Set 4 – due 16 February

"Diffraction problems are amongst the most difficult ones encountered in optics." (From "Principles of Optics," Born and Wolf)

1) [20 points] Jackson 10.9a. Use the Born approximation. The integral peaks strongly in the forward direction, so you can replace $qa \simeq ka\theta$, $d\cos\theta = \theta d\theta$, and take the range of θ from 0 to infinity. You'll get an integral

$$\sigma \simeq 2\pi|\epsilon - 1|^2 k^2 a^4 \int_0^\infty x dx \frac{j_1(x)^2}{x^2}. \tag{1}$$

At that point Bessel function identities near Jackson 9.90 might be useful.

Notice how the Rayleigh k^4 is softened by the extended source to $\sigma \sim k^2$.

2) [20 points] Jackson 10.11. (a,b only). Hint: expand

$$R = [(x-x')^2 + y'^2 + z^2]^{1/2} \sim z[1 + \frac{(x-x')^2 + y'^2}{2z^2} + \ldots] \tag{2}$$

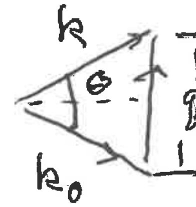
For Fresnel integrals, see Wikipedia "Fresnel Integral," Abramowitz and Stegun, p. 300, or Morse and Feshbach, p. 816. There does not seem to be a standardized notation for these functions.

3) [20 points] Jackson 10.12. In (a), work in the Fraunhofer limit. It is quite similar to the calculation done on pp. 491-492, (it is basically "solve by copy") except that the initial polarization is $\vec{\epsilon}_0 = \hat{y}$. For part (b), use the Dirichlet formula, 10.85. (drop the $+i/(kR)$)

10.9 – Scattering from a uniform dielectric sphere of radius a, permittivity $\epsilon = 1 + \delta\epsilon$. Begin with Jackson eq 10.31

$$\frac{d\sigma}{d\Omega}(\hat{\epsilon}, \hat{\epsilon}_0) = \frac{k^4}{16\pi^2} |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 \left| \int d^3x \, e^{i\vec{q}\cdot\vec{x}} \, \delta\epsilon(x) \right|^2$$

and $\vec{q} = \vec{k} - \vec{k}_0$ so $|\vec{q}| = 2k \sin\frac{\theta}{2}$



To do the integral put the z axis along q so $\vec{q}\cdot\vec{x} = qr\cos\theta$, then

$$F(q) = \int_{r<a} d^3x \, e^{iqx} = 2\pi \int_0^a r^2 dr \int_{-1}^{1} d\mu \, e^{iqr\mu}$$

$$= 2\pi \int_0^a r^2 dr \left[\frac{e^{iqr} - e^{-iqr}}{iqr} \right]$$

$$= \frac{4\pi}{q^3} \int_0^{qa} qr \sin qr \, d\,qr = \frac{4\pi}{q^3} \int_0^{qa} x \sin x \, dx$$

$$= \frac{4\pi}{(qa)^3} \cdot a^3 \left[\sin qa - qa \cos qa \right] \quad \text{from tables}$$

$$= 4\pi a^3 \frac{j_1(qa)}{qa} \quad \text{in terms of a spherical Bessel fn,}$$

$$\therefore \frac{d\sigma}{d\Omega} = k^4 a^6 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 (\delta\epsilon)^2 \left[\frac{j_1(qa)}{qa} \right]^2$$

Note as $qa \to 0$, $j_1(qa) \sim \frac{1}{3} qa$, $\frac{d\sigma}{d\Omega} = k^4 a^6 \left| \frac{\epsilon - 1}{3} \right|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2$

10.9.2

The factor $\frac{\epsilon-1}{3}$ is the $\epsilon\to 1$ limit of $\frac{\epsilon-1}{\epsilon+2}$, the term in the dipole moment of the sphere, found in eq. 10.6. In MKS it is $\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} \simeq \frac{\epsilon/\epsilon_0 - 1}{3}$.

