Set 10 – 12 April

"The secret to creativity is knowing how to hide your sources." – A. Einstein

1) [20 points] Jackson 12.6. Instead of part (a), show that a Lorentz transformation can be made, to go to a frame where $\vec{B}'$ and $\vec{E}'$ are parallel, and that the boost velocity satisfies the equation

$$\frac{\vec{\beta}}{1+\beta^2} = \frac{\vec{E}\times\vec{B}}{E^2+B^2} \tag{1}$$

As a hint, you can start by arguing that the boost velocity is $\vec{\beta} = \lambda(\vec{E}\times\vec{B})$ at the beginning of your calculation, and then determine λ.

In part (b), note the Jacksonian phrase "with appropriate constants of motion."

2) [20 points] Jackson 12.3. In part (b) I obtained

$$\frac{eE_0x}{\gamma_0 mc^2} = \cosh(\frac{eE_0y}{\gamma_0 v_0 mc}) - 1 \tag{2}$$

12.6 Hint: go to a frame where E' & B' are parallel.

The general formula is $\vec{E}' = \gamma[\vec{E} + \vec{\beta}\times\vec{B}] - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})$

$$\vec{B}' = \gamma[\vec{B} - \vec{\beta}\times\vec{E}] - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

If $\vec{E}' \parallel \vec{B}'$, $\vec{E}'\times\vec{B}' = 0$. We guess the boost velocity is $\vec{\beta} = \lambda(\vec{E}\times\vec{B})$ for some λ, find λ, show it works.

This $\vec{\beta}$ choice means $\vec{\beta}\cdot\vec{E} = 0$, $\vec{\beta}\cdot\vec{B} = 0$, no $\gamma^2/\gamma+1$ terms.

The BAC-CAB rule says

$$\vec{\beta}\times\vec{E} = -\lambda\,\vec{E}\times(\vec{E}\times\vec{B}) = -\lambda\vec{E}(\vec{E}\cdot\vec{B}) + \lambda E^2\vec{B}$$

$$\vec{\beta}\times\vec{B} = -\lambda\,\vec{B}\times(\vec{E}\times\vec{B}) = \lambda\vec{B}(\vec{E}\cdot\vec{B}) - \lambda B^2\vec{E}$$

so

$$\vec{E}' = \gamma[\vec{E}(1-\lambda B^2) + \lambda\vec{B}(\vec{E}\cdot\vec{B})]$$

$$\vec{B}' = \gamma[\vec{B}(1-\lambda E^2) + \lambda\vec{E}(\vec{E}\cdot\vec{B})],$$

that is $E' = c_1\vec{E} + c_2\vec{B}$. If $\vec{E}'$ and $\vec{B}'$ are parallel, $B' = \rho[c_1\vec{E} + c_2\vec{B}]$ for some ρ. Furthermore, the ratios of projections of E' & B' are equal along either $\vec{E}$ or $\vec{B}$.

E' & B' along E: $\quad 1-\lambda B^2 = \rho\lambda\vec{E}\cdot\vec{B} \quad$ a)

E' & B' along B $\quad \lambda E\cdot B = \rho\,(1-\lambda E^2) \quad$ b)

Take the ratios of a) & b), cross-multiply, ρ goes away,

$$(1-\lambda B^2)(1-\lambda E^2) = \lambda^2[\vec{E}\cdot\vec{B}]^2$$

$$1 - \lambda^2(E^2+B^2) = \lambda^2[(E\cdot B)^2 - E^2B^2]$$

$$= -\lambda^2[\vec{E}\times\vec{B}]^2$$

$$= -\beta^2$$

improbable vector identity...

This says $1+\beta^2 = \lambda(E^2+B^2)$ so

$$\lambda = \frac{1+\beta^2}{E^2+B^2}, \quad \beta = \lambda(\vec{E}\times\vec{B}) \text{ says}$$

$$\vec{\beta} = \left[\frac{1+\beta^2}{E^2+B^2}\right]\vec{E}\times\vec{B} \Rightarrow \boxed{\frac{\vec{\beta}}{1+\beta^2} = \frac{\vec{E}\times\vec{B}}{E^2+B^2}}$$

b) We work in the $\vec{E}' \parallel \vec{B}'$ frame and drop the primes. It's easiest to work with the proper time formulas $\frac{du^\mu}{d\tau} = \frac{e}{mc}F^{\mu\nu}u_\nu$,

$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v})$, with u^μ the 4-velocity.

