7320 MIDTERM

Begin each problem on a separate piece of paper. Take your time and think before you write. Show all your work. When an explanation is required, write complete sentences in grammatical English.

1) [40 points] Scalar radiation of wavelength λ falls with normal incidence on an infinitely long slit of width $d >> \lambda$ in a flat screen. The screen sits in the $x-y$ plane with the slit lying along the y axis. Find the intensity of the diffracted radiation through the slit. (Hint: let the observer be located at $y = 0$). Assume any boundary conditions (Dirichlet, Neumann) that you wish.

2) [40 points] A charge q rotates in an elliptical orbit in the $x-z$ plane, with

$$\vec{r}(t) = \hat{i}x_0 \sin\omega t + \hat{k}z_0 \cos\omega t. \tag{1}$$

Find the time averaged angular distribution of radiated power per unit solid angle. (Work with multipoles!)

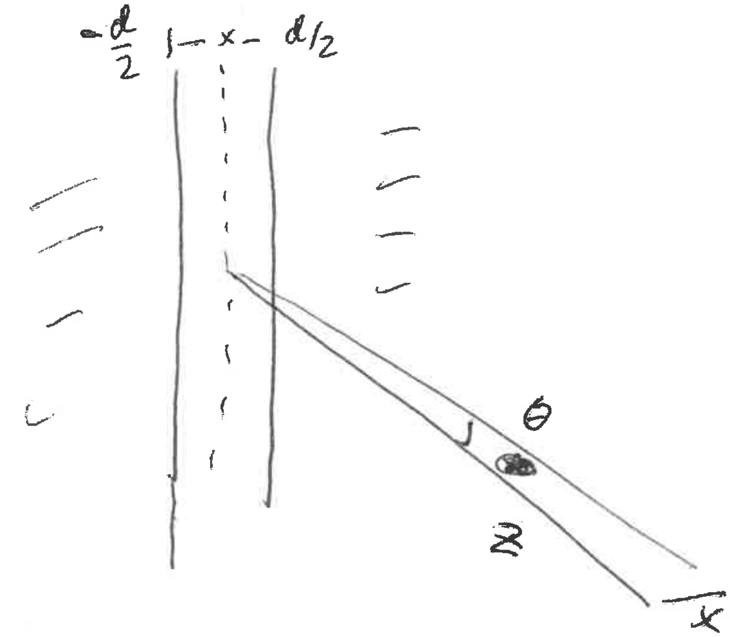
3) [20 points] "Asymmetric B factories" like the Belle facility in Japan are designed to produce pairs of B mesons (mass about 5 GeV each) from the head-on collision of a highly relativistic electron beam against a highly relativistic positron beam (about 7 GeV and 4 GeV at Belle; recall the electron mass is 0.5 MeV). For the purpose of this problem, the reaction is $e^+ + e^- \to B + \bar{B}$ and let's assume that the reaction occurs at threshold; the B and $\bar{B}$ are produced at rest in the center of mass frame. You can, of course, treat the electron and positron as massless in what follows.

(a) [10 points] If the energy of the two beams is E_1 and E_2 and the mass of the B (and $\bar{B}$) is M_B, what is the relation between E_1, E_2 and M_B for this to occur?

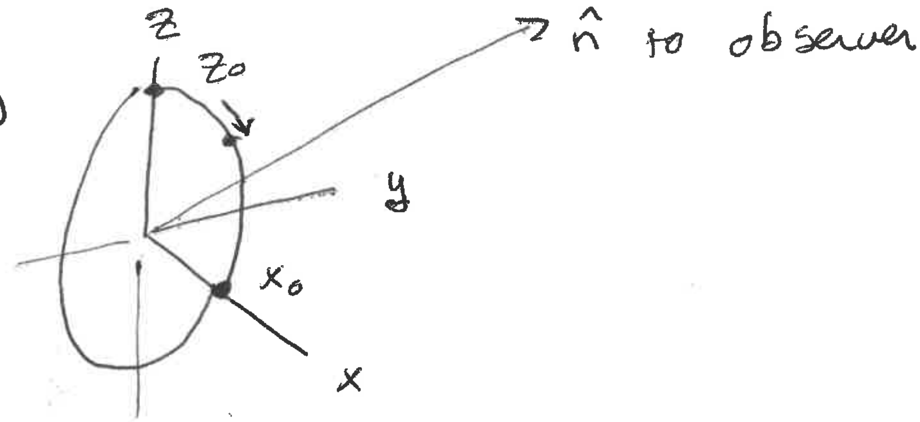
(b) [10 points] It would be easier to build a symmetric ($E_1 = E_2$) machine, but the B's are detected because they have a finite lifetime, and so, because they are not produced at rest in the lab frame, the location in space where they decay is not at the place where they are produced. If their lifetime is τ, how far (on average) is the decay vertex displaced from the production point? Your answer will involve E_1, E_2 and M_B (and τ, of course).

