Noether's theorem

Classical mechanics for a set of classical variables q_i & $\dot{q}_i$ (defined for all t)
begins with ~~an action~~ a Lagrangian L and an action S

$$S = \int dt \, L\left(q_i, \frac{dq_i}{dt}\right)$$

and $\delta S = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

Classical mechanics for ~~fields~~ classical fields begins with the fields themselves, defined for all points in space x & time t - generically $\varphi(x,t)$ and their derivatives $\frac{\partial \varphi}{\partial x_j}, \frac{\partial \varphi}{\partial t}$: generically, $\partial_\mu \varphi$.

The analog of L is the Lagrange density $\mathcal{L}$

$$L = \int \mathcal{L} \, d^3x$$

and the action is ($dt \cdot d^3x = d^4x$)

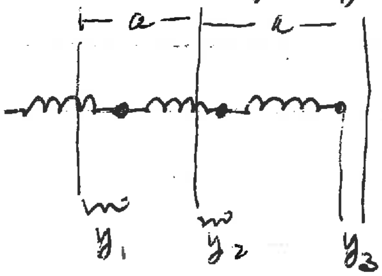
$$S = \int d^4x \, \mathcal{L}(\varphi_j, \partial_\mu \varphi_j)$$

The bottom line, if $\delta S = 0$:

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_j)} \right] - \frac{\partial \mathcal{L}}{\partial \varphi_j} = 0$$

is our starting point. Lets do an example to refresh memory:

1-d Lagrangian for a set of mass points, springs



$y_j = y(x_j, t)$ = displacement of jth mass from equilibrium

$$L = \sum_{j=1}^{N} \frac{m}{2}\dot{y}_j - \frac{1}{2}k(y_j - y_{j+1})^2 - \tilde{V}(y_j)$$

eqn of motion: $m\ddot{y}_j - k\left[y_{j-1} - 2y_j + y_{j+1}\right] - \frac{\partial \tilde{V}}{\partial y_j} = 0$

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_j} - \frac{\partial L}{\partial y_j} = 0\right)$$

Now suppose y varies smoothly with x:

$$y_{j+1} = y_j + a\frac{dy}{dx}\bigg|_{x=x_j} + \frac{a^2}{2}\frac{d^2y}{dx^2}\bigg| + \cdots$$

$$m\ddot{y} + ka^2\frac{d^2y}{dx^2} + \frac{\partial \tilde{V}}{\partial y_j} = 0$$

$$= \cancel{\frac{a}{a}\frac{m}{ka^2}}\left[\frac{m}{ka^2}\frac{d^2y}{dt^2} - \frac{d^2y}{dx^2} - \frac{\frac{1}{(ka^2)}\partial\tilde{V}}{\partial\left(\frac{y}{ka^2}\right)}\right] ka^2$$

$$\varphi(x,t) = \frac{y}{ka^2}, \quad \frac{1}{c^2} \equiv \frac{m}{ka^2}, \quad V = \frac{\tilde{V}}{ka^2}$$

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial V}{\partial \varphi} = 0$$

$$\partial_\mu\partial^\mu\varphi - \frac{\partial V}{\partial\varphi} = 0$$

$$= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right] - \frac{\partial\mathcal{L}}{\partial\varphi}, \quad \mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi)$$

Now back to $\mathcal{L}$. Suppose we imagine only a change in the field variables

$$\varphi(x,t) \Rightarrow \varphi(x,t) + \delta\varphi(x,t) \qquad (1)$$

and at the same time

$$\partial_\mu \varphi(x,t) \Rightarrow \partial_\mu \varphi(x,t) + \partial_\mu \delta\varphi(x,t)$$
$$= \partial_\mu \varphi(x,t) + \delta\left[\partial_\mu \varphi(x,t)\right]$$

~~These are called "global" symmetry transformations when $\delta\varphi$ is independent of x, t~~

: The change in $\mathcal{L}$ is

$$\delta\mathcal{L} = \sum_j \frac{\partial \mathcal{L}}{\partial \varphi_j}\delta\varphi_j + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_j)}\delta(\partial_\mu \varphi_j)$$

Recall the equation of motion $\dfrac{\partial \mathcal{L}}{\partial \varphi_j} = \partial_\mu\left[\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_j)}\right]$

so

$$\delta\mathcal{L} = \sum_j \left[\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_j)}\right)\right]\delta\varphi_j + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_j)}\,\partial_\mu(\delta\varphi_j)$$

$$= \partial_\mu\left[\sum_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_j)}\delta\varphi_j\right]$$

$$\equiv \partial_\mu J^\mu \quad \text{defining } [\;] \text{ as a current.}$$

Now if it happens that $\delta\mathcal{L} = 0$, the Lagrangian is invariant under the change of φ's

then $\partial_\mu J^\mu = 0$

That is, J^μ is a conserved current.