Because $qa = 2ka\sin\frac{\theta}{2}$, large ka implies large qa

Now $\frac{j_1(x)}{x} = \frac{\sin x}{x^3} - \frac{\cos x}{x^2}$. For small angles,

$x = qa \simeq ka\theta$ ($\sin\theta \sim \theta$). A picture

$$\sigma = 2\pi\int_{-1}^{1} d\cos\theta \, \frac{d\sigma}{d\Omega} = 2\pi\,|\epsilon-1|^2 k^4 a^6 \times \int d\cos\theta \, |\hat{\epsilon}^*\cdot\hat{\epsilon}_0|^2 \, \frac{j_1(qa)^2}{(qa)^2}$$

Averaging initial polarizations, summing final ones,

$$\frac{1}{2}\sum_{pol} |\hat{\epsilon}^*\cdot\epsilon_0|^2 = \frac{1+\cos^2\theta}{2} \simeq 1 \text{ at small } \theta.$$

The integral is strongly peaked at $\theta\sim 0$. Calling $x = qc = ka\theta$, $d\cos\theta = \theta\,d\theta = \frac{x\,dx}{(ka)^2}$, $\int_0^\pi d\theta \simeq \int_0^\infty d\theta$ to get a nicer integral (very small error doing this! if ka is large)

$$\sigma = 2\pi |\epsilon - 1|^2 \frac{k^4 a^6}{(ka)^2} \int_0^\infty x\,dx \frac{j_1(x)^2}{x^2}$$

$$= 2\pi |\epsilon - 1|^2 k^2 a^4 I \quad \text{where } I = \int_0^\infty dx \frac{j_1(x)^2}{x}$$

There are spherical Bessel function identities

$$\frac{2 j_1(x)}{x} = j_0(x) - j_1'(x)$$

$$j_1(x) = -j_0'(x)$$

so
$$I = \int_0^\infty dx\, j_1(x) \left[\frac{j_0(x) - j_1'(x)}{2} \right]$$

$$= -\frac{1}{2} \int_0^\infty dx \left[j_0'(x) j_0(x) + j_1'(x) j_1(x) \right]$$

$$= -\frac{1}{4} \left[j_0(x)^2 + j_1(x)^2 \right] \Big|_0^\infty$$

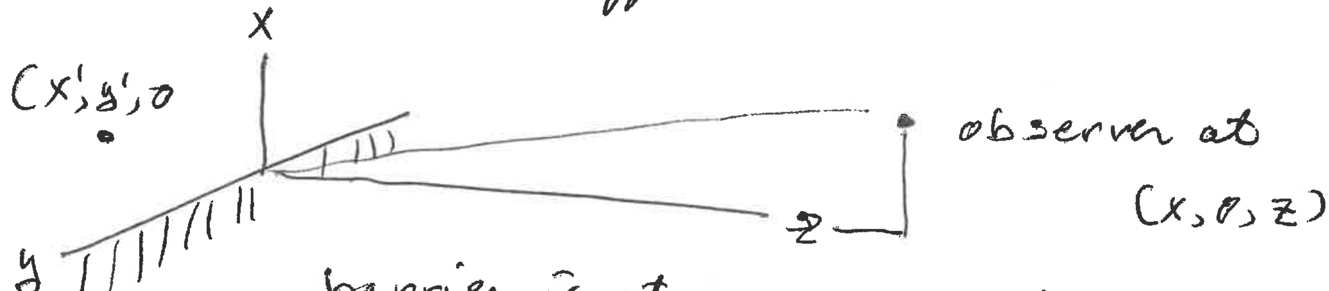
$$= \frac{1}{4} [1 + 0] = \frac{1}{4}$$

and so
$$\sigma = \frac{\pi}{2} |\epsilon - 1|^2 k^2 a^4 \qquad \text{CGS}$$

$$\frac{\pi}{2} \left| \frac{\epsilon}{\epsilon_0} - 1 \right|^2 k^2 a^4 \qquad \text{MKS}$$

Note how the form factor suppresses the Rayleigh k^4 factor when ka gets large.

