"Spontaneous Breaking of Gauge Symmetry"

We can get a taste for the modern discussion of the photon mass (and related issues) if we diverge from pure electrodynamics to consider two related topics (which we will do by example):

Goldstone's theorem

the Higgs effect

Let's consider once again a classical field with n components

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{n} (\partial_\mu \varphi_j)(\partial^\mu \varphi_j) - V(\varphi)$$

where the "potential" is taken for simplicity to be

$$V(\varphi) = \frac{\mu^2}{2} \varphi^2 + \frac{\lambda}{4!} (\varphi^2)^2$$

and $\varphi^2 = \sum_{j=1}^{n} \varphi_j^2$

These models arise in a variety of contexts

- In condensed matter physics φ might represent the average value of the spin of a patch of a system of atoms (in a magnet). The term $V(\varphi)$ measures the energy of the patch due to self-interactions (do φ want to be large or small?) The $(\partial_\mu\varphi)(\partial^\mu\varphi)$ term measures the interaction between patches of spins, separated in space. ["Ginzberg-Landau model"]

- In macroscopic systems showing quantum behaviors: as a description of the condensate in a Bose Einstein gas or in liquid helium: here Ψ ($\equiv \varphi$) is complex, the wave-fn.

- As models for real fundamental particles - the Higgs

What does the potential term do to the field equations?

$$-\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial V}{\partial \varphi} \quad \text{provides a "force" which}$$

attempts to drive φ towards minima of V. A particularly interesting and simple case to study is the case when V has a minimum in φ, and we simply linearize the equations of motion (or expand $\mathcal{L}$ quadratically) about that minimum. Then

$$V(\varphi) = V(\varphi_0) + \frac{1}{2} V''(\varphi_0)(\varphi-\varphi_0)^2$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} V''(\varphi_0)(\varphi-\varphi_0)^2 + \text{constant } V(\varphi_0)$$

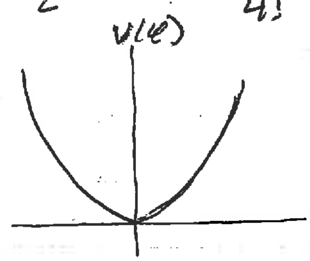
$$= \frac{1}{2}\partial_\mu(\varphi-\varphi_0)\partial^\mu(\varphi-\varphi_0) - \frac{1}{2}\underbrace{V''(\varphi_0)}_{\mu^2}(\varphi-\varphi_0)^2$$

$\boxed{\therefore \text{ mass}^2 = \text{2nd derivative of } V \text{ at minimum.}}$

— Hierarchy of examples —

<u>example</u> $\quad D=1 \quad V = \frac{1}{2}\mu_0^2 \varphi^2 + \frac{\lambda}{4!}\varphi^4$

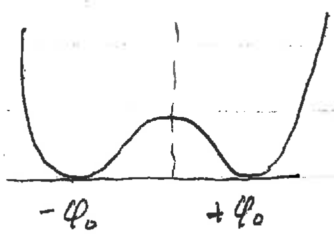
a) $\mu_0^2 > 0$



minimum is at $\varphi_0 = 0$

$$\mu^2 = \mu_0^2$$

b) $\mu_0^2 < 0 \quad \Leftarrow$ it's just a parameter $\left\{ \begin{array}{l} \text{in a magnet we might} \\ \text{parametrize it as } \mu_0^2 \propto T-T_c \\ \text{~~in a magnet } \mu_0^2 \propto T-T_c~~ \end{array}\right.$

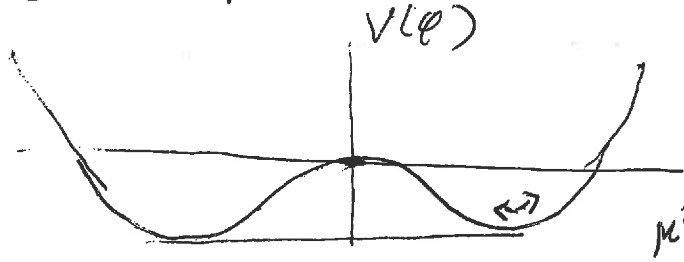


$$\left.\frac{\partial V}{\partial \varphi}\right|_{\varphi_0} = 0 = \mu_0^2 \varphi_0 + \frac{\lambda}{6}\varphi_0^3$$

$$\varphi_0^2 = \frac{-6\mu_0^2}{\lambda}$$

$$\frac{\partial^2 V}{\partial \varphi^2}\Big|_{\varphi_0} = \left[\mu_0^2 + \frac{\lambda}{2}\varphi_0^2\right] = (\mu_0^2 - 3\mu_0^2)$$
$$= -2\mu_0^2$$