Symmetry implies conservation law
The "charge" associated with the current is also conserved

$$Q \equiv \int d^3x\, J_0$$

Changes in φ could be "internal" - redefine φ at every space-time point.

ex: $\varphi(x,t) \to e^{i\theta(x,t)}\varphi(x,t)$

$\theta(x,t)$ = constant: ~~[illegible]~~ in x,t
example of a global symmetry transf.

or $\theta(x,t)$ varies w/ x,t: "local symmetry transformation"

or $\delta\varphi$ could involve changes in coordinates.
Latter ones lead to "~~[illegible]~~ conserved currents" associated with energy & momentum.

Internal ones are more interesting for the moment

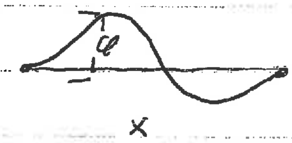
~~Crucial point: global symmetry transf"~~

~~S...~~

$\cancel{\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right] - \frac{\partial \mathcal{L}}{\partial \varphi} = 0}$

Example

Let us suppose for the sake of argument that we have a classical physical system described by a complex field $\varphi(x) = \varphi_1(x) + i\,\varphi_2(x)$ (or we could think of as a pair of real fields). A Boulder-style realization of φ could be the QM wave function of a Bose-Einstein condensate, ~~or the other possible~~ be it of atoms or of liquid helium. Suppose that its Lagrange density does not have any explicit dependence on space and time - instead, that $\mathcal{L}$ only depends on x and t through φ and $\partial_\mu \varphi$.* That would be the case for a stretched "string" where $\varphi(x,t)$ is the local displacement of the string, and you recall [$c=1$]

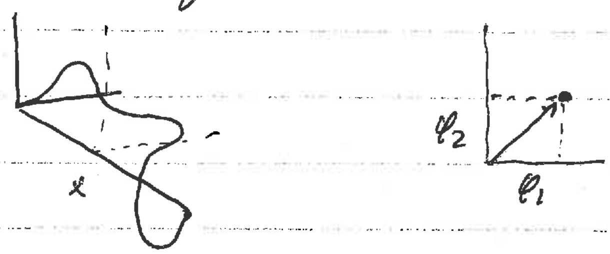


$$\mathcal{L} = \left(\frac{\partial \varphi}{\partial t}\right)^2 - (\nabla \varphi)^2$$

$$\equiv (\partial_\mu \varphi)^2$$

A complex field would have a pair of φ's

so imagine in 1-d



~~slice at one (x, t)~~

* turns out, to construct relativistically covariant theories, you have to do this

Now suppose that we imagine making a ~~smooth~~ global ~~change in the~~ rotation in the field variables

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

or $\quad \varphi' = e^{i\theta}\varphi \quad$ if we call $\varphi' \equiv (\varphi_1 + i\varphi_2)$

and suppose that under this change the $\mathcal{L}$ is invariant. For the case of the string, this is just a rotation in the field space (φ_1, φ_2).

Lets suppose that θ is infenitesimal, $\theta = g\epsilon$

$$\varphi' = \varphi + \delta\varphi$$

$$\delta\varphi = i g \epsilon \varphi$$

ϵ is the same everywhere in space. How does $\mathcal{L}$ change?

