Set 3 – due 9 February

"If you can't be a giant, you want to be the giant killer"–M. Perl

1) [15 points] Jackson 10.4, parts a (5 points) and b (10 points) only. $\delta \gg R$ means that the size of the sphere is much less than the attenuation length, so the scatterer can just be treated as a dielectric sphere. In part b, to find absorption, calculate the power dissipated due to conductivity σ.

2) [15 points] Jackson 10.5. Part a (10 points) only. It might be useful to show that the induced magnetic dipole moment is

$$\vec{m} = \frac{ik\sigma}{2c} \int \vec{x}(\vec{x} \cdot \vec{B}) d^3x. \tag{1}$$

In lieu of part (b), compute [5 points] the differential and total cross sections for scattering in the long wavelength limit including both the induced magnetic moment term from this problem and the electric dipole term from problem 1. Explicitly work through the average over initial polarizations ande sum over final ones. Notice that both the electric and magnetic dipole moments are complex.

The optical theorem is an exact relation when used with the exact scattering amplitude. It is very tricky to apply when you only have an approximate scattering amplitude.

3) [10 points] This is the simple mathematics behind the Weisskopf article. Suppose you model the atom as an electron on a spring of natural frequency ω_0. Include a damping force $F = -\Gamma d\vec{x}/dt$. Put the spring in an external electric field $\vec{E}\exp(-i\omega t)$, solve for the steady state motion, compute the resulting dipole moment, and drop the result into the appropriate formula for the differential cross section. Now do physics: note how you recover Rayleigh scattering for $\omega \ll \omega_0$. At high frequency, the cross section becomes independent of frequency. This is called the Thomson cross section. (It is the same formula for scattering off a free electron.) There is a dimensionful constant (in CGS)

$$r_0 = \frac{e^2}{m_e c^2} \tag{2}$$

where m_e is the electron mass, which characterizes the scale of elastic light scattering on electrons and hence on atoms. Find a number for this in cm. This says that the natural scale for light scattering on the electrons in atoms is $\sigma \sim r_0^2$ (or maybe more correctly, $r_0^2(\omega/\omega_0)^4$, which is even smaller). Notice how all the interesting polarization behavior is frequency-independent. This isn't generally true, but it is true in dipole approximation. There are other ways to derive this result which we'll encounter later on in the course.

10.4: Scattering from a "cloudy sphere" starts with a complex dielectric constant $\epsilon = \epsilon_r + i\epsilon_i$

$= \epsilon + \frac{4\pi\sigma}{\omega} i$ in CGS, $\lambda >> R$ means we treat the sphere as a solid dielectric. Then for incident $\vec{E}_0 = \hat{\epsilon}_0 E_0$,

$$\vec{p} = R^3 \frac{\epsilon - 1}{\epsilon + 2} \vec{E}_0$$

and the scattered E field is

$$\vec{E}_{sc} = \frac{e^{ikr}}{r} k^2 [(\hat{n} \times \hat{p}) \times \hat{n} - \hat{n} \times \hat{m}],$$

copying Eq 10.2 and temporarily including the magnetic moment. Continue copying eq 10.3,

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{r^2 \frac{c}{8\pi} |\hat{\epsilon}^* \cdot \vec{E}_{sc}|^2}{\frac{c}{8\pi} |\hat{\epsilon}_0^* \cdot \vec{E}_0|^2} = \frac{k^4}{E_0^2} |\hat{\epsilon}^* \cdot \vec{p} + (\hat{n} \times \hat{\epsilon}^*) \cdot \vec{m}|^2.$$

Setting $\vec{m} = 0$,

$$\frac{d\sigma_{sc}}{d\Omega} = k^4 R^6 \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2$$

which is eq 10.7 generalized to complex $\vec{p}$.
After averaging initial polarizations, summing final ones (for details, see page 10.5.3 below)

$$\sigma_{sc} = \frac{8\pi}{3} k^4 R^6 \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2$$

– note the absolute square!

