SET 14–DUE 29 APRIL

The final will be Saturday, 2 May, 1:30-4:00 PM

"The sciences do not explain, they hardly ever try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification for such a mathematical construct is solely and precisely that it is expected to work." (J. von Neumann)

1) [15 points] Find the lifetime of the 2P state (with $m_l = 0$) of hydrogen, due to electric dipole decay $2P \to 1S$. Note that this will involve similar manipulations as in Prob. 1 from last week, except that you can justify all your hard work "from first principles." Your final answer should be a number in seconds.

Useful facts:

$$\psi_{1S} = \frac{1}{(\pi a_0^3)^{1/2}} e^{-r/a_0}$$

$$\psi_{2P} = \frac{1}{8} \frac{1}{(\pi a_0^3)^{1/2}} \frac{r}{a_0} e^{-r/2a_0} \sqrt{2} \cos\theta$$

2) For one mode of the electromagnetic field, the photon field is

$$\vec{A} = \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} \vec{\epsilon}_{k\sigma} [a_{k\sigma} e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{k\sigma} e^{-i\vec{k}\cdot\vec{x}}].$$

The electric field is

$$\vec{E}(x,t) = \frac{1}{c}\frac{\partial \vec{A}}{\partial t}.$$

It is easy to show that the expectation value of $\vec{E}$ in an n-photon state, $\langle n|\vec{E}|n\rangle = 0$, while $\langle n|E^2|n\rangle = \hbar\omega(n+1/2)/V$. The interpretation is that the state has n photons, but their phases are random, so that when we average over phases, $\langle \vec{E} \rangle = 0$.

In a coherent state, $\vec{E}$ behaves more like a classical field. Let c be a complex number and define the state $|c\rangle$ by

$$|c\rangle = \sum_{n=0}^{\infty} b_n |n\rangle$$

with

$$b_n = c^n \frac{e^{-1/2|c|^2}}{\sqrt{n!}}.$$

(a) [5 points] Show that $\langle c|a|c\rangle = c$ and $\langle c|a^\dagger|c\rangle = c^*$. Then

$$\langle c|\vec{E}|c\rangle = -i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\vec{\epsilon}_{k\sigma}[ce^{i\vec{k}\cdot\vec{x}} - c^* e^{-i\vec{k}\cdot\vec{x}}]$$

has the usual form of a classical electromagnetic wave.

(b)[10 points] Show

$$\langle n\rangle = \langle c|a^\dagger a|c\rangle = |c|^2$$

$$\langle c|aa^\dagger|c\rangle = |c|^2 + 1$$

$$\langle n^2\rangle = \langle c|a^\dagger a a^\dagger a|c\rangle = |c|^4 + |c|^2$$

and $\langle c|a^2|c\rangle = c^2$ and $\langle c|a^{\dagger 2}|c\rangle = c^{*2}$. This means that $\langle n^2\rangle - \langle n\rangle^2 = \langle n\rangle$; i.e. the number of photons in the coherent state is Poisson-distributed.

(c) [5 points] Finally, show

$$\langle c|E^2|c\rangle - \langle c|\vec{E}|c\rangle^2 = \frac{2\pi\hbar\omega}{V}$$

which vanishes in the classical $\hbar \to 0$) limit.