(recall μ_0^2 was set < 0) $\therefore$ mass$^2 = \mu^2 = -2\mu_0^2$



μ^2 measures "rocking"

Notice that the original model was invariant under the <u>discrete</u> symmetry $\varphi(x,t) \to -\varphi(x,t)$.

the system chooses one vacuum φ_0

If the system chooses to minimize its potential by sitting in one of the minima, we say the symmetry is "broken" (by the vacuum) - and indeed if we write

$$\varphi(x,t) = \varphi_0 + \chi(x,t)$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi - V(\chi)$$

$$V(\chi) = \frac{1}{2}\mu_0^2(\chi+\varphi_0)^2 + \frac{\lambda}{4!}(\chi+\varphi_0)^4$$
$$= -\mu_0^2\chi^2 + \frac{\lambda}{6}\varphi_0\chi^3 + \frac{\lambda}{4!}\chi^4 + \text{constants.}$$

(again $\mu_0^2 < 0$) (see next page)

It's not obvious that $\chi \to -\varphi_0 - (\chi+\varphi_0)$ $(\varphi \to -\varphi)$ is a symmetry. "An ant living in a magnetized magnet has a hard time realizing that the underlying system is rotationally invariant"

<u>Word</u> "spontaneous symmetry breaking"

$$V(\chi+\varphi_0) = \frac{1}{2}\mu_0^2\left[\chi+\varphi_0\right]^2 + \frac{\lambda}{24}\left[\chi+\varphi_0\right]^4$$

$$= \frac{1}{2}\mu_0^2\left[\chi^2 + 2\chi\varphi_0 + \cdots\right]$$

$$+ \frac{\lambda}{24}\left[\chi^4 + 4\chi\phi_0^3 + 6\chi^2\phi_0^2 + \cdots\right]$$

$$= \chi^2\left[\frac{1}{2}\mu_0^2 + \frac{\lambda}{4}\phi_0^2\right] \qquad \chi^2\left(\frac{1}{2}\mu_0^2 - \frac{6}{4}\mu_0^2\right)$$

$$\left.+ \chi\left[\mu_0^2\varphi_0 + \frac{\lambda}{6}\phi_0^3\right] \longrightarrow 0 \;\right\} \doteq \chi^2\mu_0^2$$

+ higher order

$$\phi_0^2 = -\frac{6\mu_0^2}{\lambda}$$

2nd example: $j=2$, or φ is 2-dimensional

$$V(\varphi) = \frac{1}{2}\mu_0^2\left[\varphi_1^2+\varphi_2^2\right] + \frac{\lambda}{4!}\left[\varphi_1^2+\varphi_2^2\right]^2$$

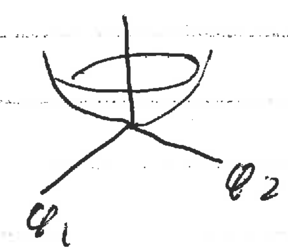
Note the symmetry

$$\begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix} = \begin{bmatrix}\cos\Omega & \sin\Omega\\ -\sin\Omega & \cos\Omega\end{bmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

or $\varphi' = R\varphi$. A global rotation of $(\varphi_1, \varphi_2$ leaves $\mathcal{L}$ invariant. There is an associated conse Noether current, as we found earlier.

~~Now if $\mu_0^2 < 0$ the potential surface looks a sombrero~~

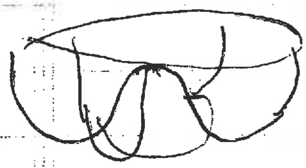
If $\mu^2>0$ the potential is concave up a minimum at $\varphi_1=\varphi_2=0$ -



$$\frac{1}{2}\frac{\partial^2 V}{\partial\varphi_1^2}\Big|_{\varphi_1=\varphi_2=0} = \frac{1}{2}\frac{\partial^2 V}{\partial\varphi_2^2}\Big|_{\varphi_1=\varphi_2=0} =$$