b) The absorption cross section is

$$\sigma_{abs} = \frac{P_{abs}}{|\vec{S}_{incident}|} = \frac{P_{abs}}{\frac{c}{8\pi} |E_0|^2}$$

where $P_{abs} = \frac{1}{2}\int_{sphere} d^3x\, \vec{J}\cdot\vec{E}^*$

$$= \frac{\sigma}{2}\int d^3x\, |E|^2$$

using $\vec{J} = \sigma\vec{E}$ and including a $\frac{1}{2}$ for the time average. We need $\vec{E}$ inside the sphere and Eq 4.39 tells us that

$$\vec{E}_{in} = \frac{3}{2+\epsilon}\vec{E}_0 \quad \text{in CGS, in MKS, } \epsilon \to \frac{\epsilon}{\epsilon_0}.$$

Thus $P_{abs} = \frac{\sigma}{2}\cdot\frac{9}{|2+\epsilon|^2}\cdot E_0^2\cdot\frac{4\pi}{3}R^3$

$$\sigma_{abs} = \frac{1}{2}\cdot\frac{8\pi}{c}\cdot\frac{4\pi R^3}{3}\cdot\frac{9\sigma}{|2+\epsilon|^2}$$

$$= \frac{48\pi^2 R^3\sigma}{c\,|2+\epsilon|^2} = \frac{48\pi^2 R^2\left(\frac{R\sigma}{c}\right)}{\left[(2+\epsilon)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2\right]^2}$$

In the long wavelength limit, $kR \ll 1$,

$\sigma_{sc} \sim (kR)^4 \times \sigma_{abs}$ or $\sigma_{sc} \ll \sigma_{abs}$

By the way, if you are curious, here is part (c). 10.4.3

$$\vec{E}_{sc} = -k^2 \frac{e^{ikr}}{r} \hat{n} \times (\hat{n} \times \vec{p})$$

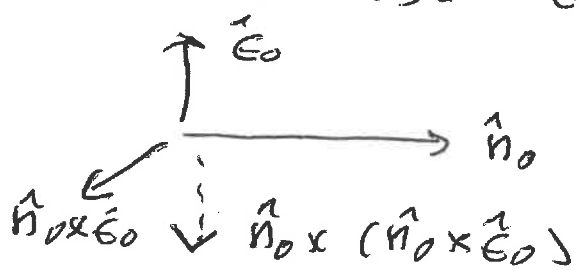
$$= -\left(\frac{\epsilon - 1}{\epsilon + 2}\right) k^2 R^3 \frac{e^{ikr}}{r} \hat{n} \times (\hat{n} \times \vec{E}_{in}),$$

and $\vec{E}_{in} = \hat{\epsilon}_0 E_0$. The scattering amplitude is

$$\vec{f}(\hat{n}) = \frac{\vec{E}_{sc}}{\left[E_0 \frac{e^{ikr}}{r}\right]} = -\left(\frac{\epsilon - 1}{\epsilon + 2}\right) k^2 R^3 \hat{n} \times (\hat{n} \times \hat{\epsilon}_0).$$

$$\mathrm{Im}\, f(\hat{n} = \hat{n}_0) = \mathrm{Im}\left[\frac{\epsilon - 1}{\epsilon + 2}\right] k^2 R^3 \hat{\epsilon}_0 \equiv \hat{\epsilon}_0 \,\mathrm{Im}\, f$$

because $\hat{n}_0 \times (\hat{n}_0 \times \hat{\epsilon}_0) = -\hat{\epsilon}_0$



The optical theorem says

$$\sigma_{TOT} = \frac{4\pi}{k} \mathrm{Im}\, f(\hat{n} = \hat{n}_0) = 4\pi k R^3 \,\mathrm{Im} \frac{\epsilon - 1}{\epsilon + 2}$$

If $\epsilon = \epsilon_r + i\epsilon_i$, $\frac{\epsilon - 1}{\epsilon + 2} = \frac{\epsilon_r - 1 + i\epsilon_i}{\epsilon_r + 2 + i\epsilon_i} \times \frac{\epsilon_r + 2 - i\epsilon_i}{\epsilon_r + 2 - i\epsilon_i}$

$$= \frac{\cdots + i\left[(\epsilon_r + 2)\epsilon_i - \epsilon_i(\epsilon_r - 1)\right]}{(\epsilon_r + 2)^2 + \epsilon_i^2}$$

so

$$\sigma_{TOT} = 4\pi R^2 \cdot kR \cdot \frac{3\epsilon_i}{[\epsilon_r+2]^2+\epsilon_i^2}$$