~~$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)$~~

~~but $\delta\varphi = ig\epsilon\varphi$~~

~~$\delta(\partial_\mu\varphi) = ig\epsilon\,\partial_\mu\varphi$~~

~~$\delta\mathcal{L} = ig\epsilon\left[\frac{\partial\mathcal{L}}{\partial\varphi}\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\varphi\right]$~~

~~But use the equation of motion~~

~~$\frac{\partial\mathcal{L}}{\partial\varphi} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right]$~~

and

$$\delta\mathcal{L} = ig\epsilon\left[\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right]\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\varphi\right]$$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2 \right] - V(\varphi_1, \varphi_2)$$

$$\begin{bmatrix} \varphi_1' \\ \varphi_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \underset{\text{small } \theta}{\sim} \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

i.e. $\delta\varphi_1 = \theta\varphi_2$; $\delta\partial_\mu\varphi_1 = \theta\partial_\mu\varphi_2$

$\delta\varphi_2 = -\theta\varphi_1$ $\quad\delta\partial_\mu\varphi_2 = -\theta\partial_\mu\varphi_1$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_1)}\delta(\partial_\mu\varphi_1) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_2)}\delta(\partial_\mu\varphi_2) + \underbrace{\frac{\partial\mathcal{L}}{\partial\varphi_1}\delta\varphi_1 + \frac{\partial\mathcal{L}}{\partial\varphi_2}\delta\varphi_2}$$

Suppose $V(\varphi_1, \varphi_2)$ is a function only of $\varphi_1^2 + \varphi_2^2$ – obviously, "potential" part of $\mathcal{L}$ is unchanged under this transf- and last terms give zero. Kinetic term

$$\delta\mathcal{L} = (\partial^\mu\varphi_1)[\theta\partial_\mu\varphi_2] + (\partial^\mu\varphi_2)[-\theta\partial_\mu\varphi_1]$$

$= 0$, too!

Yes, it's a symmetry. What is the associated conserved current?

$$J^\mu = \sum_j \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_j)}\delta\varphi_j = (\partial^\mu\varphi_1)\theta\varphi_2 + (\partial^\mu\varphi_2)(-\theta\varphi_1)$$

$$= (\varphi_2\partial^\mu\varphi_1 - \varphi_1\partial^\mu\varphi_2)$$ up to an overall constant.

A lot like Schrodinger eqn ...

So far we considered a variation of ϕ

$$\delta\phi = iq\epsilon\phi$$

with ϵ a constant over space, and we saw that if $\mathcal{L}(\phi') = \mathcal{L}(\phi)$, with $\phi' = \phi + \delta\phi$, then the system has a conserved current. What if we now let ϵ vary from point to point in space? That is, suppose that we asked for the following transformation to be a symmetry:

$$\delta\phi(x) = iq\epsilon(x)\phi(x) \qquad \text{infinitesimal}$$

or $\qquad \phi(x') = \exp iq\epsilon(x)\,\phi(x) \qquad$ finite

and want $\delta\mathcal{L} = 0$

We know that the variation in $\mathcal{L}$ is still

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)$$

but now $\delta(\partial_\mu\varphi) = \partial_\mu\delta\varphi$

$$= \underbrace{iq\epsilon\,\partial_\mu\varphi}_{\text{old term}} + \underbrace{iq\phi\partial_\mu\epsilon}_{\text{new term!}}$$

the equations of motion

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial\mathcal{L}}{\partial\varphi}$$

again allow us to write

$$\delta\mathcal{L} = \epsilon(x)\,\partial_\mu J^\mu(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}iq\varphi\,\partial_\mu\epsilon$$

with $J^\mu = iq\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\varphi$ is the same current we had before - it is conserved

$$\delta\mathcal{L} = \epsilon(x)\,\overset{\nearrow 0}{\partial_\mu J^\mu(x)} + J^\mu(x)\,\partial_\mu\epsilon(x)$$

What about the second term?

The only way to make it zero is to introduce new ~~two~~ fields into $\mathcal{L}$, whose variation cancels the $\partial_\mu \epsilon$ term. The current is a 4-vector, so let's add in a vector field $A_\mu(x)$, whose variation is

$$\delta A_\mu(x) = \partial_\mu \epsilon(x)$$

i.e. $$\vec{A}(x,t)' = \vec{A}(x) + \vec{\nabla}\epsilon$$

$$A_0(x,t)' = A_0(x,t) + \frac{\partial \epsilon}{c\,\partial t}$$

Now $$\delta \mathcal{L}' = \delta \mathcal{L}_{matter} + \underbrace{\frac{\partial \mathcal{L}}{\partial A_\lambda}\delta A_\lambda} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)}\partial_\nu \partial_\lambda \epsilon$$

$$\delta \mathcal{L}' = \underset{(\partial_\lambda \epsilon) J_\lambda}{\delta \mathcal{L}} + \frac{\partial \mathcal{L}}{\partial A_\lambda}\partial_\lambda \epsilon + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)}\partial_\nu \partial_\lambda \epsilon$$

