Set 1– due 26 January

"The world of our imagination is narrower and more special in its logical structure than the world of physical things." (Max Born)

1) [10 points] Jackson 9.3. In CGS, the discontinuity in $E \cdot \hat{n}$ across a conductor is $4\pi\sigma$ where σ is the surface charge density. Or, just do the problems in MKS and let us deal with the units...

2) [20 points] Consider an array of $2N + 1$ dipoles. Each has a time-dependent dipole moment $\vec{p}(t) = \hat{z}pe^{-i\omega t}$, and they are spaced a distance D apart along the $\hat{z}$ axis, centered on the origin. What is the resulting antenna pattern? Explicitly work out what happens at θ (the polar angle measured with respect to the $\hat{z}$ axis) near $\pi/2$.

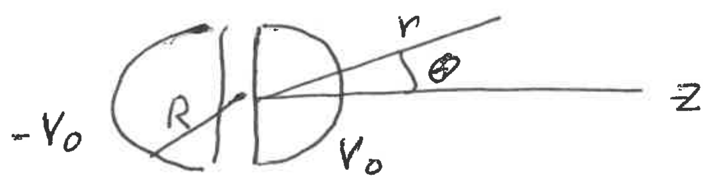
This is an iconic problem: many similar radiators distributed in space, so the antenna pattern tells you things about the spatial distribution of radiators.

3) (a) [15 points] The surface of a charge distribution of uniform density is almost a sphere: the radius as a function of the polar angle is $R(\theta) = R_0(1 + \gamma \cos\theta)$. The quantity γ oscillates in time with frequency ω, so we are describing something like surface waves on a water balloon. Working to lowest nontrivial order in the small parameter γ, and in the long wavelength limit, find an expression for the antenna pattern of emitted radiation and the total power radiated.
(b) [5 points] Part (a) is a variation of Jackson 9.12, except in that problem $R(\theta) = R_0(1 + \beta P_2(\cos\theta))$. What is the leading multipole behavior of the radiation in that case? Why?

Jackson 9.3. The potential between the hemispheres is

$$V(t) = V_0 \cos\omega t = \mathrm{Re}\, V_0 e^{-i\omega t}$$

Since these are conductors, a charge is induced on them, and there is an induced time dependent dipole moment which radiates.



$$V(r,\theta) = \sum_\ell a_\ell \left(\frac{r}{R}\right)^\ell P_\ell(\cos\theta) \quad \text{if } r<R$$

$$= \sum_\ell b_\ell \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta) \quad \text{if } r>R$$

V is continuous at $r=R$ so $a_\ell = b_\ell$. In CGS

$$4\pi\sigma = -\left.\frac{\partial V}{\partial r}\right)_{r=R^+} + \left.\frac{\partial V}{\partial r}\right)_{r=R^-} \qquad (1)$$

We only want the dipole term:

$$4\pi\sigma_D = \frac{a_1 + 2b_1}{R}\cos\theta = \frac{3a_1}{R}\cos\theta \qquad (2)$$

$$\int V(R,\theta)\, P_\ell(\cos\theta)\, d\cos\theta = \frac{2}{2\ell+1} a_\ell$$

$$V_0\left[\int_0^1 \cos\theta\, d\cos\theta - \int_{-1}^{0} \cos\theta\, d\cos\theta\right] = \frac{2}{3} a_1$$

or $V_0 = 2/3\, a_1$ or $a_1 = \frac{3}{2} V_0$ so

$$\sigma_D = \frac{3}{4\pi R} \cdot \frac{3}{2} V_0 \cos\theta$$

The dipole moment is $\vec{p} = \hat{z} \int \sigma \cdot R\cos\theta \, dA$

$$\vec{p} = \hat{z}\, \frac{9}{8\pi} V_0 \frac{R^3}{R} \int_{-1}^{1} d\cos\theta \, \cos^2\theta \cdot 2\pi$$

$$= \hat{z}\, V_0 R^2 \cdot \frac{9}{8\pi} \cdot 2\pi \cdot \frac{2}{3} = \frac{3}{2} V_0 R^2 \hat{z} \qquad (3)$$

and we are done! $\vec{B} = k^2 (\hat{n} \times \vec{p}) \frac{e^{ikr}}{r}$

$$\vec{E} = -\hat{n} \times \vec{B}$$

power $P = \frac{ck^4}{3} |\vec{p}|^2 = \frac{3}{4} c V_0^2 (kR)^4$

In MKS $\sigma_D / \epsilon_0 = \frac{3a_1}{R} \cos\theta \quad (2')$

$a_1 = \frac{3}{2} V_0$, the same as for CGS;