10.11 – Fresnel diffraction



barrier is at $z=0,\ x<0$, for all y

The incident wave is $\psi_0(x,t) = I_0^{1/2}\exp[i(kz-\omega t)]$

The observer is at $(x,0,z)$ with $z \gg x$, $kz \gg 1$.

as The Kirchhoff integral is

$$\psi(x) = \frac{k}{2\pi i}\int_0^\infty dx'\int_{-\infty}^\infty dy'\,\frac{e^{ikR}}{R}\,\frac{\hat{n}'\cdot\vec{R}}{R}\,\psi_0(x',y')$$

$$\psi_0(x',y') = I_0^{1/2}\exp(ikz') = I_0^{1/2} \text{ at } z'=0$$

$$\hat{n}' = \hat{z},\quad R = \vec{x}-\vec{x}',\quad R = [(x-x')^2 + y'^2 + z^2]^{1/2}$$

$$\simeq z\left[1 + \frac{(x-x')^2 + y'^2}{2z^2} + \cdots\right]$$

and $\dfrac{\hat{n}\cdot\vec{R}}{R} = \dfrac{\hat{z}\cdot(\vec{x}-\vec{x}')}{R} = \dfrac{z}{R} \sim 1$. Note $\vec{x}'$ is at $z'=0$.

Assembling all the parts gives

$$\psi(x,0,z) = \frac{kI_0^{1/2}}{2\pi i}\int_0^\infty dx'\int_{-\infty}^\infty dy'\,\frac{e^{ikz}}{z}\,e^{\frac{ik(x-x')^2}{2z}}\,e^{\frac{iky'^2}{2z}}$$

$$= \frac{kI_0^{1/2}e^{ikz}}{2\pi i z}\int_0^\infty dx'\exp\frac{ik(x-x')^2}{2z}\int_{-\infty}^\infty dy'\exp\frac{iky'^2}{2z}$$

The y' integral is a Gaussian, giving

$$\sqrt{\frac{2\pi z}{-ik}} = (1+i)\sqrt{\frac{\pi z}{k}}.$$

The x-integral gives the Fresnel formula. 10-11.2

To get Jackson's answer, begin with

$$J = \int_0^\infty dx' \exp ik(x-x')^2/2z.$$

set $t = \sqrt{\frac{k}{2z}}(x'-x)$ so $dx' = \sqrt{\frac{2k}{z}}\,dt$. Notice that $x'=0$ at $t = -x\sqrt{\frac{k}{2z}} \equiv -\xi$ and

$$\psi(x,0,z) = I_0^{1/2} e^{ikz}\left[\frac{1+i}{2i}\right]\sqrt{\frac{2}{\pi}}\int_{-\xi}^{\infty} dt\, e^{it^2}.$$

b) $I = |\psi|^2 = I_0 \cdot \frac{2}{4}\cdot\frac{2}{\pi}\left|\int_{-\xi}^{\infty} dt\, e^{it^2}\right|^2$

$= \frac{I_0}{\pi}\left|\int_{-\xi}^{\infty} \cdots\right|^2$. The integral is the famous Fresnel integral, with various conventions. Abramowitz + Stegun p. 300 define

$C_1(u) = \sqrt{\frac{2}{\pi}}\int_0^u dt \cos t^2$, $S_1(u) = \sqrt{\frac{2}{\pi}}\int_0^u dt \sin t^2$

Wikipedia defines $C(u) = \sqrt{\frac{\pi}{2}}\,C_1(u)$, $S(u) = \sqrt{\frac{\pi}{2}}\,S_1(u)$.