This is zero as long as

$$\frac{\partial \mathcal{L}}{\partial A_\lambda} = -J_\lambda \quad \longrightarrow$$ unique specification of coupling of new field to the conserved current

$$\mathcal{L} = \mathcal{L} - J_\mu A^\mu]$$

and $$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)} = -\frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\nu)} \longrightarrow$$ so that the $\partial\, \partial_\nu \partial_\lambda \epsilon$ and $\partial_\lambda \partial_\nu \epsilon$ bits cancel

says $\mathcal{L}$ involves $\partial_\mu A_\nu - \partial_\nu A_\mu$.

The new condition does not give conservation laws, but instead conditions on the couplings of fields.

1) Suppose a global symmetry

$$\delta\varphi \sim i\epsilon\varphi$$

leaves $\mathcal{L}$ invariant. There is a conserved current

$$\partial_\mu J^\mu = 0$$

2) replace global symmetry by local symmetry

$$\delta\varphi(x,t) \sim i\epsilon(x,t)\,\varphi(x,t)$$

Force $\delta\mathcal{L} = 0$ discover

a) need new gauge field, a 4-vector $A_\mu(x)$

Coupling b) $\mathcal{L}_I = -J^\mu A_\mu$ – the same ~~...~~ conserved current as in (1)

c) $\mathcal{L}$ includes $\partial_\mu A_\nu - \partial_\nu A_\mu \equiv F_{\mu\nu}$.

Symmetry transformation is

$$\varphi'(x) = \exp i q \epsilon(x)\, \phi(x)$$

$$A'_\mu(x) = A_\mu + \partial_\mu \epsilon(x)$$

A way to encode invariance of this transformation in $\mathcal{L}$ is to replace the usual derivative

$$\partial_\mu \varphi$$

by the covariant derivative

$$D_\mu \varphi = \partial_\mu \varphi - i q A_\mu \varphi$$

and to write $\mathcal{L}$ as a function of $(D_\mu \varphi)^* D^\mu \varphi$, not

$$\partial_\mu \varphi\, \partial^\mu \varphi$$

$$D'_\mu \varphi' = \partial_\mu \varphi' - i q A'_\mu \varphi' = \partial_\mu \left[e^{iq\epsilon} \phi \right] - i q \left[A_\mu + \partial_\mu \epsilon \right] e^{iq\epsilon} \phi$$

$$= e^{iq\epsilon} \left(\partial_\mu \varphi + i q \varphi \partial_\mu \epsilon \right) - i q A_\mu \varphi e^{iq\epsilon} - i q \varphi\, \partial_\mu \epsilon\, e^{iq\epsilon} \phi$$

$$= e^{iq\epsilon} \left[\partial_\mu \varphi - i q A_\mu \varphi \right] = e^{iq\epsilon} D_\mu \varphi$$

$$\mathcal{L} = (D'_\mu \varphi')^* (D'_\mu \varphi) - V(|\varphi'|^2)$$

$$= e^{-iq\epsilon} e^{iq\epsilon} (D_\mu \varphi)^* (D_\mu \varphi) - V(|\phi|^2)$$

is invariant

$$|\varphi'|^2 = |e^{iq\epsilon} \varphi|^2 \text{ of course}$$

And finally, what would be a good candidate $\mathcal{L}$ for the new electromagnetic degrees of freedom?

To get a world scalar, contract indices made of tensors

$$\mathcal{L} = c_1 \left[F_{\mu\nu}^2 \right] + c_2 \left[F_{\mu\nu}^2 \right]^2 + \cdots$$

$$F_{\mu\nu}^2 = F_{\mu\nu} F^{\mu\nu}$$

and indeed, the Lagrange density which produces ~~classical~~ Maxwell's equations is "the simplest thing you can write down"

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu \;+\; \text{terms independent of } A.$$

obviously rel covariant made of 4 vectors by construction

The coefficients and signs are chosen arbitrarily so that the field equations have CGS conventions.

— Why isn't there more? Lets defer that for a while. Lets first show how this $\mathcal{L}$ produces Maxwell's equations as its equations of motion.

