Set 2 – due 2 February

"Simple experiments are always best."–E. Fermi

Three "typical" Jackson quadrupole problems this week. To get going on the first one, I calculated the time dependent quadrupole moment in the time domain before trying to pick off the coefficients of complex exponentials. In all the problems, be alert to the possibility of recycling calculations Jackson already has done for you.

1) [20 points] Jackson 9.2

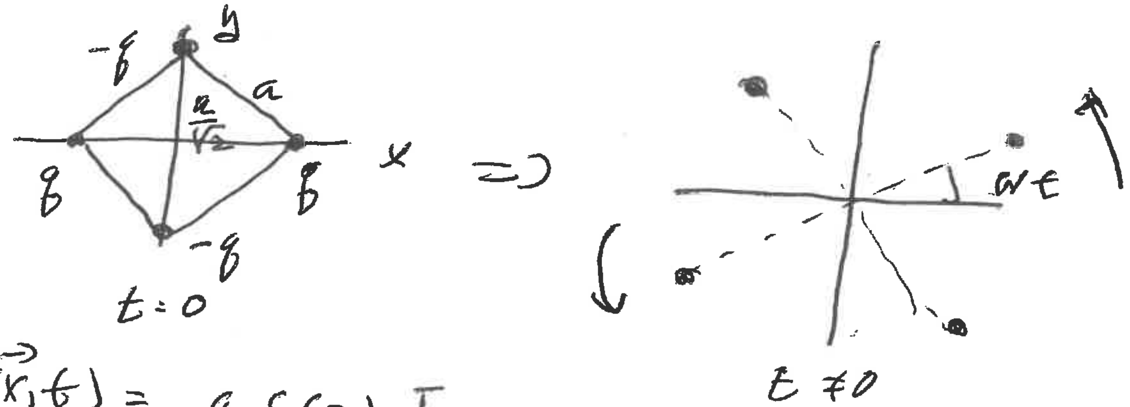
2) [20 points] Jackson 9.16. Consider the "far field" case only. Use $I = I_0 \sin(2\pi z/d) \exp(-i\omega t)$. In part (a), sketch the angular distribution, do not give a detailed plot. I evaluated the integral

$$\int_0^1 dx \frac{\sin^2 \pi x}{1 - x^2} \tag{1}$$

numerically.

3) [20 points] Jackson 9.17. You can borrow formulas 9.50-9.52.
If you are working in CGS; R in ohms (in MKS) is R in CGS times $30c$.

9.2. To begin, compute $Q_{ij}(t)$ before going into the frequency domain. With charges oriented as shown



$$
\begin{aligned}
\rho(\vec{x},t) = q\,\delta(z)\Big[&\delta\left(x-\tfrac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y-\tfrac{a}{\sqrt{2}}\sin\omega t\right) \\
&-\delta\left(x-\tfrac{a}{\sqrt{2}}\cos\left(\omega t+\tfrac{\pi}{2}\right)\right)\delta\left(y-\tfrac{a}{\sqrt{2}}\sin\left(\omega t+\tfrac{\pi}{2}\right)\right) \\
&+\delta\left(x-\tfrac{a}{\sqrt{2}}\cos(\omega t+\pi)\right)\delta\left(y-\tfrac{a}{\sqrt{2}}\sin(\omega t+\pi)\right) \\
&-\delta\left(x-\tfrac{a}{\sqrt{2}}\cos\left(\omega t+\tfrac{3\pi}{2}\right)\right)\delta\left(y-\tfrac{a}{\sqrt{2}}\sin\left(\omega t+\tfrac{3\pi}{2}\right)\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
= q\,\delta(z)\Big[&\delta\left(x-\tfrac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y-\tfrac{a}{\sqrt{2}}\sin\omega t\right) \\
&-\delta\left(x+\tfrac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y-\tfrac{a}{\sqrt{2}}\cos\omega t\right) \\
&+\delta\left(x+\tfrac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y+\tfrac{a}{\sqrt{2}}\sin\omega t\right) \\
&-\delta\left(x-\tfrac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y+\tfrac{a}{\sqrt{2}}\cos\omega t\right)\Big]
\end{aligned}
$$

