Set 12 – due 26 April

"All composite objects decay. Strive diligently." – the Buddha

1) [15 points] Jackson 14.21 Use Bohr-model values for the radius and velocity.

2) [20 points] Jackson 15.4. Begin with eqns 15.1-15.2 and set the β_j's to zero, and let the β'_j be the velocity of either particle in the decay. Part (a) is trivial. Watch the sign of the interference term carefully! You'll have to make a reasonable assumption to get from $dI/d\omega$ to the fraction of energy radiated as photons. Also, does Jackson tell you that $m_\pi = 140$ MeV, $m_e = 0.511$ MeV?

14.21. The correspondence principle relates the classical power radiated, P_n, to the lifetime of the excited state τ_n via

$$P_n = \frac{\hbar\omega}{\tau_n}$$ where the energy difference is $\hbar\omega$

So $$\frac{1}{\tau_n} = \frac{P_n}{\Delta E}$$

The electron in the nth Bohr orbital has a classical acceleration $a_n = \frac{v^2}{r}$ which goes into the Larmor formula:

$$P_n = \frac{2}{3}\frac{e^2 a_n^2}{c^3} = \frac{2}{3}\frac{e^2}{\hbar c}\frac{a_n^2 \cdot \hbar c}{c^3}; \quad \frac{e^2}{\hbar c} = \alpha = \frac{1}{137}$$

In the Bohr atom $\frac{mv^2}{r} = \frac{Ze^2}{r^2}$ or $mv^2 = \frac{Ze^2}{r}$,

the energy is $E = \frac{1}{2}mv^2 - \frac{Ze^2}{r} = -\frac{1}{2}\frac{Ze^2}{r} = -\frac{1}{2}Z\frac{e^2}{\hbar c}\frac{\hbar c}{r}$.

The quantization condition is $mvr = n\hbar$, so

$$mv^2 = m\left[\frac{n\hbar}{mr}\right]^2 = \frac{Ze^2}{r} \quad \text{or}$$

$$r = r_n = \frac{n^2\hbar^2}{Ze^2 m} = \frac{n^2}{Z}\frac{\hbar c}{e^2}\frac{\hbar c}{mc^2} = \frac{n^2}{Z\alpha}\frac{\hbar c}{mc^2}$$

and $$E_n = -\frac{1}{2}Z\alpha\frac{\hbar c}{r_n} = -\frac{1}{2}(Z\alpha)^2\frac{mc^2}{n^2}.$$

The familiar Bohr result. The acceleration is

$$a_n = \frac{v^2}{r_n} = \left(\frac{n\hbar}{mr_n}\right)^2\frac{1}{r_n} = \frac{n^2\hbar^2}{mr_n^3} = \frac{n^2(\hbar c)^2 c^2}{m^2c^4}\left[\frac{Z\alpha\, mc^2}{n^2\hbar c}\right]^3$$

so $a_n = \frac{(Z\alpha)^3}{n^4} \frac{mc^2}{\hbar c} \cdot c^2$

$$P_n = \frac{2}{3}\frac{\hbar c \alpha}{c^3} a_n^2 = \frac{2}{3}\frac{\hbar c\alpha}{c^3}\frac{(Z\alpha)^6 (mc^2)^2 c^4}{n^8 (\hbar c)^2}$$

$$= \frac{2}{3}\alpha \frac{(Z\alpha)^6}{n^8}\frac{(mc^2)^2}{\hbar c}\cdot c$$

Now we consider transitions $n \to n-1$.

$$\Delta E_n = \Delta n \frac{dE}{dn} = -(Z\alpha)^2 \frac{mc^2}{n^3} \quad \text{so}$$

$$\frac{1}{\tau_n} = \frac{P_n}{\Delta E_n} = \frac{n^3}{(Z\alpha)^2 mc^2} \cdot \frac{2}{3}\alpha\frac{(Z\alpha)^6}{n^8}\frac{(mc^2)^2}{\hbar c}\cdot c$$

$$= \frac{2}{3}\alpha (Z\alpha)^4 \frac{mc^2}{\hbar c}\cdot c \cdot \frac{1}{n^5} \equiv \frac{1}{\tau_0}\frac{1}{n^5}.$$

That is, $\tau_n = \tau_0 n^5$ and for $Z=1$

$$\tau_0 = \frac{3}{2}\frac{1}{\alpha^5}\frac{[\hbar c = 2000 \text{ eV-Å}]}{[mc^2 = 5.11\times 10^5 \text{ eV}]} \frac{1}{[c = 3\times 10^{18} \text{ Å/sec}]}$$

$$= 9.4 \times 10^{-11} \text{ sec}$$

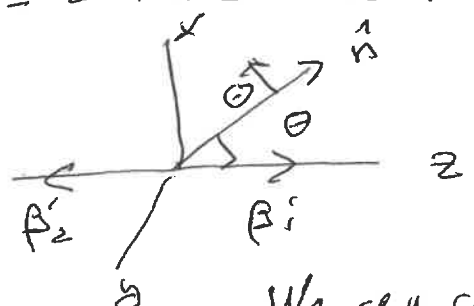
n	$\tau_{classical} = \tau_0 n^5$	Jackson's QM table
2	3×10^{-9} sec	1.6×10^{-9} sec
4	9.68×10^{-8} sec	7.3×10^{-8} sec
6	7.34×10^{-7} sec	6.1×10^{-7} sec

The classical formula is only off by a factor of 2 at $n=2$ and gets better and better as n rises!