Let $\vec{E}$ & $\vec{B}$ lie along $\hat{z}$. Then

$$\frac{du_0}{d\tau} = \frac{e}{mc}Eu_z \qquad \frac{du_z}{d\tau} = \frac{e}{mc}Eu_0$$

$$\frac{du_y}{d\tau} = -\frac{e}{mc}Bu_x \qquad \frac{du_x}{d\tau} = \frac{e}{mc}Bu_y$$

Introduce the cyclotron frequency $\omega = eB/mc$, the x & y equations are $\frac{d}{d\tau}(u_x+iu_y) = -i\omega(u_x+iu_y)$

so $u_x+iu_y = A'e^{-i\omega\tau}$ or $\begin{cases} u_x = A'\cos\omega\tau \\ u_y = -A'\sin\omega\tau \end{cases}$

Integrate again-

$$x(\tau) = \frac{A'}{\omega}\sin\omega\tau, \quad y(\tau) = \frac{A'}{\omega}\cos\omega\tau$$

Jackson writes $\frac{A'}{\omega} \equiv AR$, $R = \frac{c}{\omega}$, $\phi = \omega\tau$ so for him $x(\tau) = AR\sin\phi$, $y(\tau) = AR\cos\phi$.

Next, define $e = E/B$ so $\frac{eE}{mc} = e\frac{eB}{mc} = e\omega$.

$$\frac{d^2u_0}{d\tau^2} = \omega^2 e^2 u_0 \quad \text{and} \quad \frac{d^2u_z}{d\tau^2} = \omega^2 e^2 u_z \quad \text{so}$$

$$u_0 = C\cosh \omega e\tau + D\sinh \omega e\tau$$
$$u_z = E\cosh \omega e\tau + F\sinh \omega e\tau.$$

Integrating the u_z equation gives

$$z(\tau) = \frac{E}{\omega e}\sinh \omega e\tau + \frac{F}{\omega e}\cosh \omega e\tau$$
$$= \frac{F}{\omega e}\cosh \omega e\tau \quad \text{"with appropriate constants of integration"}$$

Then $\frac{du_0}{d\tau} = e\omega u_z = e\omega F\sinh \omega e\tau$

$$u_0(\tau) = F\cosh \omega e\tau.$$

We still need F. To get it, use $u_\mu u^\mu = c^2$ or

$$c^2 = u_0^2 - u_x^2 - u_y^2 - u_z^2$$
$$= F^2\cosh^2 \omega e\tau - A'^2 - F^2\sinh^2 \omega e\tau$$
$$= F^2 - A'^2$$

or $F^2 = c^2 + A'^2$.

$$F^2 = c^2 + A'^2 = c^2 + (\omega A R)^2$$

and $\omega R = c$ so $F = c\sqrt{1+A^2}$

Recall $\phi = \omega \tau$, $\omega e = \frac{c\,e}{R}$, so in

Jackson's notation

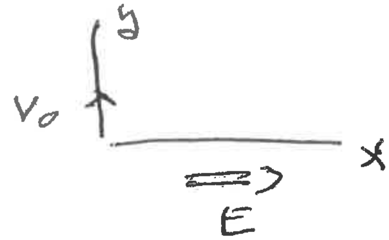
$$x(\tau) = AR \sin\phi$$
$$y(\tau) = AR \cos\phi$$
$$z(\tau) = \sqrt{1+A^2}\,\frac{R}{e}\cosh e\phi$$
$$ct(\tau) = \sqrt{1+A^2}\,\frac{R}{e}\sinh e\phi.$$

12.3 Relative motion in an E-field.

At time $t=0$ $\vec{V}_0 = \hat{y} V_0$, $\vec{E} = \hat{x} E_0$, $\vec{B}=0$.

Motion is confined to the x-y plane. We use

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m \vec{v}) = e\vec{E}$$



$$\frac{dp_y}{dt} = 0 \text{ so } p_y = \text{constant} = m\gamma_0 V_0, \quad \gamma_0 = \frac{1}{\sqrt{1-V_0^2/c^2}}.$$

$$\frac{dp_x}{dt} = eE_0 \text{ so } p_x(t) = eE_0 t.$$

The energy is $E(t) = \gamma(t) mc^2 = \left[(p(t)c)^2 + (mc^2)^2\right]^{1/2}$

or $$E(t) = mc^2\left[1 + \left(\frac{eE_0 t}{mc}\right)^2 + \left(\frac{\gamma_0 V_0}{c}\right)^2\right]^{1/2}$$

the $[\ ]^{1/2}$ is $\gamma(t)$.

Next $$\frac{V_y(t)}{c} = \frac{c\,p_y(t)}{E(t)}, \quad \frac{V_x(t)}{c} = \frac{c\,p_x(t)}{E(t)}.$$

Start with V_x. $$\frac{V_x(t)}{c} = \frac{eE_0 t}{mc}\frac{1}{\left[1 + \left(\frac{eE_0 t}{mc}\right)^2 + \left(\frac{\gamma_0 V_0}{c}\right)^2\right]^{1/2}}$$

Note $1 + \frac{\gamma_0^2 V_0^2}{c^2} = 1 + \frac{\beta_0^2}{1-\beta_0^2} = \frac{1}{1-\beta_0^2} = \gamma_0^2$ so

$$\frac{V_x(t)}{c} = \frac{eE_0 t}{mc}\frac{1}{\left[\gamma_0^2 + \left(\frac{eE_0 t}{mc}\right)^2\right]^{1/2}}.$$