Recalling that $\epsilon_r = \epsilon$, $\epsilon_i = \frac{4\pi\sigma}{\omega}$, $\omega = ck$, this is

$$\sigma_{TOT} = \frac{48\pi^2 R^2 \left(\frac{\sigma R}{c}\right)}{(\epsilon+2)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}$$

which is equal to the σ_{abs} of part (b). This seems particular - the total cross section is equal to the absorption cross section? Where is the real scattering? The answer is that our scattering is not exact, it is only an approximation - the dipole approximation.

So the cross section from the optical theorem is only approximate. Still,

$$\frac{\sigma_{sc}}{\sigma_{abs}} \sim (kR)^4 \ll 1$$ in the long wavelength limit.

10.5 - Magnetic scattering - $\vec{m}$ comes from $\vec{J}$ which comes from $\sigma\vec{E}$. 2 ways to get it.

i) $\vec{m}$ from the hint: $\vec{m} = \frac{1}{2c}\int d^3x \left[\vec{r} \times \vec{J}(r)\right]$

or, by components, $m_i = \frac{1}{2c}\int d^3x \, (\nabla x_i)\cdot(\vec{r}\times\vec{J})$

We introduce the tricky gradient to integrate by parts

$$m_i = -\frac{1}{2c}\int x_i \, \vec{\nabla}\cdot(\vec{r}\times\vec{J})\, d^3x$$

and then use $\vec{J} = \sigma\vec{E}$ so

$$m_i = -\frac{\sigma}{2c}\int x_i \, \vec{\nabla}\cdot(\vec{r}\times\vec{E})\, d^3x$$

$$= \frac{-\sigma}{2c}\int x_i \left[\vec{r}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{r})\right] d^3x$$

$\vec{\nabla}\times\vec{r} = 0$ and $\vec{\nabla}\times\vec{E} = ik\vec{B}$, so the induced dipole moment is given by the hint:

$$\vec{m} = \frac{ik\sigma}{2c}\int \vec{x}\left[\vec{x}\cdot\vec{B}\right] d^3x$$

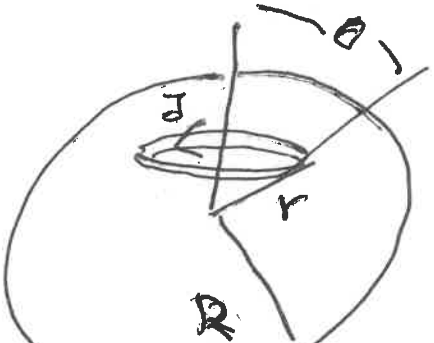
Pick axes so $\vec{B} = \hat{z}B_{in}$, $\vec{m}$ points along B,

$$\vec{m} = \hat{z}B_{in}\left[\frac{ik\sigma}{2c}\right]\int_0^R 2\pi r^2 dr \int_{-1}^{1} d\cos\theta \; r^2\cos^2\theta$$

(replacing $\hat{x}$ by $z = r\cos\theta$ to project along B)

$$\vec{m} = \hat{z}B_{in}\left[\frac{ik\sigma}{2c}\right]2\pi\frac{R^5}{5}\cdot\frac{2}{3} = i\,4\pi\sigma\frac{kR^2}{c}\frac{R^3}{30}\vec{B}_{in}$$