Write

$$\mathcal{L} = -\frac{1}{16\pi} g_{\lambda\kappa} g_{\nu\sigma} \left[\partial^\kappa A^\sigma - \partial^\sigma A^\kappa \right]\left[\partial^\lambda A^\nu - \partial^\nu A^\lambda \right] - \frac{1}{c} J_\alpha A^\alpha$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{16\pi} g_{\lambda\kappa} g_{\nu\sigma} \Big[\delta_\beta{}^\kappa \delta_\alpha{}^\sigma F^{\lambda\nu} - \delta_\beta{}^\sigma \delta_\alpha{}^\kappa F^{\lambda\nu} + \delta_\beta{}^\lambda \delta_\alpha{}^\nu F^{\kappa\sigma} - \delta_\beta{}^\nu \delta_\alpha{}^\lambda F^{\kappa\sigma} \Big]$$

$$= \frac{-4}{16\pi} F_{\beta\alpha} = \frac{1}{4\pi} F_{\alpha\beta}$$

we need $\displaystyle \partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} - \frac{\partial \mathcal{L}}{\partial A^\alpha} = 0$

The ~~[illegible]~~ transformation

$$A'_\mu = A_\mu + \partial_\mu \epsilon(x) \qquad (**)$$

is called a local gauge transformation. Theories which are invariant under ~~gauge~~ local gauge transformations are called "gauge theories" and include

Electrodynamics

and (actually under a matrix generalization of

$$\phi(x) \to e^{ig\epsilon(x)}\phi(x) \Rightarrow \phi \to R\phi \quad)$$

QCD (strong interactions)

Weinberg-Salem model (weak + electromagnetic interactions).

i.e. all nature (gauge invariance for GR = coordinate reparametrization

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How to satisfy local gauge invariance?

$$\mathcal{L} = \mathcal{L}(\varphi, D_\mu \varphi, F_{\mu\nu})$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D_\mu \varphi = \partial_\mu \varphi - i g A_\mu \varphi \qquad \text{covariant derivative}$$

~~for then,~~

$$\cancel{\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{\partial \mathcal{L}}{\partial (D_\mu \varphi)}(-ig\varphi)}$$

$$\cancel{= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)}(-ig\varphi) = -J_\mu}$$

$$\cancel{\delta \mathcal{L}' = \partial_\mu \epsilon \, J_\mu - J_\mu (\partial_\mu \epsilon) = 0}$$

$$\cancel{\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta(\partial_\mu \varphi) + \frac{\partial \mathcal{L}}{\partial A_\mu} \delta A_\mu}$$

And $$\frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c} J_\alpha$$

or $$\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c} J_\alpha$$

Now the field strength tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ is nothing more than the E and B fields:

$$\vec{B} = \vec{\nabla}\times\vec{A} = \epsilon_{ijk}\,\partial_j A_k$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \qquad E_i = -\partial_i A_0 - \partial_0 A_i$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & +B_z & 0 & -B_x \\ -E_z & -B_y & +B_x & 0 \end{bmatrix}$$

so the inhomogeneous field equation is just $(J^\kappa = (c\rho, \vec{J}))$

$\alpha = 0 \quad \nabla\cdot E = 4\pi\rho$

$\alpha = i \quad \nabla\times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$ 2 of 4 maxwell eqns!

The homogeneous equations vanish by construction
$(\nabla\cdot B = 0 \to B = \nabla\times A)$

~~[illegible]~~ In covariant ~~constant~~ language they are

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0 \qquad \alpha \neq \beta \neq \gamma$$

or $\partial_\alpha \mathcal{F}^{\alpha\beta} = 0$

where $\mathcal{F}^{\alpha\beta} \equiv$ "dual field strength tensor"

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ & 0 & E_z & -E_y \\ & & 0 & E_x \\ & & & 0 \end{bmatrix}$$

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\partial_\alpha F_{\mu\nu}$$

$$= \frac{1}{2}\epsilon^{\beta\mu\alpha\nu}\partial_\alpha(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$= 0$ because for each term, ϵ is antisymmetric

While we're at it, lets think about the Lagrangian ~~&~~ Hamiltonian.

$$L = \int d^3x\, \mathcal{L}$$

$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$: look at 4x4 tensors, sum up

$$\mathcal{L} = \frac{1}{8\pi}(E^2 - B^2)$$

Hamiltonian? Write $\mathcal{L} = -\frac{1}{8\pi}\left[\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right]\frac{\partial A^\mu}{\partial x^\nu}$

Need canonical coordinates – obvious choice the 4 A_μ's

Canonical momenta?

