Quantum Electrodynamics

We've spent a whole year on classical electrodynamics, but of course the world is fundamentally quantum-mechanical and most of you are going to be involved in research in areas which are intrinsically quantum-mechanical. So it's appropriate to end the course by asking

What is a photon?

How does quantum mechanical matter interact with the quantum-mechanical electromagnetic field?

Answers to these questions can be given on many levels of complexity. My goal - ~~the minimum I think a well-educated experimentalist should know~~ every day AMO

1) semiclassical radiation theory (~~[illegible]~~) $\left.\right\}$ QM matter, classical E+M

2) "quantization of the EM field"

3) ~~From quantum back to classical E+M~~ (coherent states) ~~- homework problems?~~

With as many examples as time permits!

1) logically inconsistent but gives correct answers to typical AMO problems - with interpretation

~~Old [illegible] Dirac 1927, Fermi 1932, [illegible] because~~

Schaff[illegible] ~~many subtle points, which I'll try to steer around~~

Heitler, ~~"[illegible] Approach to QFT"~~

Right off the mark - easiest to work in Coulomb Gauge, $\vec{\nabla}\cdot\vec{A}=0$.

"physical" - 2 polarizations

"unphysical" - instantaneous Coulomb int.

Covariance not apparent, gauge invariance not apparent.

Coulomb gauge: $\nabla \cdot A = 0$

Why? $\mathcal{L} = -\frac{1}{8\pi} F_{\mu\nu} F^{\mu\nu} = \mathcal{L}\left[A_\mu, \partial_\nu A_\mu\right]$

$$\pi^k = \frac{\partial \mathcal{L}}{\partial \dot{A}_k} = \frac{1}{4\pi}\left[-\frac{\partial A_0}{\partial x_k} + \frac{\partial A_k}{\partial x_0}\right]$$

$$= \frac{E_k}{4\pi}$$

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

A_0 or φ is not a dynamical variable, in ~~[illegible]~~ the absence of sources it is zero.

$$\mathcal{H} = \sum_{k=1,2,3} \Pi_k \dot{A}_k - \mathcal{L} = \frac{1}{8\pi}\left[E^2 + (\vec{\nabla}\times\vec{A})^2\right]$$

Also, in free space, $\vec{\nabla}\cdot\vec{E} = 0$, only 2 independent components of $\vec{E}$ ~~[illegible]~~ vs 3 A's!
To avoid difficulties (more/less p's than q's) easiest simply to pick Coulomb gauge from the start.

$$\vec{\nabla}\cdot\vec{A} = 0$$

$\mathcal{H}$ involves 2 sets of canonically conjugate variables

2 A's, 2 E's.

and φ doesn't participate - it is constrained

$$\nabla^2\varphi = -4\pi\rho$$

$$\varphi(x,t) = \int \frac{\rho(x',t)\,d^3x'}{|\vec{x}-\vec{x}'|}$$

Semiclassical Radiation Formulas

Treat A as classical plane wave field.

$$\vec{A}(x,t) = q\,\vec{\epsilon}\left[e^{i(k\cdot x-\omega t)} + e^{-i(k\cdot x-\omega t)}\right]$$

Consider H for charged particle in field

$$H = \frac{1}{2m}\left(\vec{p} - e\frac{\vec{A}}{c}\right)^2 + V(r)$$

Expand
$$H = \underbrace{\frac{p^2}{2m} + V(r)}_{H_0} \underbrace{- \frac{e}{mc}\vec{A}\cdot\vec{p}}_{H_1} + \underbrace{\frac{e^2A^2}{2mc^2}}_{H_2}$$

($\vec{p} = \frac{\hbar}{i}\vec{\nabla}$ but in Coulomb gauge $\nabla\cdot A = 0$)

group

Treat H_1 and maybe H_2 as a perturbation, evaluate transition prob./unit time by Golden Rule — for spontaneous emission

$$d\Gamma_{i\to f} = \frac{2\pi}{\hbar}\left|\langle f|\hat{H}_I|i\rangle\right|^2 \delta(E_f - E_i \pm \hbar\omega) \times \left[V\frac{d^3k_\gamma}{(2\pi)^3}\right]$$

and

1) $|i\rangle, |f\rangle$ eigenstates of H_0
2) H_I is a harmonic perturbation because A is harmonic — each term in $A \to \pm\hbar\omega$ in $\delta(\ )$
3) $\hat{H}_I = -\frac{e}{mc}q\exp(\pm ik\cdot x)\,\hat{\epsilon}\cdot\left(\frac{\hbar}{i}\vec{\nabla}\right)$
4) phase space — to make sense of δ fn.

~~in box of volume V~~

Though classical, A is like a free particle WF, have to count # of modes w/ freq ω, wave # between k & $k+dk$.

