Set 6 – due 1 March

The midterm exam will be on Thursday evening, 14 March, place and time to be determined.

"What a pity that I have to die in the age of relativity's development." – H. Minkowski (1909)

1) [20 points] "Why, these monsters had star travel!" In Heinlein's classic novel, "Have Space Suit, Will Travel," the earth is menaced by aliens from Proxima Centauri (distance 4.3 light years), who travel in ships which accelerate for half the trip at $A = 8$ g, and decelerate for the other half. What is the elapsed time of a one-way trip from there to here, as seen by the crew, and as measured by observers on either planet?

I can think of two ways to proceed. The first one is go into the moving frame (of velocity v) and imagine an object with velocity u' and acceleration du'/dt'. In the stationary frame the object has a velocity w and an acceleration dw/dt. Lorentz transformation and velocity addition lead to

$$\frac{dw}{dt} = \frac{1}{\gamma}\frac{du'}{dt'}\frac{1 - v^2/c^2}{(1 + vu'/c^2)^3}. \tag{1}$$

In the limit $u' \to 0$, this reduces to $dv/dt = A/\gamma^3$ for this problem, and you can show $At = \gamma v$, which you can use to find $v(t)$ or $\gamma(t)$, then the total distance and time, integrating using $dt' = dt/\gamma$.

The second way is to realize that the ship's acceleration is a proper acceleration which can be boosted into the earth's frame, and then (in terms of proper time) $A^\mu = dU^\mu(\tau)/d\tau$ where $U^\mu(\tau)$ is the four velocity and τ is the proper time. Then $U^\mu(\tau) = dx^\mu(\tau)/d\tau$. You can integrate everything to get $x^\mu(\tau)$ which gives the same relations between A, the distance traveled, and the elapsed time in both frames, as the first way gives.

2) [10 points] A set of invariants which describe the scattering of unequal mass relativistic particles ($1 + 2 \to 3 + 4$) are $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$. (a) [5 points] Show $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$. (b) [5 points] Now assume all the particles have equal mass m. From (a) only two of s, t, u are independent. Draw a two dimensional plot of s vs t showing the kinematically allowed region(s)– i.e., ANY scattering experiment will populate only your allowed region. ($c = 1$ here.) The reason why this is interesting: scattering amplitudes are most usefully computed in terms of invariants, because then it is easy to evaluate them in different frames. (For example, $d\sigma/dt$ rather than $d\sigma/d\Omega$.) It is useful to know the ranges of s, t, etc, when you examine the amplitudes to look for "interesting" behavior.

3) [20 points] Two equal mass particles (of mass m) scatter elastically with a scattering angle α in the center of mass frame. Show that the scattering angle in a frame where one particle is at rest and the other has energy E is given by

$$\cos^2\theta = \frac{\cos^2\alpha/2}{1 - \frac{E-mc^2}{E+mc^2}\sin^2\frac{\alpha}{2}} \tag{2}$$

Comment on the nonrelativistic and extreme relativistic limits.