15.4. a) Eqns 15.1 and 15.2 say

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{1}{4\pi^2 c} \left| \hat{\epsilon}^* \cdot \sum_j e_j\, e^{-i\omega \hat{n}\cdot \vec{r}_j(0)/c} \times \left(\frac{\vec{\beta}_j'}{1-\hat{n}\cdot\vec{\beta}_j'} - \frac{\vec{\beta}_j}{1-\hat{n}\cdot\vec{\beta}_j} \right) \right|^2$$

b) $j=1,2$ label the decay products, which are produced back to back with $\vec{\beta}_j = 0$. Use coordinates as in the figure - the particles move ~~along~~ along the $\pm z$ axis and the γ is produced in the x-z plane -



$$\hat{n} = \hat{z}\cos\theta + \hat{x}\sin\theta$$
$$\vec{\beta}_1' = \hat{z}\beta = -\vec{\beta}_2'$$

We can choose polarizations to be

$\hat{\epsilon}_1 = \hat{y}$, $\hat{\epsilon}_2 = -\hat{z}\sin\theta + \hat{x}\cos\theta$ - so $\hat{\epsilon}_i \cdot \hat{n} = 0$,

$\hat{\epsilon}_1 \cdot \vec{\beta}_j = 0$, $\hat{\epsilon}_2 \cdot \vec{\beta}_j = \mp \beta\sin\theta$, $\hat{n}\cdot\vec{\beta}_j = \pm\beta\cos\theta$.

Then $$\hat{\epsilon}_2 \cdot \left[\frac{\pm \vec{\beta}_1'}{1 \pm \vec{\beta}_1' \cdot \hat{n}} \right] = \frac{\mp \beta \sin\theta}{1 \pm \beta \cos\theta}$$

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c} \sin^2\theta \left[\frac{\beta}{1-\beta\cos\theta} + \frac{\beta}{1+\beta\cos\theta} \right]^2$$

The relative sign is $+ =$ (-direction) x (-charge)

Now write $\frac{dI}{d\omega} = \frac{e^2\beta^2}{2\pi c} J$ where

$$J = \int_{-1}^{1} d\mu\,(1-\mu^2)\left[\frac{1}{(1-\beta\mu)^2} + \frac{2}{1-\beta^2\mu^2} + \frac{1}{(1+\beta\mu)^2}\right]$$

Set $\mu = -x$ in the last term, $\int_{-1}^{1} d\mu = -\int_{1}^{-1} dx = \int_{-1}^{1} dx$

$$J = 2\int_{-1}^{1} d\mu\,\frac{(1-\mu^2)}{(1-\beta\mu)^2} + 2\int_{-1}^{1} d\mu\,\frac{1}{1-\beta^2\mu^2} - 2\int_{-1}^{1}\frac{\mu^2\,d\mu}{1-\beta^2\mu^2}$$

(splitting the middle term)

$$J \equiv 2\left[K_1 + K_2 - K_3\right]$$

Now for the three integrals –

$$K_1 = \int_{-1}^{1} d\mu\,\frac{(1-\mu^2)}{(1-\beta\mu)^2} = \frac{1}{\beta}\frac{1}{(1-\beta\mu)}\Bigg|_{-1}^{1} - \int\frac{\mu^2\,d\mu}{(1-\beta\mu)^2}$$

$$= \frac{1}{\beta}\left[\frac{1}{1-\beta} - \frac{1}{1+\beta}\right] - \frac{(-1)}{\beta^3}\left[1-\beta\mu - 2\ln(1-\beta\mu) - \frac{1}{1-\beta\mu}\right]\Bigg|_{-1}^{1}$$

(tables for the 2nd integral

$$K_1 = \frac{2}{1-\beta^2} + \frac{1}{\beta^3}\left[-2\beta + 2\ln\frac{1+\beta}{1-\beta} - \frac{2\beta}{1-\beta^2}\right]$$

$$= \frac{2}{\beta^3}\ln\frac{1+\beta}{1-\beta} + \frac{2}{1-\beta^2} - \frac{2}{\beta^2} - \frac{2}{\beta^2(1-\beta^2)}$$

$$= \frac{2}{\beta^3}\ln\frac{1+\beta}{1-\beta} + \frac{2\beta^2 - 2(1-\beta^2) - 2}{\beta^2(1-\beta^2)}$$