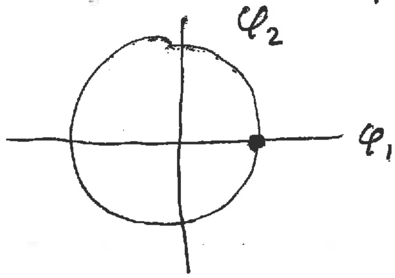
and $$\frac{1}{2}\frac{\partial^2 V}{\partial\varphi_1 \partial\varphi_2}\Big)_{\varphi_1=\varphi_2} = 0$$

But if $\mu_0^2<0$, the potential surface looks like sombrero or the bottom of a wine bottle --



Defining $\varphi_1^2+\varphi_2^2=\rho^2$, the potential has a minimum at any point on t circle $\rho^2 = -\frac{6\mu_0^2}{\lambda}$

Let's arbitrarily suppose that the vacuum chooses to break the symmetry by setting $\varphi_1 = \sqrt{-\frac{6\mu_0^2}{\lambda}}$, $\varphi_2 = 0$



$$\equiv \varphi_0$$

$$\varphi_0^2 = -\frac{6\mu_0^2}{\lambda}$$

(recall $\mu_0^2 < 0$)

What is the spectrum?

Write $\varphi_1 = \varphi_0 + \chi_1$

$\varphi_2 = \chi_2$

$$V = \frac{1}{2}\mu_0^2\left[(\varphi_0+\chi_1)^2 + \chi_2^2\right] + \frac{\lambda}{24}\left[(\varphi_0+\chi_1)^2+\chi_2^2\right]^2$$

$$= \frac{1}{2}\mu_0^2\left[\varphi_0^2 + 2\varphi_0\chi_1 + \chi_1^2 + \chi_2^2\right]$$ next page

$$+ \frac{\lambda}{24}\left[\varphi_0^2 + 2\varphi_0\chi_1 + \chi_1^2 + \chi_2^2\right]^2$$

$$= \chi_1\left[+\varphi_0\mu_0^2 + \frac{4\lambda}{24}\varphi_0^3\right]$$ ← zero: $\mu_0^2 + \frac{\lambda\varphi_0^2}{6} = 0$

$$+ \chi_1^2\left[\frac{1}{2}\mu_0^2 + \frac{\lambda}{24}(4\varphi_0^2 + 2\varphi_0^2)\right]$$ ← nonzero $\frac{1}{2}\mu_0^2 + \frac{\lambda}{4}\varphi_0^2$

$$= \frac{1}{2}\mu_0^2 - \frac{3}{2}\mu_0^2 = -\mu_0^2$$

$$+ \chi_2^2\left[\frac{1}{2}\mu_0^2 + \frac{2\lambda}{24}\varphi_0^2\right]$$ ← zero!

$$+ \chi_1\chi_2\left[0\right]$$

$+$ higher order terms

Quadratic part $= \underbrace{(-\mu_0^2)}_{\frac{m^2}{2}}\chi_1^2 + 0\cdot\chi_2^2$

While the χ_1 mode is massive
The χ_2 mode is massless. This is a general feature:

$$\frac{1}{2}\mu_0^2\left[(\varphi_0+\chi_1)^2+\chi_2^2\right]+\frac{\lambda}{24}\left[(\varphi_0+\chi_1)^2+\chi_2^2\right]^2$$

$$=\frac{1}{2}\mu_0^2\left[\varphi_0^2+2\varphi_0\chi_1+\chi_1^2+\chi_2^2\right]$$

$$+\frac{\lambda}{24}\left[\varphi_0^2+2\varphi_0\chi_1+\chi_1^2+\chi_2^2\right]^2$$

$$=\chi_1\left[\varphi_0\mu_0^2+\frac{\lambda}{24}\cdot 2\cdot 2\varphi_0\cdot\varphi_0^2\right]$$

$$+\chi_1^2\left[\frac{1}{2}\mu_0^2+\frac{\lambda}{24}(2\varphi_0^2+4\varphi_0^2)\right]$$

$$+\chi_2^2\left[\frac{1}{2}\mu_0^2+\frac{2\lambda}{24}(\varphi_0^2)\right]$$

$$+\chi_1\chi_2\cdot 0$$

+ higher order

Now $\varphi_0^2 = -\frac{6\mu_0^2}{\lambda}$

$$\chi_1\varphi_0(\mu_0^2+6\lambda\varphi_0^2) = \chi_1\, 0$$

$$+\chi_1^2\left(\frac{1}{2}\mu_0^2+\frac{\lambda}{4}\left(\frac{-6\mu_0^2}{\lambda}\right)\right) = -\frac{1}{2}\mu_0^2\chi_1^2 \text{ (no sym}$$

$$+\chi_2^2\left[\frac{1}{2}\mu_0^2+\frac{\lambda}{12}\varphi_0^2\right] = \chi_2^2\cdot 0$$