Our integral is $\tilde{I} = \int_{-\xi}^{\infty} dt\, e^{it^2} = \int_{-\xi}^{0} e^{it^2} dt + \int_0^\infty e^{it^2} dt$

$\tilde{I} = \int_0^{\xi} dt\,[\cos t^2 + i \sin t^2] + \frac{\sqrt{i\pi}}{2}$

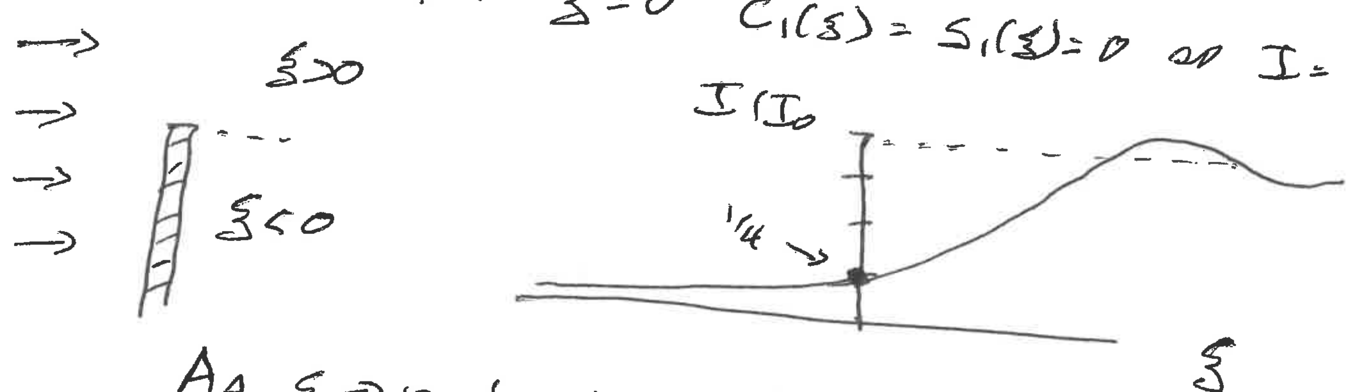
(another Gaussian!)

so -

$$\tilde{J} = \sqrt{\frac{\pi}{2}}\left[C_1(\xi) + i S_1(\xi) + \frac{1+i}{2}\right]$$ and

the intensity is $$I = \frac{I_0}{2}\left[\left(C_1(\xi)+\frac{1}{2}\right)^2 + \left(S_1(\xi)+\frac{1}{2}\right)^2\right]$$

$\xi = \sqrt{\frac{k}{2z}}\,x$ so $\xi > 0$ is the "bright region" and $\xi < 0$ is the "dark region" below the screen. At $\xi = 0$ $C_1(\xi) = S_1(\xi) = 0$ so $I = \frac{1}{4} I_0$



As $\xi \to \infty$ both C_1 and S_1 approach $\frac{1}{2}$ and $I/I_0 = 1$. We can write ($\xi > 0$, Wikipedia)

$$C_1(\xi) = \sqrt{\frac{2}{\pi}}\left[\sqrt{\frac{\pi}{8}} - \frac{\sin \xi^2}{2\xi} - \frac{\cos \xi^2}{4\xi^3}\right] \sim \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\frac{\sin \xi^2}{\xi}$$

$$S_1(\xi) = \sqrt{\frac{2}{\pi}}\left[\frac{\pi}{8} - \frac{\cos \xi^2}{2\xi} - \frac{\sin \xi^2}{4\xi^3}\right] = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\frac{\cos \xi^2}{\xi} \ldots$$