$\sigma_D = \frac{9}{2} \frac{V_0}{R} \epsilon_0 \cos\theta$, $\vec{p} = \hat{z} \cdot \frac{9}{2} V_0 R^2 \epsilon_0 \cdot 2\pi \cdot \frac{2}{3}$

$\vec{p} = \frac{3}{2} V_0 R^2 (4\pi\epsilon_0) \hat{z} \Rightarrow 6\pi\epsilon_0 V_0 R^2 \hat{z}$

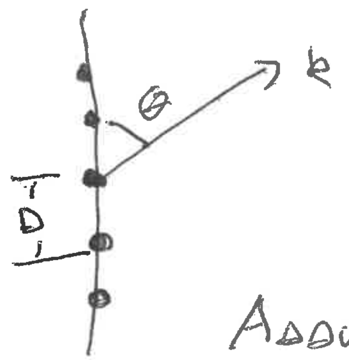
Power radiated $= \frac{c^2 Z_0 k^4}{12\pi} (6\pi\epsilon_0 V_0 R^2)^2$

$$P = 3\pi V_0^2 (kR)^4 c^2 Z_0 \epsilon_0^2 .$$

With $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$, $c^2 Z_0 \epsilon_0^2 = \frac{\epsilon_0^2}{\mu_0 \epsilon_0} \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{1}{Z_0}$

and $P = \frac{V_0^2}{Z_0} \cdot 3\pi (kR)^4$. Z_0 is a resistance, so the units are correct.

2) An array of dipoles, the jth source offset by
$\vec{r}_j = \hat{z}(jD)$ for $j = -N$ to N.



The vector potential is

$$\vec{A} = \frac{e^{ikr}}{r} \int d^3x \, J(x) \, e^{ik \cdot x}$$

Assume that J peaks near $\vec{x} = \vec{x}_j$. Write
$\vec{x} = \vec{x}_j + \vec{y}$. Assume the range of y is
much less than x_j (or Δx_j). Then

$$A_j \sim \int d^3x \, J(x) \, e^{ik_j \cdot x} = e^{ik \cdot x_j} \int d^3y \, J(y) \, e^{ik \cdot y}$$

and
$$\vec{A}_j = \left[\frac{e^{ikr}}{r} \right] \vec{p} \, e^{i\vec{k} \cdot \vec{x}_j}.$$

Here $\vec{p}$ is the individual dipole moment measured
with respect to location $\vec{x}_j$, or to $\vec{y} = 0$.

For all the dipoles

$$\vec{A} = \left[\frac{e^{ikr}}{r} \right] \hat{z} p \sum_{j=-N}^{N} e^{i\vec{k} \cdot \vec{r}_j} \equiv \frac{e^{ikr}}{r} \hat{z} p \, F(\hat{n}).$$

The antenna pattern is

$$\frac{dP}{d\Omega} = \frac{dP_0}{d\Omega} \left| F(\hat{n}) \right|^2$$ where $\frac{dP_0}{d\Omega}$ is the
antenna pattern for a single dipole, $\frac{dP_0}{d\Omega} = \frac{c k^4 p^2 \sin^2\theta}{8\pi}$.

Note the factorization – this occurs again & again dealing
with the physics of similar objects.