pictures on next page (on back)

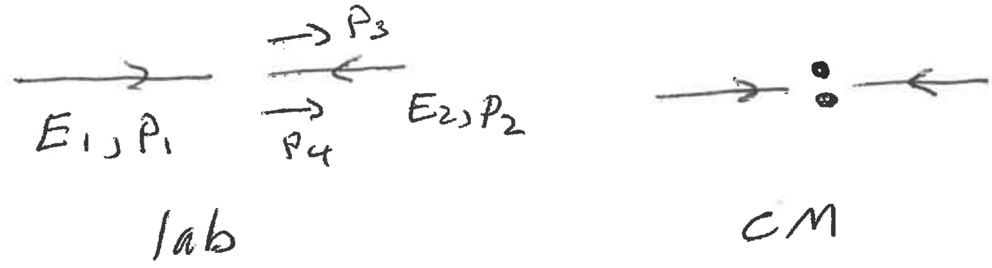
1)



2)



3)



1

1.1 Let's use Dirichlet b.c.'s

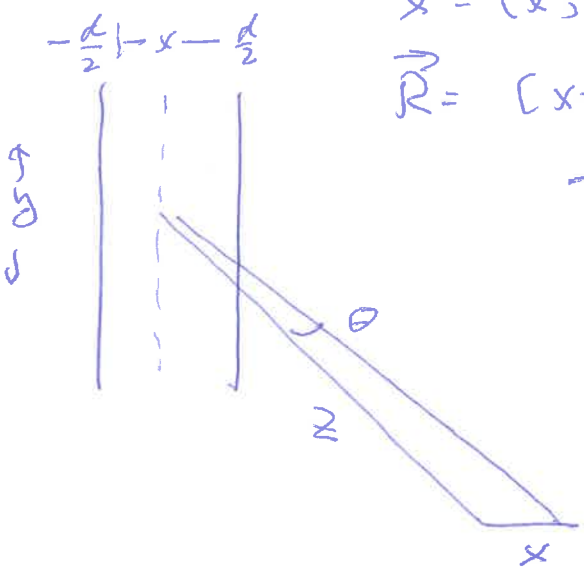
$$\psi_{diff} = \frac{k}{2\pi i}\cos\theta \int \frac{e^{ikR}}{R}\,\psi(x')\,dA'$$

$\hat{n} = \hat{z} = (0,0,1)$ normal to slit

$x' = (x', y', 0)$ on the plane

$x = (x, 0, z)$ observer on the y-axis

$\vec{R} = (x - x', -y', z)$



$$-\infty < y < \infty, \quad -\frac{d}{2} < x < \frac{d}{2}$$

$$R = \left[(x-x')^2 + y'^2 + z^2\right]^{1/2}$$

$$= \Big(\underbrace{z^2 + x^2}_{r^2} - 2x\cdot x' + y'^2\Big)^{1/2}$$

$$\simeq r - \frac{x\cdot x'}{r} + \frac{y'^2}{2r} + \cdots$$

This is basically Fraunhofer in x, Fresnel in y!

$$\psi = \frac{k}{2\pi i}\cos\theta\, \frac{e^{ikr}}{r} \int_{-\infty}^{\infty} dy' \int_{-d/2}^{d/2} dx'\, e^{-ik\frac{\vec{x}\cdot x'}{r}}\, e^{ik\frac{y'^2}{2r}}$$

$$= \frac{k\cos\theta\, e^{ikr}}{2\pi r} \int_{-\infty}^{\infty} dy\, e^{\frac{iky'^2}{2r}} \int_{-\frac{d}{2}}^{\frac{d}{2}} dx'\, e^{-ik\sin\theta\, x'}.$$

This is very similar to the Fresnel homework problem

$$I_y = \int_{-\infty}^{\infty} dy' \, e^{iky'^2/2r} = \sqrt{\frac{\pi \cdot 2r}{-ik}} = \sqrt{\frac{2\pi r}{k}} \left(\frac{1+i}{\sqrt{2}}\right)$$