For an alternate solution, integrate $\mathcal{E} = -\frac{1}{c}\dot{\Phi}$ or



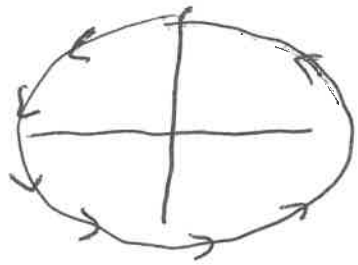
$$\int \vec{E}\cdot d\vec{l} = \frac{1}{c}\int \dot{\vec{B}}\cdot\hat{n}\, dA$$

$$2\pi(r\sin\theta)E = -\frac{\pi[r\sin\theta]^2}{c}\frac{dB}{dt}$$

$$\frac{d\vec{B}}{dt} = -i\omega\vec{B} = -i\omega\,\hat{z}\,B_{in}, \quad \vec{J} = \sigma\vec{E}$$

so $\vec{J} = \frac{\sigma}{2c} r\sin\theta\,[i\omega B_{in}]\,\hat{\varphi} = J_\varphi\,\hat{\varphi}$,

$\vec{m} = \frac{1}{2c}\int \vec{r}\times\vec{J}\, d^3x$. In rectangular coords,



top view

$$\vec{m} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ -J_\varphi\sin\varphi & J_\varphi\cos\varphi & 0 \end{vmatrix}$$

$$\vec{m} = \hat{k}J_\varphi[x\cos\varphi + y\sin\varphi] = \hat{k}J_\varphi r\sin\theta$$

$(x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ \cos^2\varphi + \sin^2\varphi = 1)$

$$\vec{m} = \hat{k}\left[\frac{i\sigma\omega}{4c^2}B_{in}\right]\int_0^R r^2 dr \cdot 2\pi\int_0^\pi \sin\theta\, d\theta\cdot r^2\sin^2\theta$$

$$= \hat{k}\left[\frac{i\pi\sigma}{2c^2}\omega B_{in}\right]\int_0^R r^4 dr\int_{-1}^{1} \underbrace{d\cos\theta\,(1-\cos^2\theta)}_{2-\frac{2}{3}}$$

$$= \hat{k}\left[\frac{i\omega}{c^2}\cdot\frac{\pi\sigma}{2}B_{in}\cdot\frac{R^5}{5}\cdot\frac{4}{3}\right]$$

$= \hat{k}B_{in}\left[\frac{ik\sigma}{2c}\right]2\pi\frac{R^5}{5}\cdot\frac{4}{3}$ as on the bottom of page 10.5.1

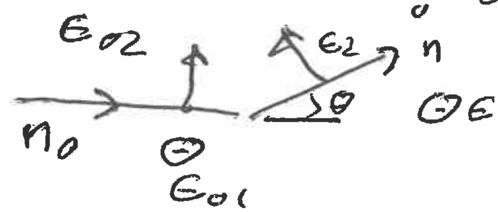
b) Write $\vec{p} = p_0 \vec{E}_0$ and $\vec{m} = m_0 \vec{B}_0$ where p_0 and m_0 are both complex, $p_0 = R^3\left[\frac{\epsilon-1}{\epsilon+2}\right]$ and we just computed m_0 - note it is pure imaginary. Then with $\vec{E}_0 = \hat{\epsilon}_0 E_0$ and $\vec{B}_0 = (\hat{n}\times\hat{\epsilon}_0)E_0$, the differential cross section is the CGS version of Eq 10.14,

$$\frac{d\sigma}{d\Omega}(\hat{n},\hat{\epsilon};\hat{n}_0,\hat{\epsilon}_0) = k^4 \left| \hat{\epsilon}^* \cdot \hat{\epsilon}_0 p_0 + (\hat{n}\times\hat{\epsilon}^*)\cdot(\hat{n}_0\times\hat{\epsilon}_0) \times m_0 \right|^2$$

We are asked to find the differential cross section averaged over initial polarizations, summed over final ones, so consider

$$R = \frac{1}{2}\sum_{\epsilon,\epsilon_0} \left| \hat{\epsilon}^* \cdot \hat{\epsilon}\, p_0 + (n\times\epsilon^*)\cdot(n_0\times\epsilon_0)\, m_0 \right|^2$$