$k = 1,2,3$ $\quad \pi^k = \frac{\partial \mathcal{L}}{\partial \dot{A}_k} = -\frac{1}{8\pi}\times 2\cdot\left[\frac{\partial A_k}{\partial x_0} - \frac{\partial A_0}{\partial x^k}\right] = \frac{E_k}{4\pi}$

$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \implies \mathcal{H} = \sum_{k=1}^{3}\pi^k \dot{A}_k - \mathcal{L} = \frac{1}{4\pi}\left(\frac{1}{2}(E^2+B^2) + \vec{E}\cdot\vec{\nabla}\Phi\right)$

At this point things get a bit tricky.
Note (first of all) that there is no momentum conjugate to A_0: A_0 has <u>no</u> independent dynamics

Second, in the absence of sources, Maxwell's equations (which we could as well derive as a set of Hamilton equations) require

$$\vec{\nabla} \cdot \vec{E} = 0$$

so the 3 E's are not independent

The system has constraints. They are related to gauge symmetry; not all the A's are independent variables.
To the best of my knowledge, this is not a problem for the classical theory, but it is a problem in constructing the quantum theory using "canonical quantization":

1) write down classical $\mathcal{L}$
2) construct classical H; identify p, q
3) Impose canonical quantization conditions
$$[q_i, p_j] = i\hbar\delta_{ij}$$
4) Construct ~~classical~~ quantum H by replacing coordinates & momenta by coord & mom. <u>operators</u>.

To proceed beyond point 3) it is necessary to fix a gauge ~~(in which eigenfunctions)~~ even

to have energy eigenfunctions →

and in order to make H time independent the gauge must be noncovariant, like Coulomb gauge. But then the formulas all look noncovariant (there is an instantaneous Coulomb interaction, for example) and then it is a lot of work to show that the <u>quantum</u> theory is covariant: all the noncovariant parts cancel.

Anyway, back to Lagrangians. The A_μ's form a four vector and transform under Lorentz transformations like any 4-vector. In matrix notation where the L.T. is

$$\Lambda = \exp(i\omega\cdot S - \zeta K)$$

$$\begin{pmatrix} A_0' \\ \vec{A}' \end{pmatrix} = \bar{\Lambda} \begin{pmatrix} A_0 \\ \vec{A} \end{pmatrix}$$

E and B are elements of the second rank tensor $F^{\mu\nu}$. It transforms as

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} F^{\mu\nu}$$

or $$F' = \bar{\Lambda} F \bar{\Lambda}^T \quad \text{(matrix notation)}$$

Let's multiply out for a boost along the x or 1 axis:

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_1' = E_1 \qquad B_1' = B_1$$

$$E_2' = \gamma(E_2 - \beta B_3) \qquad B_2' = \gamma(B_2 + \beta E_3)$$

$$E_3' = \gamma(E_3 + \beta B_1) \qquad B_3' = \gamma(B_3 - \beta E_2)$$

this I checked

$$\gamma - \frac{\gamma^2\beta^2}{\gamma+1} = \frac{\gamma^2+\gamma-\gamma^2\beta^2}{\gamma+1} = \frac{\gamma^2(1-\beta^2)+\gamma}{1+\gamma} = \frac{1+\gamma}{1+\gamma}$$

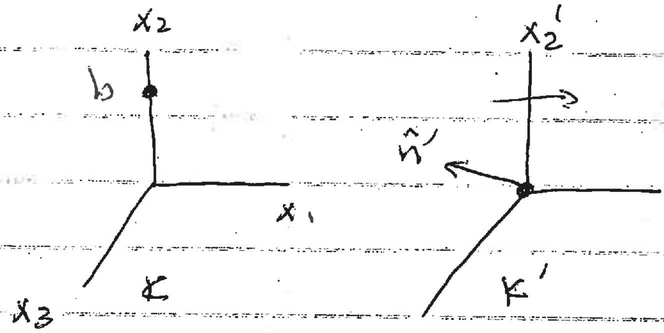
General result

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})$$ ← This I

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$ ← did not.