Also since $C_1(-\xi) = -C_1(\xi)$, $S_1(-\xi) = -S_1(\xi)$

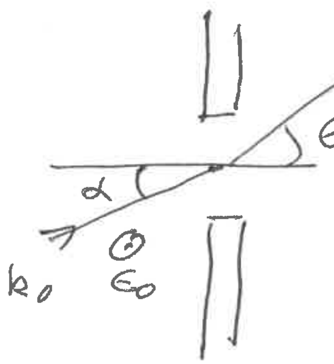
$$I(z) \xrightarrow[z \to -\infty]{} \frac{I_0}{2}\left[2 \times \left(-\frac{1}{2} + O\left(\frac{1}{\xi}\right) + \frac{1}{2}\right)^2\right] \sim \frac{I_0}{\xi^2}$$

so the dark region is really dark. Also

$$I(z) \xrightarrow[z \to \infty]{} I_0\left[1 + \frac{1}{\sqrt{2\cdot\pi}}\,\frac{\sin \xi^2 - \cos \xi^2}{\xi} + \ldots\right]$$

expressing the ripples in Fresnel diffraction.

10.12) Diffraction through a circular aperture – the side view labels angles. We begin with 10.141, the Smythe–Kirchhoff formula



$$\vec{E}(x) = \frac{1}{2\pi} \vec{\nabla} \times \int_{\text{aperture}} dA' \left[\hat{n}' \times \vec{E}\right] \frac{e^{ikR}}{R}$$

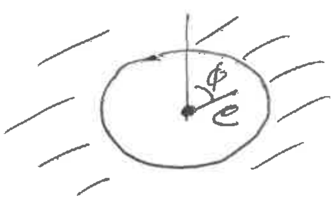
where $\vec{E}_0 = \hat{\epsilon}_0 E_0 e^{i k_0 x}$, $\hat{\epsilon}_0 = \hat{y}$, $\hat{n}' = \hat{z}$,

$|k_0| = k$, $kR = kr - k\hat{n}\cdot\vec{x}'$ in the Fraunhofer limit,

and $\vec{k}_{sc} = \hat{n} k$ — so –

$$\vec{E}(x) = ikE_0 \frac{e^{ikr}}{r} \left[\hat{n} \times (\hat{z} \times \hat{\epsilon}_0)\right] \int dA' e^{i(\vec{k}-\vec{k}_{sc})\cdot x'} .$$

Still more labels – $\int dA = \int_0^a \varrho\, d\varrho \int_0^{2\pi} d\varphi$



and $\vec{x}' = \varrho\left[\hat{x}\cos\phi' + \hat{y}\sin\phi'\right]$ with polar coordinates for the aperture,

$\hat{n} = (\hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta)$

$\vec{k}_0 = k(\hat{x}\sin\alpha + \hat{z}\cos\alpha)$ (the beam is in the x-z plane)

so $\vec{k}_{sc}\cdot\vec{x}' = k\hat{n}\cdot x' = k\varrho\sin\theta\cos(\phi-\phi')$

$\vec{k}_0\cdot\vec{x}' = k\varrho\sin\alpha\cos\phi.$

put it all together

$$\vec{E}(x) = ikE_0 \frac{e^{ikr}}{2\pi r} \left[\hat{n} \times (\hat{z} \times \hat{\epsilon}_0)\right] \tilde{I}$$

where

$$\tilde{I} = \int_0^a \rho\, d\rho \int_0^{2\pi} d\varphi' \exp\left[ik\rho\left(\sin\alpha\cos\varphi' - \sin\theta\cos(\varphi-\varphi')\right)\right]$$

This is pretty much eq. 10.112. The key to doing the integral is to write the term in parentheses as

$$\sin\alpha\cos\varphi' - \sin\theta\cos\varphi\cos\varphi' + \sin\theta\sin\varphi\sin\varphi'$$
$$= A\cos\varphi' + B\sin\varphi'$$
$$= C\cos(\varphi' - D) \quad \text{(trig identity)}$$

and to use

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi' \exp\left[-iC\cos(\varphi'-D)\right]$$
$$= \frac{1}{2\pi}\int_0^{2\pi} d\varphi'' \exp\left[-iC\cos\varphi''\right] = J_0(C)$$

where $J_0(C)$ is the order-zero cylindrical Bessel function. Yes, it's improbable, but this is why you own a copy of Jackson!