Draw a picture



$\epsilon^* \cdot \epsilon_0$

$\epsilon_0 \backslash \epsilon$	1	2
1	1	0
2	0	$\cos\theta$

$(n\times\epsilon^*)\cdot(n_0\times\epsilon_0)$

$\epsilon_0 \backslash \epsilon$	1	2
1	$\cos\theta$	0
2	0	1

$$R = \frac{1}{2}\left\{ |p_0 + \cos\theta\, m_0|^2 + |\cos\theta\, p_0 + m_0|^2 \right\}$$

$$= \frac{1}{2}\left\{ (|p_0|^2 + |m_0|^2)(1+\cos^2\theta) + 2\,\mathrm{Re}\, p_0^* m_0 \cdot 2\cos\theta \right\}$$

It's a bit messy! Call

$$\mathbb{R} = C_1 [1 + \cos^2\theta] + C_2 \cos\theta$$

$$\frac{d\sigma}{d\Omega} = k^4 \left[C_1 (1 + \cos^2\theta) + C_2 \cos\theta \right]$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi \int_{-1}^{1} d\cos\theta \, \frac{d\sigma}{d\Omega}$$

$$= 2\pi k^4 \left\{ C_1 \cdot \left(2 + \frac{2}{3}\right) + C_2 \cdot 0 \right\}$$

$$\sigma = \frac{8\pi k^4}{3} \left[|p_0|^2 + |m_0|^2 \right]$$

The interference term integrates away

Notice that since $m_0 = i M$, & p_0

$p_0 = p_{0r} + p_{0i}$,

$$C_2 = 2 \,\mathrm{Re}\, p_0^* m_0 = 2 M p_{0i}$$

3) This is simple: $\dfrac{d\sigma}{d\Omega} = \dfrac{k^4}{E_0^2} |\hat{\epsilon}^* \cdot \vec{p}|^2.$

Use $\vec{p} = e\vec{x}$ and

$$m\ddot{\vec{x}} + \Gamma\dot{\vec{x}} + m\omega_0^2\vec{x} = e\hat{\epsilon}_0 E_0 e^{-i\omega t}$$

For a steady state solution $\vec{x}(t) = \hat{\epsilon}_0\, x\, e^{-i\omega t}$

and $\left(\omega_0^2 - \omega^2 - i\omega\dfrac{\Gamma}{m}\right)x = \dfrac{e}{m}E_0$

so $(k = \omega/c)$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 \frac{\omega^4}{\left|(\omega_0^2 - \omega^2) - i\omega\frac{\Gamma}{m}\right|^2}$$

The overall scale is set by

$$r_0 = \frac{e^2}{mc^2} = \frac{e^2}{\hbar c}\frac{\hbar c}{mc^2} = \frac{1}{137}\frac{197\ \text{MeV-fm}}{\frac{1}{2}\ \text{MeV}}$$

$= $ a few fm or a few $\times 10^{-13}$ cm, 10^{-15} m.

By itself, this gives $\sigma \sim 10^{-26}\ \text{cm}^2$. This is much smaller than the geometric size of an atom, $1\ \text{Å}^2 \sim 10^{-16}\ \text{cm}^2$.

$$\bullet\ |f(\omega)|^2 = \frac{\omega^4}{[\omega^2\ \omega_0^2]^2 + \frac{\omega^2\Gamma^2}{m^2}}$$

$\boxed{\text{notice } f(\omega) \text{ is complex!}}$

At $\omega \gg \omega_0$, $|f(\omega)|^2 = 1$. This gives the formula for the differential cross section for scattering on a free electron (in the nonrelativistic limit). The Thomson cross section is

$$\sigma_{Th} = \frac{8\pi}{3} r_0^2 .$$

At $\omega \ll \omega_0$ we recover Rayleigh ~~scattering~~ scattering,

$$\frac{d\sigma}{d\Omega} = r_0^2 \, |\hat{\epsilon}^* \cdot \epsilon_0|^2 \, \frac{\omega^4}{\omega_0^2}$$

For a bound atom, ω_0 is typically in the UV so $\frac{\omega^4}{\omega_0^2}$ is small for visible light, and σ is tiny compared to σ_{Th}.

$$\sigma \sim \left[r_0 \frac{\omega^2}{\omega_0^2} \right]^2 \sim \left[10^{-15} m \times 10^{-2} (?) \right]$$

$$\sim \left[10^{-17} m \right]^2$$

See circa Eq 14.80 of my quantum notes for the real story!