It's a Gaussian $\int$, $\sqrt{i} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$$I_x = \int_{-d/2}^{d/2} dx' \, e^{-ik\sin\theta\, x'} = \frac{e^{-ik\frac{d}{2}\sin\theta} - e^{ik\frac{d}{2}\sin\theta}}{-ik\sin\theta}$$

so

r & cancel against i

$$\psi = \frac{k\cos\theta}{2\pi i r} e^{ikr} \sqrt{\frac{2\pi r}{k}} \left(\frac{1+i}{\sqrt{2}}\right) \frac{2d}{2} \left[\frac{\sin\left[\frac{kd\sin\theta}{2}\right]}{\frac{kd\sin\theta}{2}}\right]$$

The intensity is (up to overall factors)

$$\frac{1}{r}\left[\frac{\sin\left(\frac{kd}{2}\sin\theta\right)}{\frac{kd\sin\theta}{2}}\right]^2$$

$\frac{1}{r} \times$ ("sinc-function")^2

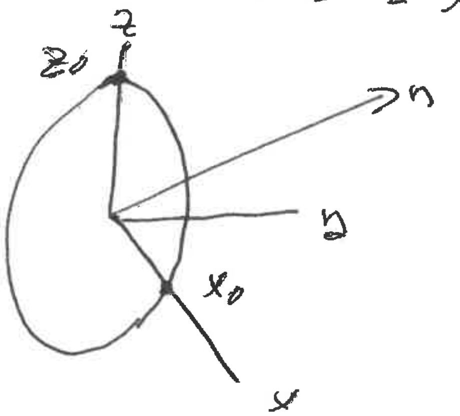
This is a cylindrical wave

$$I \sim |\psi|^2 \sim \frac{1}{r}$$

so that $\frac{dP}{d\Omega} \sim r \cdot I \sim |f(\theta)|^2$ is

r-independent. Hence ψ should be $\sim \frac{1}{\sqrt{r}}$.

2) $\vec{r}(t) = \hat{\imath}\, x_0 \sin \omega t + \hat{k}\, z_0 \cos \omega t$ (1)



We just need the dipole moment $\vec{p}_0 = e^{i\omega t}$, then go to blue Jackson 9.22

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2} \left\| (\hat{n}\times\vec{p})\times\hat{n} \right\|^2$$

or red Jackson 9.22 $\frac{dP}{d\Omega} = \frac{c k^4}{8\pi} \left| (\hat{n}\times\vec{p})\times\hat{n} \right|^2$

$|(\hat{n}\times\vec{p})\times\hat{n}|^2 = \underline{|p|^2 - |\vec{p}\cdot\hat{n}|^2}$ see 9.48 or crank it out

$[n\times(p\times n)]_i = \epsilon_{ijk}\, n_j\, \epsilon_{klm}\, p_l n_m$ (2)

$= \epsilon_{ijk}\,\epsilon_{lmk}\, n_j\, p_l\, n_m$

$= [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}]\, n_j p_l n_m$

$= p_i(n\cdot n) - n_i(n\cdot p)$

$\Rightarrow [\hat{n}\times(\vec{p}\times\hat{n})] = \vec{p} - \hat{n}(\hat{n}\cdot\vec{p})$

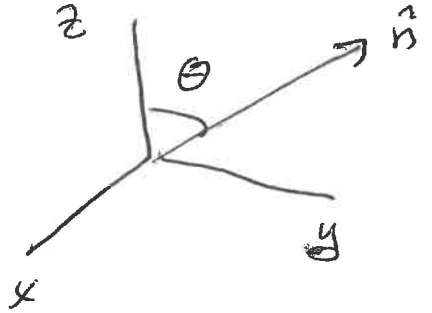
square to get (2)

From (1) $\vec{p}(t) = q[\hat{\imath}\, x_0 \sin\omega t + \hat{k}\, z_0 \cos\omega t]$

$= \mathrm{Re}\{ q(\hat{\imath}(-i x_0) + \hat{k} z_0)\, e^{i\omega t} \}$

i.e. $\vec{p}_0 = q(-i x_0\, \hat{\imath} + z_0\, \hat{k})$

and



$\hat{n} = \hat{\imath} \sin\theta \cos\varphi + \hat{\jmath} \sin\theta \sin\varphi + \hat{k} \cos\theta$ so

$$\vec{p}_0 \cdot \vec{p}_0^{\,*} = q^2 [x_0^2 + z_0^2]$$

$$\vec{p}_0 \cdot \hat{n} = q [-i x_0 \sin\theta \cos\varphi + z_0 \cos\theta]$$

$$|\vec{p}_0 \cdot \hat{n}|^2 = q^2 [x_0^2 \sin^2\theta \cos^2\varphi + z_0^2 \cos^2\theta]$$

