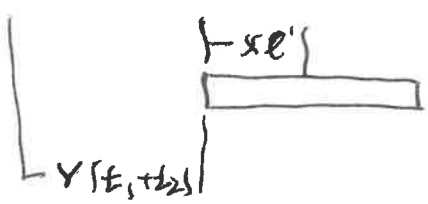
$$l_2 = ct_2$$

$$l' = vt_2 + l_2 + xl'$$

or $l_2 = l'(1-x) - vt_2 = ct_2$

$$t_2 = \frac{l'(1-x)}{c+v}$$

c) The total elapsed time is $t_1 + t_2$. At the end of that interval, the particle is found at a location xl' from the left end of the rod. This is a distance $xl' + v(t_1+t_2)$ from the starting point. The particle has velocity w, so



$$w(t_1+t_2) = xl' + v(t_1+t_2).$$

Now it's just algebra -

the particle | the photon

$$t_1 + t_2 = \frac{x\ell'}{w-v} = \frac{\ell'}{c-v} + \frac{\ell'(1-x)}{c+v}$$

$$\frac{x}{w-v} = \frac{1}{c-v} + \frac{1-x}{c+v} = \frac{c(2-x)+vx}{c^2-v^2}$$

$$w - v = \frac{x(c^2-v^2)}{(2-x)c+xv}$$

$$w = v + \frac{(c^2-v^2)x}{(2-x)c+xv}$$

$$w = \frac{(2-x)cv + \cancel{xv^2} + c^2x - \cancel{v^2x}}{(2-x)c+xv} = \frac{(2-x)cv + c^2x}{(2-x)c+xv}$$

Now recall $x = \frac{2u}{c+u}$, $2 - x = \frac{2c}{c+u}$.

Plug in - the $\frac{1}{c+u}$ is common to all terms, drops out

$$w = \frac{2c^2v + 2c^2u}{2c^2 + 2uv} = \frac{v+u}{1 + \frac{uv}{c^2}} \quad (\mathrm{V})$$

This is the well known formula for velocity addition - but in this case we only require that x and c are frame - independent.