This can be integrated

$$x(t) = \int_0^t v_x(t')\,dt' = \frac{eE_0}{m}\int_0^t \frac{t'\,dt'}{\left[\gamma_0^2 + \left(\frac{eE_0 t'}{mc}\right)^2\right]^{1/2}}$$

$$= \underbrace{\frac{eE_0}{m}\left(\frac{mc}{eE_0}\right)^2}_{\frac{mc^2}{eE_0}} \left[\sqrt{\gamma_0^2 + \left(\frac{eE_0 t}{mc}\right)^2} - \gamma_0\right]$$

or

$$\frac{eE_0\,x(t)}{mc^2} + \gamma_0 = \sqrt{\gamma_0^2 + \left(\frac{eE_0 t}{mc}\right)^2} \qquad (1')$$

Next,

$$y(t) = \int \frac{c^2 p_y(t')}{E(t')}\,dt' = \frac{\gamma_0 v_0 m}{m}\int_0^t \frac{dt'}{\sqrt{\gamma_0^2 + \left(\frac{eE_0 t}{mc}\right)^2}}$$

Tables give

$$y(t) = \gamma_0 v_0 \left[\frac{mc}{eE_0}\right] \sinh^{-1} \frac{eE_0 t}{\gamma_0 mc} \qquad (2)$$

b) To plot x vs y, invert Eq. 2

$$\frac{eE_0 t}{mc} = \gamma_0 \sinh \frac{eE_0 y}{\gamma_0 v_0 mc}$$

and use (1'):

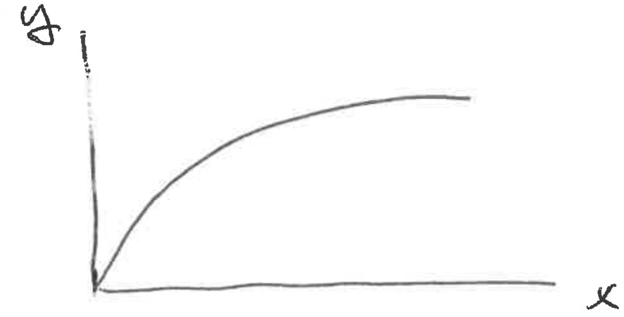
$$\left[\frac{eE_0\,x(t)}{mc^2} + \gamma_0\right]^2 = \gamma_0^2 + \left(\frac{eE_0 t}{mc}\right)^2$$

$$= \gamma_0^2\left[1 + \sinh^2 \frac{eE_0 y}{\gamma_0 v_0 mc}\right]$$

$$= \gamma_0^2 \cosh^2 \frac{eE_0 y}{\gamma_0 v_0 mc}.$$

This gives $x(t)$ vs $y(t)$ as

$$\frac{eE_0\, x(t)}{mc^2} = \gamma_0 \left[\cosh \frac{eE_0\, y(t)}{\gamma_0 v_0 mc} - 1 \right] \qquad (3)$$



Check: At very short times, when $\frac{eE_0 t}{\gamma_0 mc} \ll 1$,

(2) says $y(t) = \frac{\gamma_0 v_0 mc}{eE_0} \cdot \frac{eE_0 t}{\gamma_0 mc} = v_0 t$ –

of course!

Eq (1) is $x(t) = \frac{mc^2}{eE_0} \cdot \frac{1}{2\gamma_0} \left(\frac{eE_0 t}{mc} \right)^2$

$$= \frac{1}{2}\left(\frac{eE_0}{\gamma_0 m}\right) t^2 = \frac{1}{2} a t^2.$$

Eq (3) = $\frac{eE_0 x}{mc^2} = \gamma_0 \left(\frac{eE_0 y}{\gamma_0 v_0 mc} \right)^2$ –

parabolic motion, $x \propto y^2$.

At late times, $\frac{eE_0 t}{\gamma_0 mc} \gg 1$, eq. (1) is

$$x(t) \sim \frac{mc^2}{eE_0} \frac{eE_0 t}{mc} = ct$$

The particle has accelerated to (near) the speed of light.

Eq 2 is $\sinh \frac{eE_0 y}{\gamma_0 v_0 mc} = \frac{eE_0 t}{mc\gamma_0}$

For big X, $\sinh X \sim \frac{1}{2} e^X$

$\frac{1}{2} e^X = Y$ means $X = \log 2Y$

or $$\frac{eE_0 y(t)}{\gamma_0 v_0 mc} = \log \frac{2eE_0 t}{mc\gamma_0}$$

$$y(t) = \frac{\gamma_0 v_0 mc}{eE_0} \log \frac{2eE_0 t}{mc\gamma_0}$$

The short time $y(t) \propto t$ behavior flattens into a logarithm - this is very slow growth compared to $x(t) = ct$.