Put this into the Cartesian quadrupole tensor

$$Q_{ij} = \int d^3x\, \rho(x)\left[3x_i x_j - \delta_{ij}x^2\right] =$$

$$= \sum_{n=1}^{4} q_n\left[3x_i(n)\,x_j(n) - \delta_{ij}\frac{a^2}{2}\right]. \quad i \text{ or } j = x \text{ or } y.$$

The $\frac{a^2}{2}\delta_{ij}$ term has no time dependence, so there is no radiation from it. Also $Q_{iz} = Q_{zi} = Q_{zz} = 0$. The nonvanishing Q's are

$$Q_{xx} = 3q\left(\frac{a}{\sqrt{2}}\right)^2\left[2\cos^2\omega t - 2\sin^2\omega t\right]$$
$$= 3qa^2\cos 2\omega t$$
$$Q_{yy} = -Q_{xx}$$

$$Q_{xy} = Q_{yx} = 3q\left(\frac{a}{\sqrt{2}}\right)^2 \cdot 4\cos\omega t\sin\omega t = 3qa^2\sin 2\omega t.$$

Thus $\quad Q_{xx} = -Q_{yy} = \mathrm{Re}\, 3qa^2 e^{-2i\omega t}$

$$Q_{xy} = Q_{yx} = 3iqa^2 e^{-2i\omega t}.$$

In Jackson's notation, $Q_{xx}(2\omega) = -Q_{yy}(2\omega) = 3qa^2$

$$Q_{xy}(2\omega) = 3iqa^2 = Q_{yx}.$$

We can now fill in the "standard" quadrupole formulas. Call $Q_{ij} = \epsilon_i\epsilon_j\tilde{Q}_{ij}$ for unit vectors ϵ_i,

$q_i(n) = Q_{ij}\hat{n}_j$, $\hat{n}$ is a unit vector pointing from the source to the detector. Then

$$\vec{q}(n) = \hat{x}\left[Q_{xx}n_x + Q_{xy}n_y\right] + \hat{y}\left[Q_{yx}n_x + Q_{yy}n_y\right]$$
$$= 3qa^2\left[\hat{x}(\hat{n}\cdot(\hat{x}+i\hat{y})) + i\hat{y}(\hat{n}\cdot(\hat{x}+i\hat{y}))\right]$$
$$= 3qa^2\,\hat{\epsilon}_+(\hat{n}\cdot\hat{\epsilon}_+) \quad \text{with } \hat{\epsilon}_+ = \hat{x}+i\hat{y}.$$

This goes into

$$\frac{dP}{d\Omega} = \frac{cr^2}{8\pi}\hat{n}\cdot(\vec{E}\times\vec{B}^*) = \frac{ck^6}{288\pi}\left[|\vec{q}|^2 - |\hat{n}\cdot\vec{q}|^2\right]$$

and $|\vec{q}|^2 = (3qa^2)^2(\hat{\epsilon}_+\cdot\hat{\epsilon}_+)|\hat{n}\cdot\hat{\epsilon}_+|^2$

$|\vec{q}\cdot\hat{n}|^2 = (3qa^2)(|\hat{n}\cdot\hat{\epsilon}_+|^2)^2.$

Now write $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$\hat{x} = (1,0,0)$, $\hat{y} = (0,1,0) \Rightarrow \hat{\epsilon}_+ = (1,i,0)$

so $\hat{\epsilon}_+\cdot\hat{\epsilon}_+^* = 2$

$\hat{n}\cdot\hat{\epsilon}_+ = \sin\theta(\cos\varphi + i\sin\varphi) = \sin\theta\, e^{i\varphi}$

$|\hat{n}\cdot\hat{\epsilon}_+|^2 = \sin^2\theta$

note $ck = 2\omega!$

$$\frac{dP}{d\Omega} = \frac{ck^6}{288\pi}\cdot 9q^2a^4\left[2\sin^2\theta - \sin^4\theta\right]$$

$$= \frac{ck^6}{32\pi}q^2a^4\left[(2-\sin^2\theta)\sin^2\theta\right]$$

$$(2-\sin^2\theta)\sin^2\theta = (1+\cos^2\theta)\sin^2\theta$$
$$= (1+\cos^2\theta)(1-\cos^2\theta) = 1-\cos^4\theta$$