"When a global continuous symmetry is spontaneously broken, there is an accompanying massless mode"
≡ Goldstone's theorem

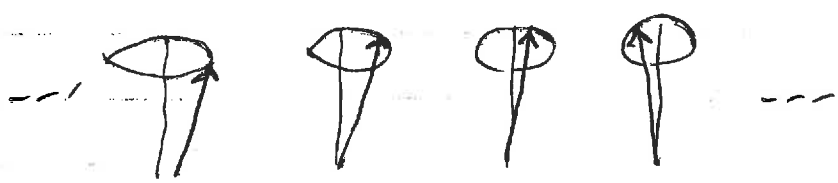
Massless ≡ $E(k) \propto k$

← rocking mode has restoring force

← rolling mode has no restoring force

Massless mode is called a "Goldstone Boson"

example: spin wave in a magnet - system can support arbitrarily long-wavelength, low energy excitations, with spins precessing around the ordered direction, ~~$E(k) \propto k$~~



This is $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with $\langle \varphi_3 \rangle = v$

Interesting group theoretic stories ...

Higgs Effect.

This is still not a massive photon. Recall that electrodynamics appeared when we tried to enforce <u>Local</u> phase rotations as a symmetry.

$$\phi(x) \to e^{i\Lambda(x)}\phi(x)$$

or
$$\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \to \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$$

Let's treat ϕ as <u>complex</u>, write

$$\mathcal{L} = [(\partial_\mu + ieA_\mu)\phi][(\partial^\mu - ieA^\mu)\phi^*] + \mu_0^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

and ~~try to~~ look at the broken symmetry case.

Note: $+\mu_0^2$ corresponds to sombrero (to avoid annoying sign. Factors and μ, λ convention of Ryder, Quantum Field Theory p.301

The minimum of V is at $|\phi| = \sqrt{\frac{\mu_0^2}{2\lambda}} \equiv \frac{a}{\sqrt{2}}$

Now write $\phi(x) = \dfrac{a + \chi_1(x) + i\chi_2(x)}{\sqrt{2}}$

i.e. move slightly off the minimizing circle and re-express in terms of the (fluctuating) physical fields χ_1 & χ_2.

Re-insert: (aside) and look only at the quadratic terms

$$\cancel{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2a^2A_\mu A^\mu + \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2}$$

$$-2\lambda a^2\phi_1^2 + \sqrt{2}\,ea\,A^\mu\partial_\mu\phi_2 + \underset{[\mathcal{O}^* A\ldots]}{\text{cubic}} + \text{quartic}$$

Lets check what we have

a) potential term

$$V(\phi) = -\mu_0^2 \left| \frac{a + \chi_1 + i\chi_2}{\sqrt{2}} \right|^2 + \frac{\lambda}{4}\left(\left| a + \chi_1 + i\chi_2 \right|^2 \right)^2$$

$$= -\frac{\mu_0^2}{2}\left[a^2 + 2a\chi_1 + \chi_1^2 + \chi_2^2 \right]$$

$$+ \frac{\lambda}{4}\left(a^2 + 2a\chi_1 + \chi_1^2 + \chi_2^2 \right)^2$$

expand, keep linear + quadratic, discard the rest

$$a^2 = \mu_0/\lambda$$

$$V(\varphi) = \chi_1 \left[-a\mu_0^2 + \frac{4a^3\lambda}{4} \right] \longrightarrow 0$$

$$+ \chi_1^2 \left[-\frac{\mu_0^2}{2} + (4a^2 + 2a^2)\frac{\lambda}{4} \right] \longrightarrow -\frac{\mu_0^2}{2} + \frac{3}{2}\lambda a^2$$

$$= \frac{\lambda a^2}{2}$$

$$+ \chi_1\chi_2 \cdot 0$$

$$+ \chi_2^2 \cdot \left[-\frac{\mu_0^2}{2} + 2a^2\frac{\lambda}{4} \right] \longrightarrow 0$$

$$V(\varphi) = \frac{\lambda a^2}{2}\chi_1^2 + 0 \cdot \chi_2^2$$

$D_\mu = \partial_\mu - ieA_\mu \quad , \quad \varphi = \frac{a + \chi_1 + i\chi_2}{\sqrt{2}}$