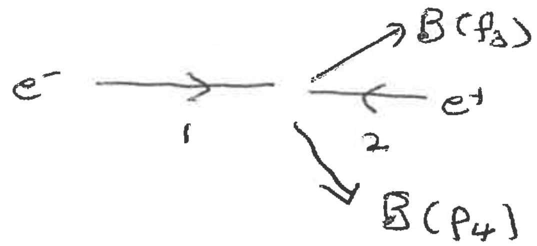
$$\frac{dP}{d\Omega} = \frac{q^2 c k^4}{8\pi} \left[x_0^2 (1 - \sin^2\theta \cos^2\varphi) + z_0^2 (1 - \cos^2\theta) \right]$$

$$= \frac{q^2 c k^4}{8\pi} \left[x_0^2 (1 - \sin^2\theta \cos^2\varphi) + z_0^2 \sin^2\theta \right]$$

Sanity check: $x_0 \to 0$ is normal linear dipole $\sin^2\theta$!

3) $e^+ + e^- \to B + \bar{B}$ at threshold, effectively

a) massless electrons.



in lab, $p_1 = (E_1, E_1, 0, 0)$

and $|p| = E$

$p_2 = (E_2, -E_2, 0, 0)$

$$p_1 + p_2 = p_3 + p_4$$

Threshold: in the CM $E_{cm} = 2M_B$ or (1)

$$p_3' = (M_B, \vec{0}), \quad p_4' = (M_B, 0)$$

square 1: $(p_1+p_2)^2 = (p_3+p_4)^2 = 4M_B^2$

$(p_1+p_2)^2 = p_1^2 + 2p_1 \cdot p_2 + p_2^2 = 0 + 2(E_1E_2 + E_1E_2)$
$= 4E_1E_2$

or $p_1+p_2 = (E_1+E_2, E_1-E_2, 0, 0)$

and again $(p_1+p_2)^2 = (E_1+E_2)^2 - (E_1-E_2)^2$
$= 4E_1E_2$

either way

$$4E_1E_2 = 4M_B^2$$

or $\boxed{E_1E_2 = M_B^2}$

b) First way: produced B's share momentum equally, share energy equally, so in lab frame

$$\left.\begin{array}{l} P_3 + P_4 = 2P_3 = E_1 - E_2 \\ E_3 + E_4 = 2E_3 = E_1 + E_2 \end{array}\right\} P_3 = \left(\frac{E_1+E_2}{2}, \frac{E_1-E_2}{2}, 0, 0\right)$$

and $P_3 = (\gamma M_B, \gamma\beta M_B, 0, 0)$

B's go a distance $d = v \cdot \gamma\tau$ before decaying due to time dilation, $d = c\beta\gamma\tau$ (and $c=1$)

Then

a) $\beta\gamma = \dfrac{E_1 - E_2}{2M_B} \rightarrow d = \dfrac{E_1 - E_2}{2M_B}\tau$

b) $\beta = \dfrac{P_3}{E_3} = \dfrac{E_1 - E_2}{E_1 + E_2}$ $\quad \gamma = \dfrac{E_3}{M_B} = \dfrac{E_1 + E_2}{2M_B}$

$$d = \left[\frac{E_1 - E_2}{E_1 + E_2}\right]\left[\frac{E_1 + E_2}{2M_B}\right]\tau = \frac{E_1 - E_2}{2M_B}\tau \text{ again}$$

c) - awkward! Boost e^+ & e^- to CM, call their 4 momenta in CM k_1 and k_2. Recall $E = |P|$

$$k_1^0 = \gamma\left[E_1^0 - \beta P_1\right] = \gamma(1-\beta)E_1$$

$$k_2^0 = \gamma(E_2 + \beta P_1) = \gamma(1+\beta)E_2$$

In CM $k_1^0 = k_2^0 = M_B = \gamma(1-\beta)E_1 = \gamma(1+\beta)E_2$

$$(1-\beta)E_1 = (1+\beta E_2) \Rightarrow E_1 - E_2 = \beta(E_1 + E_2) \Rightarrow \beta = \frac{E_1 - E_2}{E_1 + E_2}$$

$$\gamma = \frac{M_B}{E_1(1-\beta)} = \frac{M_B}{E_1\left[1 - \left(\frac{E_1 - E_2}{E_1 + E_2}\right)\right]} = \frac{M_B(E_1 + E_2)}{2E_1E_2} = \frac{E_1 + E_2}{2M_B}$$

etc ...

7320, 2023 Midterm ave = 51 σ = 15

vs	2020	72	19
	2019	66	18
	2016	77	11
	2015	73	15
	2009	70	16
	2008	71	20