$$= \frac{2}{\beta^3}\ln\frac{1+\beta}{1-\beta} - \frac{4(1-\beta^2)}{\beta^2(1-\beta^2)} = \frac{2}{\beta^3}\ln\frac{1+\beta}{1-\beta} - \frac{4}{\beta^2}$$

$$K_2: \int_{-1}^{1} d\mu \, \frac{1}{1-\beta^2\mu^2} = \frac{1}{2}\int_{-1}^{1} d\mu \left[\frac{1}{1-\beta\mu} + \frac{1}{1+\beta\mu}\right]$$

$$= \frac{1}{2\beta}\Big[-\ln(1-\mu\beta) + \ln(1+\mu\beta)\Big]_{-1}^{1} = \frac{1}{2\beta}\ln\frac{1+\mu\beta}{1-\mu\beta}\Bigg|_{-1}^{1}$$

$$= \frac{1}{\beta}\ln\frac{1+\beta}{1-\beta}$$

And $K_3 = \int_{-1}^{1}\frac{\mu^2 d\mu}{1-\mu^2\beta^2} = \int_{-1}^{1}\frac{d\mu}{1-\mu^2\beta^2}\left[-\frac{(1-\mu^2\beta^2)}{\beta^2} + \frac{1}{\beta^2}\right]$

$$= \frac{-\mu}{\beta^2}\Bigg|_{-1}^{1} + \frac{1}{\beta^2}\int_{-1}^{1}\frac{d\mu}{1-\mu^2\beta^2} = -\frac{2}{\beta^2} + \frac{1}{\beta^3}\ln\frac{1+\beta}{1-\beta}$$

$$K_2 - K_3 = -\frac{1}{\beta^3}(1-\beta^2)\ln\frac{1+\beta}{1-\beta} + \frac{2}{\beta^2}$$

$$K_1: \quad \frac{2}{\beta^3}\ln\frac{1+\beta}{1-\beta} - \frac{4}{\beta^2}$$

$$K_1 + K_2 - K_3 = \frac{1}{\beta^2}\left[-2 + \left(\ln\left(\frac{1+\beta}{1-\beta}\right)\right)\left(-\frac{(1-\beta^2)}{\beta} + \frac{2}{\beta}\right)\right]$$

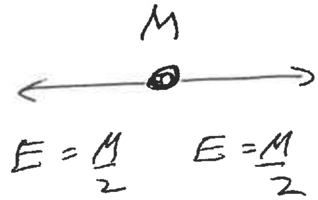
$$= \frac{1}{\beta^2}\left[-2 + \left(\frac{1+\beta^2}{\beta}\right)\ln\left(\frac{1+\beta}{1-\beta}\right)\right]$$

so $\dfrac{dI}{d\omega} = \dfrac{e^2}{\pi c}\left[\left(\dfrac{1+\beta^2}{\beta}\right)\ln\dfrac{1+\beta}{1-\beta} - 2\right]$

We can take the extreme-relativistic limit

$\beta \sim 1$, $1-\beta = \dfrac{1}{2\gamma^2}$, $\dfrac{1+\beta^2}{\beta} = 2$, $\ln\dfrac{1+\beta}{1-\beta} = \ln 4\gamma^2 = 2\ln 2\gamma$.

15.4.4

The kinematics is 

so $\gamma = \frac{E}{m} = \frac{M}{2m}$

$$\frac{dI}{d\hbar\omega} = \frac{e^2}{\hbar c} \cdot \frac{4}{\pi}\left[\ln\frac{M}{m} - \frac{1}{2}\right]$$

$I = \int_0^{\Lambda} \frac{dI}{d\hbar\omega}\, d\hbar\omega$. The upper cutoff is $\Lambda = \hbar\omega_{max}$

Our calculation assumed $\hbar\omega_{max} << Mc^2$ and $\frac{dI}{d\hbar\omega}$ = constant. To find $\hbar\omega_{max}$, a better calculation would be needed (it is some fraction of Mc^2). We'll just proceed naively:

$$\frac{I}{Mc^2} = \frac{4\alpha}{\pi}\left[\ln\frac{M}{m} - \frac{1}{2}\right] \cdot \frac{\Lambda}{Mc^2}.$$

The important part of this is the logarithm: the RHS is not $I/Mc^2 \sim \alpha$, it is bigger (logarithmically in the mass ratio).

Numbers for Jackson: set $\frac{\Lambda}{Mc^2} = \frac{1}{2}$

$Mc^2 = 784$ MeV
$m_\pi c^2 = 140$ MeV
$m_e c^2 = 0.511$ MeV

$$\frac{I}{Mc^2} = \frac{1}{2} \times 0.00929 \times \begin{cases} \left(\ln\frac{784}{.511} - \frac{1}{2}\right) \text{ for } e \\ \left(\ln\frac{784}{140} - \frac{1}{2}\right) \text{ for } \pi \end{cases} = \begin{matrix} 0.032 \\ 0.0055 \end{matrix}$$