so the radiated power is

$$P = \frac{2\pi}{32\pi}ck^6q^2a^2\int_{-1}^{1}d\cos\theta\,\underbrace{[1-\cos^4\theta]}_{2-\frac{2}{5}=\frac{8}{5}}$$

$$= \frac{1}{10}cq^2a^4k^6$$

To convert to MKS

$$\left.\frac{dP}{d\Omega}\right)_{CGS} = \frac{ck^6}{288\pi}\left[|\vec{q}|^2 - |\vec{q}\cdot\hat{n}|^2 \right]$$

$$\left.\frac{dP}{d\Omega}\right)_{MKS} = \frac{c^2 Z_0\, k^6}{1152\,\pi^2}\left[|\vec{q}|^2 - |\vec{q}\cdot\hat{n}|^2 \right]$$

i.e. $\dfrac{MKS}{CGS} = \dfrac{288}{1152\pi}\, c Z_0 = \dfrac{c\, Z_0}{4\pi}$

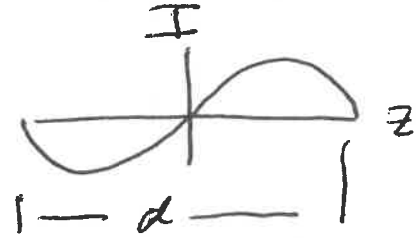
so $P_{MKS} = \dfrac{1}{10}\, c^2\, \dfrac{Z_0}{4\pi}\, q^2 a^4 k^6.$

9.16 – the full wave antenna in far field

$I: \; I_0 \sin\frac{2\pi z}{d}\, e^{-i\omega t}$ for $-\frac{d}{2} < z < \frac{d}{2}$

$\lambda = d$, $kd = 2\pi$

(or $k = \frac{2\pi}{d}$)



This uses the "exact formula"

$$\vec{A}(x) = \hat{z}\,\frac{I_0}{c}\,\frac{e^{ikr}}{r}\int_{-d/2}^{d/2} dz \cdot \sin\frac{2\pi z}{d}\, e^{-ikz\cos\theta}$$

$$= \hat{z}\,\frac{I_0}{c}\,\frac{e^{ikr}}{r}\,\frac{1}{2i}\int_{-d/2}^{d/2} dz \left[\exp\left(iz\left(\frac{2\pi}{d} - k\cos\theta\right)\right) - \exp\left(iz\left(-\frac{2\pi}{d} - k\cos\theta\right)\right)\right]$$

$$= \hat{z}\,\frac{I_0}{c}\,\frac{e^{ikr}}{r}\,\frac{1}{2i}\left[\frac{e^{iz\left(\frac{2\pi}{d} - k\cos\theta\right)}}{i\left(\frac{2\pi}{d} - k\cos\theta\right)} - \frac{e^{iz\left(-\frac{2\pi}{d} - k\cos\theta\right)}}{i\left(-\frac{2\pi}{d} - k\cos\theta\right)}\right]_{-\frac{d}{2}}^{\frac{d}{2}}$$

Define $f = \frac{kd}{2}\cos\theta$. Since $k = \frac{2\pi}{d}$, $f = \pi\cos\theta$

$$\vec{A} = \hat{z}\,\frac{I_0}{c}\,\frac{e^{ikr}}{2ir}\left[\frac{-e^{-if} + e^{if}}{i\frac{2\pi}{d}(1-\cos\theta)} + \frac{-e^{-if} + e^{if}}{i\frac{2\pi}{d}(1+\cos\theta)}\right]$$

$$= -i\hat{z}\,\frac{I_0}{c}\,\frac{e^{ikr}}{r}\,\frac{\sin(\pi\cos\theta)}{k}\left(\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}\right)$$

$$= -i\hat{z}\,\frac{I_0}{c}\,\frac{e^{ikr}}{r} \cdot \frac{2}{k}\,\frac{\sin(\pi\cos\theta)}{\sin^2\theta}$$