Example: point charge at origin in frame K'; point charge moving with $\vec{v} = +\hat{x}v$ in frame K

In K, observer is at $(x,y,z) = (0,b,0)$ but



in K', observer is at $(x',y',z') = (-vt', b, 0)$

Also in K', $\vec{B}' = 0$, $\vec{E}' = q\frac{\hat{n}'}{(r')^2}$

or $E_1' = -\frac{qvt'}{(r')^3}$ $\quad E_2' = -\frac{qb}{(r')^3}$ $\quad E_3' = 0$

and $(r')^2 = b^2 + v^2t'^2$.

We have to convert both the coordinates and the fields. To convert the coords, $t' = \gamma t$ so

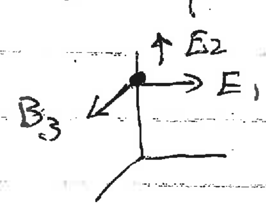
$$E_1' = \frac{q(-\gamma vt)}{[b^2 + (\gamma vt)^2]^{3/2}} \qquad E_2' = \frac{qb}{[b^2 + (\gamma vt)^2]^{3/2}}$$

Now we transform the fields

$$E_1 = E_1' = -\frac{q\gamma vt}{[b^2 + (\gamma vt)^2]^{3/2}}, \quad E_2 = \frac{qb\gamma}{[b^2 + (\gamma vt)^2]^{3/2}}$$

$(= \gamma E_2')$

$$B_3 = \beta\gamma E_2' = \beta E_2$$

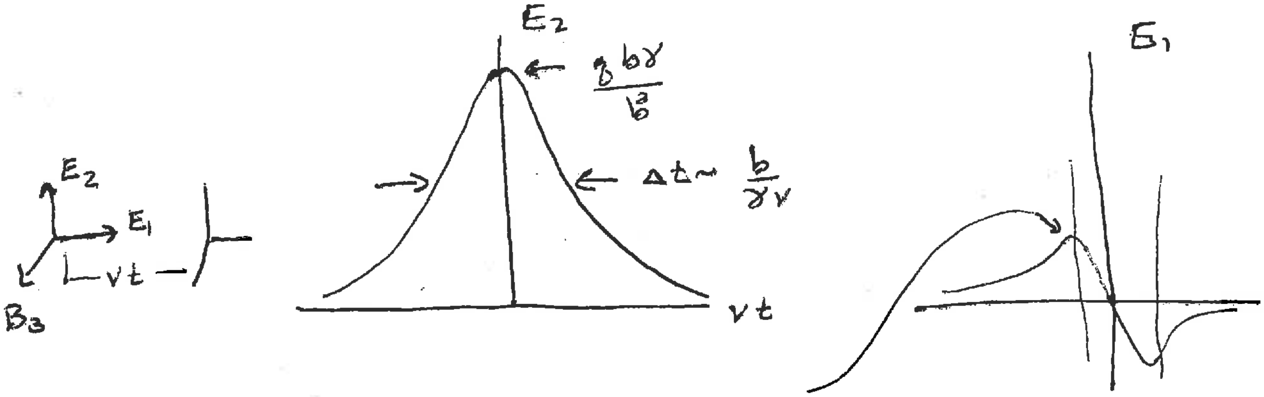


Note transverse B (B_3) and as $v \to c$ $|B_3| \approx |E_2|$

Note NR limit: $\vec{B} = \dfrac{q}{c}\dfrac{\vec{v}\times\vec{r}}{r^3}$ ($\frac{v}{c}$ in CGS) $q\dfrac{(\frac{\vec{v}}{c}\times\vec{r})}{r^3}$

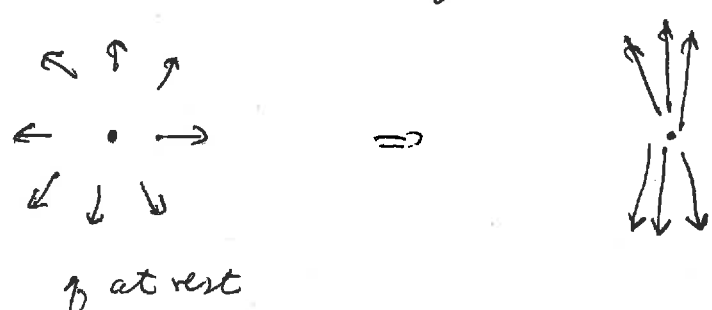
(Biot-Savart: $q\vec{v} \to I\,d\vec{l}$)

Note extreme relativistic limit $E_2 \propto \gamma$ but $E_1 \sim 1$



at $\gamma v t = b$ $E_1 = \dfrac{qb}{b^3\,2^{3/2}} \sim \dfrac{q}{b^2}$

Fields become a pulse of plane wave



indeed if detector averages over times $T > \frac{b}{\gamma v}$, $\langle E_1 \rangle = 0$!

