Set 11 – due 19 April

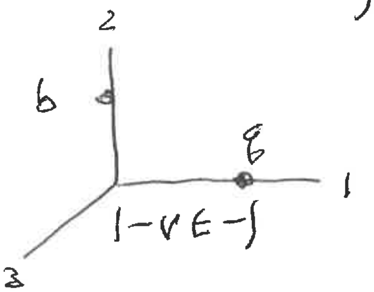
"The sciences do not explain, they hardly ever try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification for such a mathematical construct is solely and precisely that it is expected to work." (J. von Neumann)

1) [10 points] Jackson 14.1. You can borrow a lot of the solution from the book and from notes in class, but it is still instructive–for me, the instruction is not to use the Lienard-Wiechart potentials when you can do a Lorentz transformation!

2) [10 points] Jackson 14.4. The answers should all look very familiar.

3) [20 points] Jackson 14.9, parts a, b, c only.

14.1) Eq. 11.152 gives the fields — point charge with velocity $\vec{v} = \hat{x} v$ in frame K, at rest in K';



at observation point $\vec{x} = (0, b, 0)$

$$E_1 = E_1' = -\frac{q \gamma v t}{\left[b^2 + (\gamma v t)^2\right]^{3/2}}$$

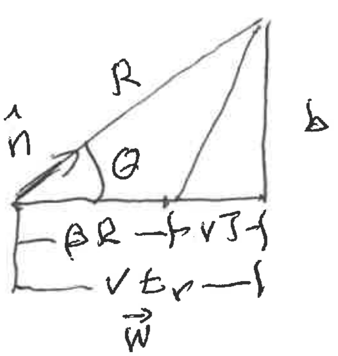
$$E_2 = \gamma E_2' = \frac{\gamma q b}{\left[b^2 + (\gamma v t)^2\right]^{3/2}}, \quad B_3 = \gamma \beta E_2' = \beta E_2.$$

This is from a Lorentz transformation — now we use Lienard-Wiechart. Eq 14.14 says

$$\vec{E} = \frac{q}{R^2} \frac{\hat{n} - \beta \hat{x}}{\left[1 - \beta \hat{x} \cdot \hat{n}\right]^3}, \quad \vec{B} = \hat{n} \times \vec{E}.$$

$\vec{R} = \vec{x} - \vec{W}(t_r)$ where $\vec{W}(t_r)$ is the location of the charge at the retarded time $t_r = t - R/c$,

$$\hat{n} = \vec{R}/R, \quad \vec{W}(t_r) = -v t_r \hat{x} = -v\left(t - \frac{R}{c}\right)\hat{x}$$



In the picture, t is negative so call $t = -T$, so $\vec{W} = \hat{x}\left[vT + \beta R\right]$.

Furthermore, $\vec{R} \cdot \hat{y} = b = R \sin\theta$.

And in class we found (this is Eq. 14.16)

$$(1 - \beta \hat{x} \cdot \hat{n})^2 R^2 = \frac{1}{\gamma^2}\left[b^2 + (\gamma v t)^2\right]$$

So — $$E_2 = \frac{\gamma q \vec{R} \cdot \hat{y}}{\left[b^2 + (\gamma v t)^2\right]^{3/2}} = \frac{\gamma q b}{\left[b^2 + (\gamma v t)^2\right]^{3/2}}$$ as from L.T.

$$E_1 = \frac{\gamma q\,[\vec{R}\cdot\hat{x} - \beta R]}{[b^2 + (\gamma v t)^2]^{3/2}}$$ takes a bit more work.

Use $\beta R = v(t - t_r)$, $\vec{R}\cdot\hat{x} = -v t_r \Rightarrow$

$$\vec{R}\cdot\hat{x} - \beta R = -v t_r - v(t - t_r) = -vt.$$

So $$E_1 = \frac{-\gamma q v t}{[b^2 + (\gamma v t)^2]^{3/2}}$$ as we saw before.

$$\vec{B} = \hat{n}\times\vec{E} = \frac{\vec{R}\times\vec{E}}{R} = [b\hat{y} - v t_r \hat{x}] \times \frac{\vec{E}}{R}$$

$$= \begin{vmatrix} i & j & k \\ -v t_r & b & 0 \\ \frac{E_1}{R} & \frac{E_2}{R} & 0 \end{vmatrix} = \frac{\hat{k}}{R}\left[-v t_r E_2 - b E_1\right]$$

$$B_3 = \frac{\gamma q}{[b^2 + (\gamma v t)^2]^{3/2}} \left\{ \frac{-v t_r b - b(-vt)}{R} \right\}$$