This is $\vec{A} = -ik \dfrac{e^{ikr}}{r} \vec{p}$ and

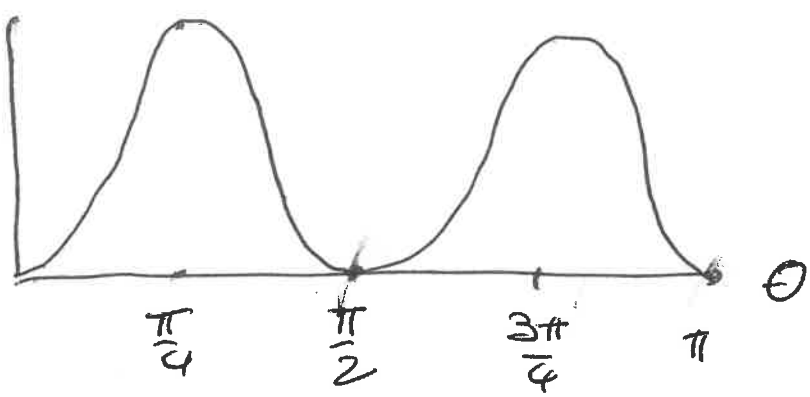
$$\vec{p} = \hat{z} \left[-\frac{2 I_0 i}{c k^2} \frac{\sin(\pi \cos\theta)}{\sin^2\theta} \right]$$

so we can plug it into the dipole formula even though it isn't a real dipole -

$$\left.\frac{dP}{d\Omega}\right)_{CGS} = \frac{c k^4}{8\pi} |\vec{p}|^2 \sin^2\theta = \frac{I_0^2}{2\pi c} \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}$$

$$MKS \; * = \frac{c Z_0}{4\pi} = \frac{c^2 Z_0 k^4}{32\pi^2} |\vec{p}|^2 \sin^2\theta$$

$\dfrac{\sin^2(\pi\cos\theta)}{\sin^2\theta}$



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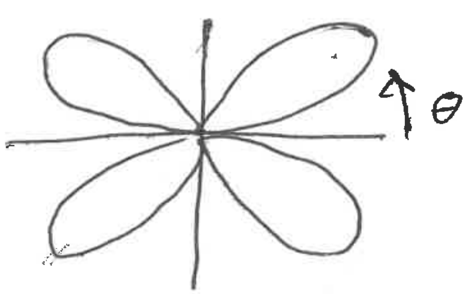


- the antenna pattern is roughly quadrupolar

$$P = \frac{1}{2} I_0^2 R = \int \frac{dP}{d\Omega}\, d\Omega$$

$$= \frac{I_0^2}{2\pi c}\, 2\pi \int_{-1}^{1} d\mu\, \frac{\sin^2 \pi\mu}{1-\mu^2}$$

R in ohms $= 30\,c \times R_{cgs}$

$$R = 30 \times 4 \times \int_0^1 d\mu\, \frac{\sin^2 \pi\mu}{1-\mu^2}$$

I did the integral numerically, got 0.78, so

$120 \times 0.78 = 93.6\,\Omega$

is the radiation resistance

Repeat 9.16 with multipoles, use

$$\vec{J}(x) = \hat{z}\, I_0\, \delta(x)\delta(y) \sin\frac{2\pi z}{d}\, \Theta\left(\frac{d}{2} - |z|\right).$$

Useful fact: $i\omega\rho(x) = \vec{\nabla}\cdot\vec{J} = \frac{\partial J_z}{\partial z}$

$$i\omega\rho = \delta(x)\,\delta(y)\,\frac{2\pi I_0}{d}\cos\frac{2\pi z}{d}.$$

1) Observe that ρ has even parity in z. This means the electric dipole moment is zero –

$\int d^3x\, \rho(x)\, \vec{x} = 0 \Rightarrow$ No E1 radiation

2) $\frac{1}{2c}\int \vec{J}(x) \times \vec{r}\; d^3x = 0$ because $J = 0$ unless $x = y = 0$. Also $\hat{z}\times\hat{z} = 0 \Rightarrow$ No M1 radiation

3) But there is E2. $Q_{ij} = \int d^3x\, \rho(x)\left[3x_i x_j - \delta_{ij} x^2\right]$

By symmetry, $Q_{ij} = 0$ if $i \neq j$. Q is traceless, so $Q_{xx} = Q_{yy} = -\frac{1}{2} Q_{zz}$ like last semester's problems. Define $Q_{zz} \equiv Q_0$.