We need to compute the form factor:

$$F(\hat{n}) = \sum_{j=-N}^{N} \exp ijDk\,\hat{n}\cdot\hat{z} = \sum_{j=-N}^{N} \exp ijDk\cos\theta.$$

Defining $x = \exp ikD\cos\theta$,

$$F(\hat{n}) = \sum_{j=-N}^{N} x^j = x^{-N}\sum_{k=0}^{2N} x^k$$

$$= x^{-N}\left(\sum_{k=0}^{\infty} x^k - x^{2N+1}\sum_{k=0}^{\infty} x^k\right)$$

$$= x^{-N}\left[\frac{1-x^{2N+1}}{1-x}\right] = \left(\frac{x^{-N} - x^{N+1}}{1-x}\right)\frac{\sqrt{x}}{\sqrt{x}}$$

$$= \frac{x^{-N-\frac{1}{2}} - x^{N+1/2}}{\frac{1}{\sqrt{x}} - \sqrt{x}} = \frac{e^{-i(N+\frac{1}{2})kD\cos\theta} - e^{i(N+\frac{1}{2})kD\cos\theta}}{e^{-i\frac{kD}{2}\cos\theta} - e^{i\frac{kD}{2}\cos\theta}}$$

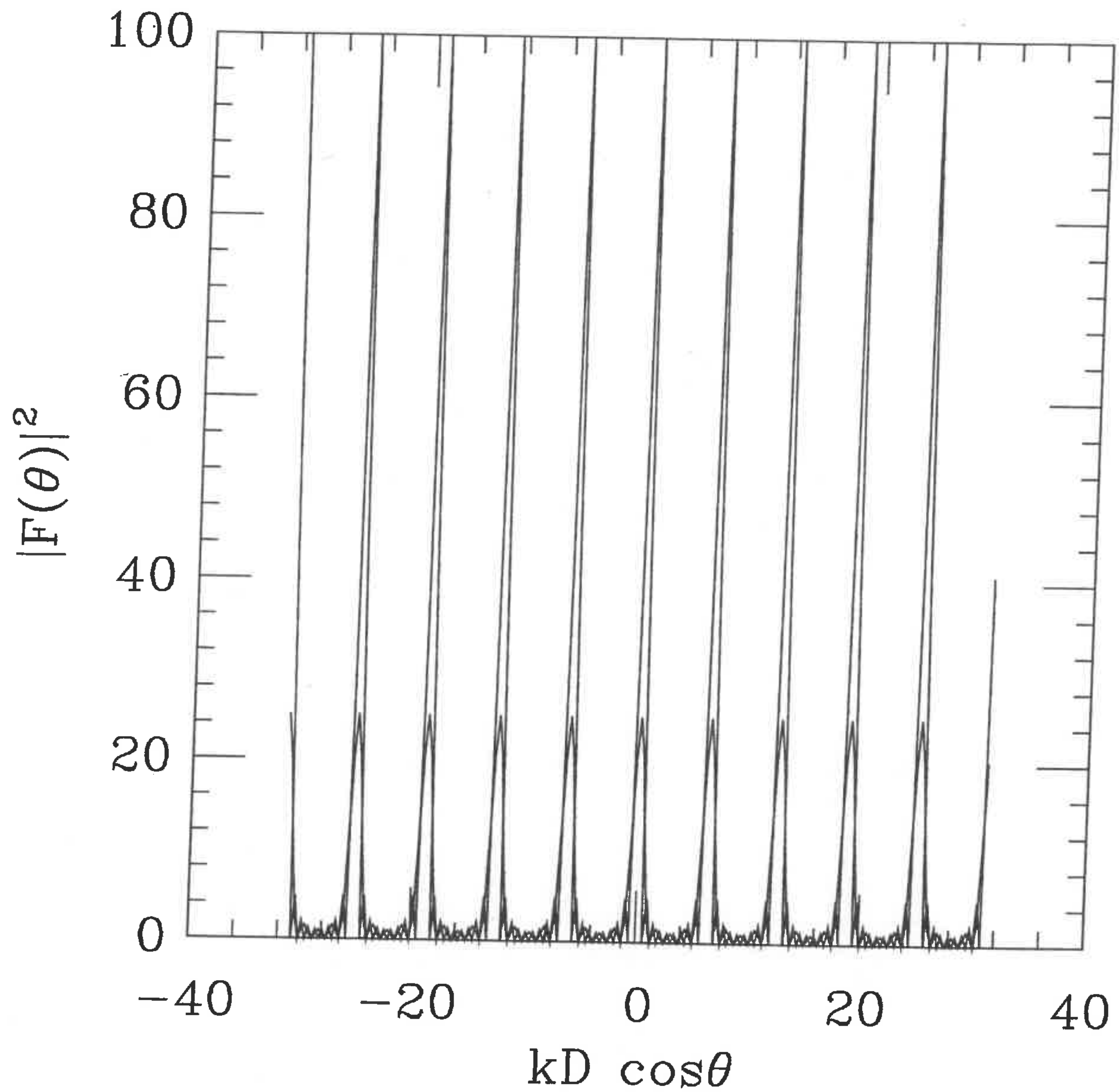
$$= \frac{\sin\left[(N+\frac{1}{2})kD\cos\theta\right]}{\sin\left[\frac{kD}{2}\cos\theta\right]}$$