$$D_\mu \varphi = \frac{1}{\sqrt{2}}\Big[\partial_\mu \chi_1 - ieA_\mu \chi_1 - ieA_\mu a + i\partial_\mu \chi_2 - eA_\mu \chi_2\Big]$$

$$(D_\mu \varphi)^*(D^\mu \varphi) \doteq \frac{1}{2}\left[\partial_\mu \chi_1 \partial^\mu \chi_1 + \partial_\mu \chi_2 \partial^\mu \chi_2\right] + \frac{1}{2}e^2a^2 A_\mu A^\mu + \frac{2}{\sqrt{2}}\, ea A_\mu \partial^\mu \chi_2 + \text{non quadratic}$$

Hmm - $A_\mu A^\mu$ - photon seems to have acquired a mass! Also χ_1 is massive.

But - what is the funny $A_\mu \partial^\mu \chi_2$ term?

To make the answer less confusing, make a change of variables - a local gauge transformation

$$\phi' = \exp[i\Lambda(x)]\cdot\phi \qquad A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \Lambda$$

$$\phi' \sim [1+i\Lambda]\phi = (1+i\Lambda)\left[\frac{a+\chi_1+i\chi_2}{\sqrt{2}}\right]$$

$$= \frac{a + (\chi_1 - \Lambda\chi_2) + i(\chi_2 + \Lambda\chi_1 + a\Lambda)}{\sqrt{2}}$$

$$= \frac{a + \chi_1' + i\chi_2'}{\sqrt{2}}$$

Choose Λ so that $\chi_2' = 0$

In that gauge

$$\mathcal{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}e^2a^2A'_\mu A'^\mu$$

$$+ \frac{1}{2}\left|D'_\mu \chi_1\right|^2 - \frac{\lambda a^2}{2}\chi_1^2$$

+ non-quadratic.

Relabel A' as A, χ' as χ. We have a Lagrangian for A_μ - massive gauge boson, $M_A^2 \sim e^2a^2 = e^2\frac{\mu_0^2}{\lambda}$

χ_1 massive scaler boson - the Higgs field or particle - $M_H^2 = \lambda a^2$

this procedure is called the Higgs effect

<u>Contrast</u>

Goldstone mode: spontaneous breaking of global $U(1)$

2 massive scalar fields when symmetry unbroken $\Rightarrow$ 1 massive scalar, 1 massless scalar when broken

Higgs mode (SB of <u>local</u> $U(1)$)

2 massive scalars + 1 massless photon $\Rightarrow$ 1 massive scalar, 1 massive photon (3 pol-states)

$2 + 1 \cdot \underset{\text{states of photon}}{2\,\text{helicity}} = 4$ modes $= 1 + 1 \cdot 3\ m_J\text{'s} = 4$ modes

"the photon has eaten the scalar mode and acquired a mass"

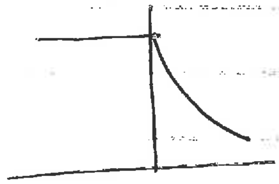
N.B. The original gauge invariance is still a symmetry of the Lagrangian, though <u>not</u> of the ~~[illegible]~~ ~~[illegible]~~ vacuum - there is still a conserved current.

Examples: ① Meissner effect in superconductor
see S. Weinberg Prog Theor Phys Suppl <u>86</u> 43 (1986)
or Ryder p.305

② W and Z mass ($U(1) \to SU(2) \times U(1)$)
More complicated group theory: 4 higgs,
3 GB's eaten $\to m_{W^\pm}, m_Z$
1 left as physical massive higgs - ~~whose~~ ~~[illegible]~~
the particle at 125 GeV

Meissner effect

Superconductivity is always accompanied by Meissner effect: E & M fields vanish, exponentially inside sc.



($\sigma = \infty$). Exponential suppression is basically mass generation for photon.