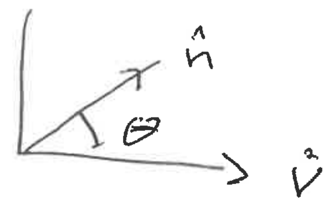
$$\{\ \} = \frac{b v (t - t_r)}{R}$$ and $t - t_r = R/c$

so $$B_3 = \frac{\gamma q b \beta}{[b^2 + (\gamma v t)^2]^{3/2}}$$

For straight line motion, the Lorentz transformation is obviously easier to carry out.

14.4. The nonrelativistic Larmor formula is

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\,|\dot{\vec{V}}|^2 \sin^2\Theta$$



a) $x(t) = \hat{z}\, a \cos\omega_0 t$

$$\dot{\vec{V}}(t) = \hat{z}\,[-\omega_0^2 a \cos\omega t]$$

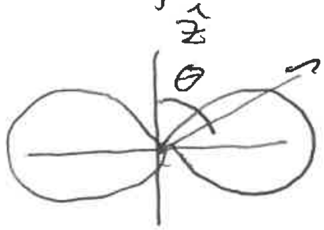
In this case $\Theta = \theta$, the usual polar angle,

$$\frac{dP}{d\Omega} = \frac{e^2 a^2 \omega_0^4}{4\pi c^3} \sin^2\theta \cos^2\omega_0 t .$$

We get the familiar dipole formulas first by time-averaging: $\langle \cos^2\omega_0 t\rangle = 1/2$, so

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2 a^2 \omega_0^4}{8\pi c^3} \sin^2\theta = \frac{c k^4}{8\pi} |\vec{p}|^2 \sin^2\theta$$

where $\vec{p} = \hat{z}\, ea$, $k = \omega_0/c$. The time averaged power radiated is



$$\langle P\rangle = \frac{c k^4}{8\pi} |\vec{p}|^2 \cdot 2\pi \cdot \left[2 - \frac{2}{3}\right] = \frac{c k^4}{3} |\vec{p}|^2.$$

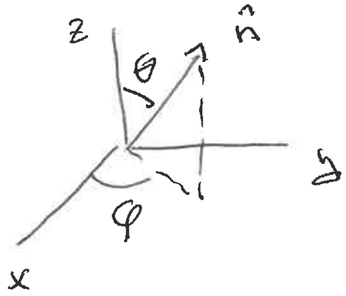
b) Next we have an orbit in the x-y plane

$$\vec{x}(t) = R\,[\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t]$$

$$\dot{\vec{V}}(t) = -\omega_0^2\, \vec{x}(t).$$

Put the observer at $\hat{n} = [\hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta]$

$$\cos\Theta = \frac{\hat{n}\cdot\dot{\vec{V}}}{|\dot{\vec{V}}|} = -\sin\theta\,[\cos\varphi\cos\omega_0 t + \sin\varphi\sin\omega_0 t]$$



$$\cos\Theta = \sin\theta\cos(\varphi - \omega_0 t)$$

$$\sin^2\Theta = 1 - \cos^2\Theta$$

$$= 1 - \sin^2\theta\cos^2(\omega_0 t - \varphi).$$

The instantaneous power radiated along $\vec{n}$ is

$$\frac{dP}{d\Omega} = \frac{e^2\omega_0^4 R^2}{4\pi c^3}\left[1 - \sin^2\theta\cos^2(\omega_0 t - \varphi)\right].$$

Now we time average: $p = eR$, $k = \omega_0/c$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2\omega_0^4 R^2}{4\pi c^3}\left[1 - \frac{1}{2}\sin^2\theta\right]$$

$$= \frac{ck^4p^2}{8\pi}\left[2 - \sin^2\theta\right]$$

$$= \frac{ck^4p^2}{8\pi}\left[1 + \cos^2\theta\right],$$

$$\langle P\rangle = \frac{ck^4p^2}{8\pi}\cdot 2\pi\left(2 + \frac{2}{3}\right) = \frac{2}{3}ck^4p^2$$

To get this directly, as in Ch 9,

$$\langle P\rangle = \frac{1}{3}ck^4|\vec{p}|^2$$

but here $\vec{p}$ is complex.