[Future: can exploit analogy between fields of rel particle and plane wave ≡ "Weisaker Williams approximation"]

Why $F_{\mu\nu}^2$?

So why Maxwell's equations? Why is

$$\mathcal{L} = -\frac{1}{8\pi} F_{\mu\nu} F^{\mu\nu} + \text{nothing else} \; ?$$

In pure classical case, only justification is observation of superposition

The real answer involves dynamics (and the answer is, really, there *is* something else!)

Lets work in units with $\hbar = c = 1$, + mass = energy, so in dimensions

$$[\text{energy}] = \frac{1}{[\text{length}]} \quad (\hbar c = \text{energy} \times \text{length}).$$

A Lagrange density $\mathcal{L}$ is $\frac{[\text{energy}]}{[\text{length}^3]} = \frac{1}{[\text{length}]^4}$

$F_{\mu\nu} \sim \vec{E}$ $(\sim q/r^2)$ ~~and~~ and q is dimensionless $\left(\frac{q^2}{\hbar c} = \frac{1}{137}\right)$

so

$$[\vec{E}] \sim \frac{1}{[\text{length}]^2} \quad \text{and so}$$

$$\underbrace{\mathcal{L}}_{\frac{1}{[\text{length}]^4}} = (\text{dimensionless } \#) \cdot \underbrace{F_{\mu\nu} F^{\mu\nu}}_{\frac{1}{[\text{length}]^4}} + \cdots$$

What about other terms? Gauge invariance + Lorentz invariance require $\mathcal{L} = \mathcal{L}(F_{\mu\nu}F^{\mu\nu}$ or $F\tilde{F})$ and so if we imagine $\mathcal{L}$ is a polynomial in F^2, each new term must be ~~accompanied~~ multiplied by a parameter of dimensions $[\text{length}]^4$ or $\frac{1}{[\text{some mass}]^4} \equiv \frac{1}{\Lambda^4}$

What is the mass? If we are thinking about ~~...~~ electrodynamics the "mass" ~~...~~ is the scale at which new physics modifies Maxwell's eqns. The simplest new physics is the fundamentally quantum process of the exchange of virtual particles as intermediate states (production of e^+e^- pairs) and so $\Lambda \sim m_e$. Now in wave numbers/momentum space, $F^2 \sim k^2$

$$\frac{F^4}{\Lambda^4} \sim \frac{k^4}{\Lambda^4}$$

(notice this is even true classically; $k \sim \Lambda$, something happens ...)

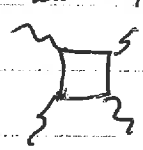
$F = \partial_\mu A_\nu - \partial_\nu A_\mu = k_\mu A_\nu - k_\nu A_\mu$

so for "typical momenta" $k \ll \Lambda$ ($= m_e \sim \frac{1}{2}$ MeV) such processes are completely negligible. And of course at $k \sim m_e$ these new processes turn on, and a classical description fails completely (of course it has already failed far below that - in the actual radiation processes in atoms). Indeed a calculation to support this statement was done by 2 of Heisenberg's students, Euler and Kockel, in 1935 - they started with QED (photons + electrons) and integrated out the electrons to derive an effective theory of photons

$$\mathcal{L} = \frac{1}{2}(E^2 - B^2) + \frac{e^4}{360\pi^2 m_e^4}\left[(E^2-B^2)^2 + 7(E\cdot B)^2\right] + \cdots$$

valid for frequencies $\hbar\omega \ll m_e c^2$. (So not even $\frac{1}{m_e^4}$, $\frac{\alpha^2}{m_e^4}$ which is 10^{-4} as big!)

The new term includes "photon splitting" 4-γ interactions as explicit nonlinear terms in $\mathcal{L}$.



QED is itself an incomplete description of Nature at very high energy, too..., it is only a part of a combined "electroweak" interaction.