(see plot on next page!)

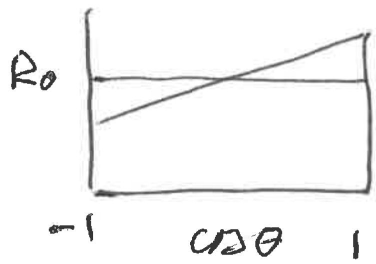
Near $\theta = \pi/2$ this is roughly $\dfrac{N+\frac{1}{2}}{\frac{1}{2}} = 2N+1$

and $\dfrac{dP}{d\Omega} = (2N+1)^2 \dfrac{dP_0}{d\Omega}$ – this is <u>coherent</u> radiation from the $2N+1$ antennas.

$2N+1 = 5 + 10$



3) $R(\theta) = R_0[1+\gamma\cos\theta]$ and $\gamma = \gamma_0 e^{-i\omega t}$



The volume of the charge is

$$\int dV = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \int_0^{R(\theta)} r^2\,dr.$$

For small γ the total charge is

$$Q = \rho_0 \int dV = \rho_0 \cdot \frac{2\pi}{3} R_0^3 \int_{-1}^{1} d\cos\theta\,[1+\gamma\cos\theta]^3$$

$$= \frac{2\pi}{3} R_0^3 \rho_0 \left[2 + 3\int_{-1}^{1} d\cos\theta\,(\gamma\cos\theta + \gamma^2\cos^2\theta) + \cdots \right]$$

$$= \frac{4\pi R_0^3}{3}\rho_0 + O(\gamma^2)$$

The lowest multipole is the electric dipole. By symmetry, $P_x = P_y = 0$ while

$$P_z = 2\pi\rho_0 \int_{-1}^{1} d\cos\theta \int_0^{R(\theta)} r^2\,dr \cdot r\cos\theta$$

$$= \frac{2\pi}{4}\rho_0 R_0^4 \int_{-1}^{1} d\cos\theta\,[1+\gamma\cos\theta]^4 \cos\theta$$

To leading order in γ,

$$P_z = \frac{2\pi}{4} R_0^4 \left[\frac{3Q}{4\pi R_0^3}\right] \cdot 4\gamma \int_{-1}^{1} \cos^2\theta\, d\cos\theta$$

$$= 2\cdot\frac{3}{4} Q R_0 \gamma \cdot \frac{2}{3} = Q R_0 \gamma.$$

This goes into the dipole formula

In CGS

$$\frac{dP}{d\Omega} = \frac{ck^4}{8\pi} |p_z|^2 \sin^2\theta$$

$$P = \frac{c}{3} k^4 |p_z|^2 = \frac{ck^4}{3} [q \ell_0 \gamma]^2$$

In MKS

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |p_z|^2 \sin^2\theta$$

$$P = \frac{c^2 Z_0}{12\pi} k^4 |p_z|^2$$

either way, $\frac{1}{P}\frac{dP}{d\Omega} = \frac{3}{8\pi} \sin^2\theta.$

b) In Jackson 9.12, $R(\theta) = 1 + \beta P_2(\cos\theta)$

and $P_2(\cos\theta) = \frac{3\cos^2\theta - 1}{2}$.

The shape remains symmetric about the equator.

This will give rise to quadrupole radiation