$$Q_0 = \int_{-d/2}^{d/2} dz\,\left[3z^2 - z^2\right] \frac{2\pi}{i\omega d} I_0 \cos\frac{2\pi z}{d}$$

$$= \frac{8\pi I_0}{i\omega d}\int_0^{d/2} z^2\, dz \cos\frac{2\pi z}{d} \;; \text{ write } u = \frac{2\pi z}{d}$$

$$Q_0 = \frac{8\pi I_0}{i\omega d}\left(\frac{d}{2\pi}\right)^3 \underbrace{\int_0^{\pi} u^2 \cos u\, du}_{\to -2\pi \text{ – tables!}}$$

$$Q_0 = -\frac{I_0 d^3 \cdot 2\pi}{i\omega d \pi^2} = \frac{i I_0 d^3 \cdot 2\pi}{\left(\frac{2\pi c}{d}\right)\cdot d \cdot \pi^2} = \frac{i I_0 d^3}{\pi^2 c}$$

using $\omega = \frac{2\pi c}{d}$.

Q_{ij} has the same tensor structure as the spheroid of Jackson's eq 9.30 - 9.52 so we just copy the formulas

$$\frac{dP}{d\Omega} = \frac{c k^6 Q_0^2}{128\pi} \sin^2\theta \cos^2\theta$$

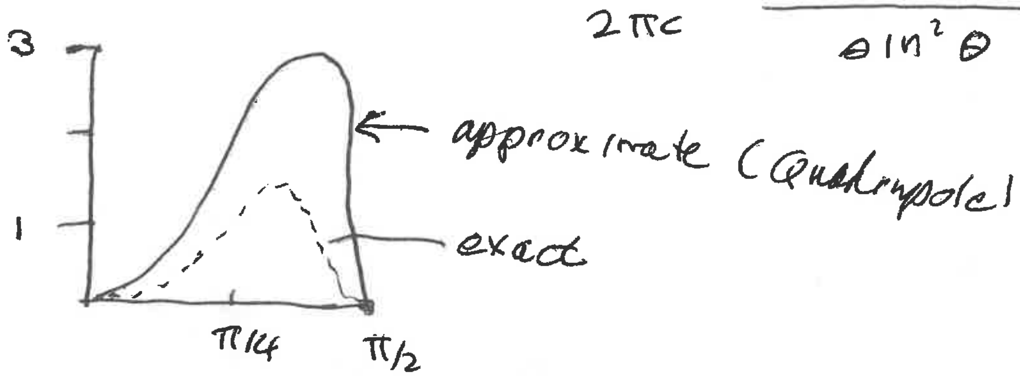
$$= \frac{c k^6 d^6 I_0^2}{128 \pi^5 c^2} \sin^2\theta \cos^2\theta$$

$$\xrightarrow[kd=2\pi]{} \frac{I_0^2 \pi}{2c} \sin^2\theta \cos^2\theta = \frac{I_0^2}{2\pi c}\left[\pi^2 \sin^2\theta \cos^2\theta\right]$$

This is CGS, multiply by $\frac{c Z_0}{4\pi}$ for MKS.

The exact answer was

$$dP_{exact}/d\Omega = \frac{I_0^2}{2\pi c} \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}$$



The quadrupole formula's total power is

$$P_Q = \frac{ck^6}{240} Q_0^2 = \frac{ck^6 d^6}{240\pi^4 c^2} I_0^2 = \frac{1}{2} I_0^2 R_Q$$

again, using $kd = 2\pi$, $R_Q = \frac{2(2\pi)^6}{240\pi^4 c^2} = \frac{128\pi^2}{240c}$

or $R_Q = \frac{8}{15}\frac{\pi^2}{c}$

R in $\Omega = 30c R_Q = 16\pi^2 \sim 160\,\Omega$ –
very different from the exact answer of $93\,\Omega$.
The issue is that the multipole expansion needs
$\frac{\text{size of radiator}}{\text{wavelength}} \ll 1$, but here $\frac{d}{\lambda} = 1$.

$$R_{MKS} = \frac{8\pi^2}{15c} \cdot \frac{c Z_0}{4\pi} = Z_0 \cdot \frac{2\pi}{15}$$

$Z_0 = 377\,\Omega$, $\pi = 3$, $\frac{2\pi}{15} = \frac{2}{5}$

$377\,\Omega \times \frac{2}{5} \sim 150\,\Omega$ or so...