We need C: $A = C\cos D = k\rho(\sin\alpha - \sin\theta\cos\varphi)$
$B = C\sin D = k\rho\sin\theta\sin\varphi$

$C^2 = A^2 + B^2$ so

$$C = k\rho\left[\sin^2\alpha - 2\sin\alpha\sin\theta\cos\varphi + \sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi\right]^{1/2}$$
$$= k\rho\left[\sin^2\alpha + \sin^2\theta - 2\sin\alpha\sin\theta\cos\varphi\right]^{1/2}$$
$$\equiv k\rho\xi.$$

This is Eq. 10.112 so we copy Eq. 10.113

$$\vec{E}(x) = ikE_0\frac{e^{ikr}}{r}\left\{\hat{n}\times(\hat{z}\times\hat{\epsilon}_0)\right\} a^2 \frac{J_1(ka\xi)}{ka\xi}$$

and $\vec{B} = \hat{n}\times\vec{E}$, $\dfrac{dP}{d\Omega} = \dfrac{c}{8\pi} r^2\, \hat{n}\cdot(\vec{E}\times\vec{B}^*)$ of course.

$$\frac{dP}{d\Omega} = \frac{c k^2 a^4}{8\pi} E_0^2 \left[\frac{J_1(ka\xi)}{ka\xi} \right]^2 \mathcal{P}.$$

$\mathcal{P}$ is the polarization piece, $\mathcal{P} = |\hat{z}\times\hat{\epsilon}_0|^2 - |\hat{n}\cdot(\hat{z}\times\hat{\epsilon}_0)|^2$

$\hat{\epsilon}_0 = \hat{y}$ so $\hat{z}\times\hat{\epsilon}_0 = \hat{x}$, $\hat{x}\cdot\hat{x} = 1$, $\hat{n}\cdot\hat{x} = \sin\theta\cos\varphi$,

$$\mathcal{P} = 1 - \sin^2\theta\cos^2\varphi = \cos^2\theta + \sin^2\theta - \sin^2\theta\cos^2\varphi$$

$$= \cos^2\theta + \sin^2\theta\sin^2\varphi.$$

In 10.114–10.115 the analog of $\mathcal{P}$ is

$$\cos^2\alpha\,[\cos^2\theta + \sin^2\theta\cos^2\varphi]$$

but in this case the polarization is in-plane, $\hat{\epsilon}_0 = \hat{x}$.
In our case $\hat{\epsilon}_0 = \hat{y}$. There is a sanity check –
if $\alpha = 0$ a rotation of φ by $\pi/2$ converts one answer to the other.

This was a lot of calculus!

The scalar version is 10.85:

$$\psi(x) = \frac{k}{2\pi i}\int dA' \frac{e^{ikR}}{R}\frac{\hat{n}'\cdot\vec{R}}{R}\psi(x')$$

If $kR \gg 1$, $\vec{R}/R = \hat{n}$, $\hat{n}\cdot\hat{n}' = \hat{n}\cdot\hat{z} = \cos\theta$, and if we write $\psi_0 = E_0 \exp[ik\cdot x']$ we would have

$$\psi(x) = \frac{kE_0}{2\pi i}\frac{e^{ikr}}{r}[\cos\theta]\int dA'\, e^{i(\vec{k}-\vec{k}_{sc})\cdot x'}$$

All that ~~happended~~ happened was that $\cos\theta$ replaced $\hat{n}\times(\hat{z}\times\hat{\epsilon}_0)$.

In the square, $\cos^2\theta$ replaces $\cos^2\theta + \sin^2\theta \sin^2\varphi$. Had we instead used 10.108, the $\cos\theta$ would be $\frac{1}{2}[\cos\theta + \cos\alpha]$.

But the only place where the signal is large is at small values of θ and α, and there, all these answers are basically identical — in fact, they are all nearly equal to $\underline{1}$. We are killing ourselves over unnecessary accuracy.