$$\vec{p}(t) = \xi R\left[\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t\right]$$

$$= \mathrm{Re}\left(\xi R[\hat{x} - i\hat{y}]e^{i\omega_0 t}\right)$$

Complex $\vec{p} = \xi R[\hat{x} - i\hat{y}]$, $\vec{p}\cdot\vec{p}^* = \xi^2R^2[1+1] = 2\xi^2R^2$

$$\langle P\rangle = \frac{2}{3}ck^4[\xi R]^2.$$

14.9. The power radiated doing relativistic motion is given by Eq. 14.24

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right)$$

and $\frac{dp_\mu}{d\tau} = \gamma\frac{dp_\mu}{dt}$ using $d\tau = \frac{dt}{\gamma}$ (below 14.24)

with $\vec{E}=0$, $\frac{d\vec{p}}{dt} = q(\vec{\beta}\times\vec{B})$, $\vec{\beta}\perp\vec{B}$ so

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\left[\gamma q\beta B\right]^2.$$

Now $\gamma^2 = \frac{1}{1-\beta^2}$, $\frac{1}{\gamma^2} = 1-\beta^2$, $\beta^2 = 1-\frac{1}{\gamma^2}$ so

$$\gamma^2\beta^2 = \gamma^2 - 1$$

$$P = \frac{2}{3}\frac{q^4}{m^2c^3}(\gamma^2-1)B^2.$$

And ~~power~~ power radiated is $-\frac{d\mathcal{E}}{dt}$, $\mathcal{E}$ = energy.

Lastly, $\gamma = \frac{\mathcal{E}}{mc^2}$, so

$$\frac{d\mathcal{E}}{dt} = -\frac{2}{3}\frac{q^4B^2}{m^2c^3}\left[\frac{\mathcal{E}^2}{(mc^2)^2}-1\right].$$

Now for the 2 limits

b) Extremely relativistic motion: ~~[illegible]~~ neglect the 1 in [ ]:

$$\int_{\varepsilon_0}^{\varepsilon} \frac{d\varepsilon'}{\varepsilon'^2} = -\frac{2}{3}\frac{g^4 B^2}{m^4 c^7}\int_0^t dt'$$

$$\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon} = -\frac{2}{3}\frac{g^4 B^2}{m^4 c^7} t$$

$$\frac{1}{\gamma_0 mc^2} - \frac{1}{\gamma mc^2} = -\frac{2}{3}\frac{g^4 B^2}{m^4 c^7} t$$

or $\quad t = \frac{3}{2}\frac{m^3 c^5}{g^4 B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right) \equiv \tilde{T}_0\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right)$

Check units: $\quad \frac{gB}{mc} = \omega_0 = \text{frequency} = \text{time}^{-1}$

$$\tilde{T}_0 = \frac{3}{2}\left(\frac{mc}{gB}\right)^2 \frac{mc^3}{g^2} = \frac{3}{2}\frac{1}{\omega_0^2}\left(\frac{\hbar c}{g^2}\right)\frac{mc^2}{\hbar c}\cdot c$$

$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$, $\hbar c$ = energy × length

$$\tilde{T}_0 = \frac{3}{2}\frac{1}{\omega_0^2}\frac{1}{\alpha}\frac{mc^2}{\hbar c}\cdot c = t^2 \times \frac{1}{\ell} \times \frac{\ell}{t} = t.$$

c) in the nonrelativistic limit, the energy is $\mathcal{E} = mc^2 + T$ where T is the kinetic energy.

$$\gamma = \frac{T + mc^2}{mc^2} = 1 + \frac{T}{mc^2}$$

$$\gamma^2 \sim 1 + \frac{2T}{mc^2} \quad \text{since } T \ll mc^2$$

$$\frac{d\mathcal{E}}{dt} = \frac{dT}{dt} = -\frac{2 g^4 B^2}{3 m^2 c^3} \cdot \frac{2T}{mc^2}$$

$$\frac{dT}{dt} = -\frac{4}{3} \frac{g^4 B^2}{m^3 c^5} T = -\frac{8}{9} \frac{T}{\tilde{T}_0}$$

so

$$T(t) = T(t=0) \exp\left[-\frac{8}{9} \frac{t}{\tilde{T}_0}\right]$$

The characteristic time $\tilde{T}_0$ is the same as in part (b) (basically dimensional analysis) but the rate of energy loss is exponential rather than power law –

$$\frac{1}{\mathcal{E}} = \frac{1}{\mathcal{E}_0} + \frac{t}{T_0} \quad \text{is} \quad \mathcal{E}(t) \sim \frac{1}{\alpha + \beta t}$$