Below T_c, electron-phonon interactions (int with lattice vibrations) lead to ~~effective~~ attractive int. of electrons - system can lower its energy from usual free electron gas by forming bound state (Cooper pairs)

$\Psi_s(r) \equiv$ wave fn of SC state

$$|\Psi_s(r)|^2 = n_{pairs}(r) = \frac{n_s(r)}{2} \quad \text{for electrons in SC state}$$

at $T > T_c$ $n_s = 0$. Write a free energy function of SC in terms of Ψ, free energy density is function of Ψ_s, $\nabla\Psi_s$. Ψ_s is a charged field (with charge $e^* = 2e$, mass $m^* = 2m$) so F depends on $|\Psi|^2$ and canonical momentum

$$\frac{\hbar}{i}\nabla \to \frac{\hbar}{i}\nabla - \frac{e^* A}{c}$$

$$F(\Psi_s) = F_{normal}(T,0) + \int d^3r \, \frac{B(r)^2}{8\pi}$$

$$+ \int d^3r \left[a|\Psi_s|^2 + \frac{b}{2}|\Psi_s|^4 + \cdots \right]$$

$$+ \int d^3r \left[\frac{1}{2m^*} \left| \left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}(r) \right) \Psi_s(r) \right|^2 + \cdots \right.$$

and $B = \nabla \times A$. We want to minimize F by varying Ψ. This is a "typical" variational problem – it should be no surprise that the min of F happens ~~at~~ if eqns of motion are satisfied, namely

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}_s \quad \text{where} \qquad 1)$$

$$\vec{J}_s = \frac{e^* \hbar}{2m^* i}\left(\psi_s^* \vec{\nabla}\psi - (\vec{\nabla}\psi_s^*)\psi\right) \qquad 2)$$

$$- \frac{e^2}{m^* c} |\psi_s|^2 \vec{A}(r)$$

diff F wrt A

$$\left\{ \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}A\right)^2 + b|\psi_s|^2 \right\} \psi_s = -a\psi_s \qquad 3)$$

Notice the 2nd term in the current; It comes from

$$\frac{\partial \mathcal{F}}{\partial A_\mu} = -J_\mu$$

where $\mathcal{F}$ is the free energy density. 1) + 3) are

$$0 = \partial_\mu \frac{\partial \mathcal{F}}{\partial^\mu A_\nu} - \frac{\partial \mathcal{F}}{\partial A_\nu} = 0$$

in (3), $-a$ is "like" an energy eigenvalue and the $b|\psi_s|^2$ is "like" a repulsive self interaction which tries to spread Ψ_s over the whole volume.

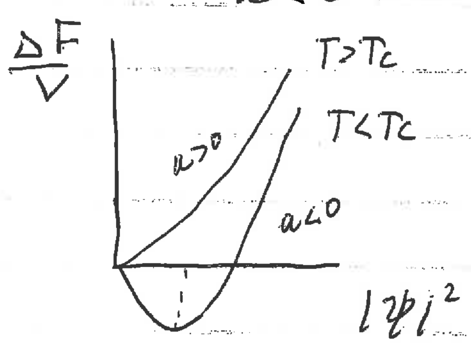
Now for solutions: deep inside the sc, $\vec{B}=0$, n_s is a constant. Then the free energy shift from the normal state is

$$\Delta F = F_s - F_N = V\cdot\left(a|\psi_s|^2 + \frac{b}{2}|\psi_s|^4\right)$$

with $\left(a + b|\psi_s|^2\right)\Psi_s = 0$

$b>0$ so if $a>0$, then $\Psi_s = 0$, the system is in its normal state

but if $T<T_c$, to get a nontrivial ground state, $a<0$



$$|\psi_s|^2 = \frac{n_s}{2} = -\frac{a}{b} > 0$$

$$\frac{\Delta F}{V} = \frac{-a^2}{2b} \text{ is } < 0$$

Now imagine adding a small magnetic field B. The sc current is (ψ_s is constant in space)

$$\vec{J}_s = \frac{-e^{*2}}{m^*c}|\psi_s|^2\vec{A} = -\frac{e^2 n_s}{m^*c}\vec{A}$$

so $\vec{\nabla}\times\vec{J} = \frac{-e^2 n_s}{m^*c}\vec{B}$

Now $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$, take curl again

$$\nabla\times(\nabla\times B) = \nabla(\cancel{\nabla\cdot B})^{0} - \nabla^2 B = \frac{4\pi}{c}\nabla\times J = \frac{-4\pi n_s e^2}{m^* c^2}\vec{B}$$

$$\nabla^2\vec{B} = \frac{1}{\lambda^2}\vec{B}, \quad \lambda = \sqrt{\frac{m^*c^2}{4\pi n_s e^2}}$$

ie - a magnetic field is screened.

λ is called the "London penetration depth"
but you can see it's basically a photon mass.