$\Rightarrow \quad V\, d^3k/(2\pi)^3$

A) What is c_λ? 2 choices possible (they are related at the end of the day)

1) processes like absorption or stimulated emission - rate $\propto$ intensity of external radiation source

Define intensity per freq interval

$$I(\omega)\Delta\omega = \frac{1}{8\pi}\left(E^2+B^2\right)\Delta\omega$$

$$\vec{E} = -\frac{1}{c}\frac{\partial A}{\partial t} = \qquad B = \nabla\times A$$

with $\vec{A} = 2 c_\lambda \vec{\epsilon} \cos(k\cdot x-\omega t)$ (take c_λ real for now!)

$$\vec{E} = -\frac{\omega}{c}\cdot 2 c_\lambda \vec{\epsilon}\sin(k\cdot x-\omega t)$$

$$\vec{B} = -\vec{k}\times\vec{\epsilon}\cdot 2 c_\lambda \sin(k\cdot x-\omega t) \qquad \omega = c|k|$$

$$I(\omega)\cancel{\Delta\omega} = \frac{4}{8\pi}|c_\lambda|^2 \underbrace{\sin^2(k\cdot x-\omega t)}_{\frac{1}{2}}\underbrace{\left\{\frac{\omega^2}{c^2}\hat{\epsilon}\cdot\hat{\epsilon} + \left[\vec{k}\times\hat{\epsilon}\right]^2\right\}}_{\frac{2\omega^2}{c^2}}$$

$$= \frac{|c_\lambda|^2\omega^2}{2\pi c^2}$$

or $$|c_\lambda|^2 = \frac{2\pi c^2}{\omega^2} I(\omega)$$

2) "Photon normalization" – idea is that ~~for~~ EM field for a photon has energy density

$$\frac{\hbar\omega}{V} = \frac{E^2+B^2}{8\pi}$$ (energy $\hbar\omega$ in volume V)

$$= |q|^2 \frac{\omega^2}{2\pi c^2}$$ same as previous calc.

$$|q|^2 = \frac{2\pi \hbar c^2}{\omega V}$$

$$\vec{A}(x,t) = \left(\frac{2\pi\hbar c}{\omega V}\right)^{1/2} \vec{\epsilon}\left[e^{i(k\cdot x-\omega t)} + e^{-i(k\cdot x-\omega t)}\right]$$

seems complicated! isn't!

B) Matrix element $\quad H_I = \frac{e}{mc}\vec{A}\cdot\vec{p}$

$$\mathcal{M} \equiv \langle f|\hat{H}_I|i\rangle = \langle f|\frac{e}{mc} q\, e^{\pm i(k\cdot x)} \hat{\epsilon}\cdot\tilde{p}|i\rangle$$

$\pm$ associated w/ $e^{\mp i\omega t}$

Also horrible. QM version of multipole expansion is actually easier than classical one!

$$e^{\pm ik\cdot x} = \underbrace{1}_{\text{dipole approx}} \pm ik\cdot x + \cdots$$

atom $r \sim \text{Å}$

$k = \frac{2\pi}{\lambda}$ $\quad \lambda = 3\times10^3\,\text{Å}$

$k\cdot r \sim 10^{-3}$

$$\mathcal{M} = \frac{qe}{mc}\,\hat{\epsilon}\cdot\langle f|\vec{p}|i\rangle$$

Still too complicated! $\vec{p} = m \frac{d\vec{x}}{dt}$

+ Heisenberg $i\hbar \frac{\partial \vec{x}}{\partial t} = \vec{x} H - H \vec{x}$

$$\langle f | \vec{p} | i \rangle = -\frac{im}{\hbar} \langle f | \vec{x} H - H x | i \rangle$$

Assume $|f\rangle$ + $|i\rangle$ eigenstates of H:

$$H|i\rangle = E_i |i\rangle$$
$$H|f\rangle = E_f |f\rangle$$

$$\langle f | \vec{p} | i \rangle = -\frac{im}{\hbar} (E_f - E_i) \langle f | x | i \rangle$$
$$= -i m \omega_{fi} \langle f | H | i \rangle$$

$$\mathcal{M} = -i q \frac{\omega_{fi}}{c} \underbrace{e \langle f | \vec{x} | i \rangle}_{\text{electric dipole operator}} \cdot \hat{\epsilon}$$

Spontaneous emission uses photon normalization

$d\Gamma \equiv$ trans prob/unit time (to emit radiation w/ wave # between $\vec{k}$ & $\vec{k}+d\vec{k}$)

$$= \frac{2\pi}{\hbar} \frac{e^2 \omega_{fi}^2}{c^2} \left| \hat{\epsilon} \cdot \langle f | \vec{x} | i \rangle \right|^2 \delta(E_f - E_i - \hbar\omega)$$

$$\times \left(\frac{2\pi \hbar c}{\omega V} \right) \quad \leftarrow \text{photon } (c_1)^2$$

$$\times \frac{V\, d^3 k_\gamma}{(2\pi)^3} \quad \leftarrow \text{phase space}$$

Note V cancels $\frac{1}{V}$! (very annoying! usual story with box normalization)

Integrate over δ-fn $\quad \omega = ck$

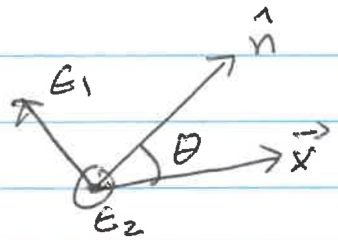
$$d^3k = \frac{\omega^2 d\omega\, d\Omega_\gamma}{c^3}$$

$$\frac{d\Gamma}{d\Omega_\gamma} = \frac{2\pi}{\hbar} \frac{e^2 \omega_{fi}^2}{c^2} \left| \hat{\epsilon} \cdot \langle f | \vec{x} | i \rangle \right|^2 \delta(\Delta E - \hbar\omega) \frac{\omega^2 d\omega}{(2\pi)^3 c^3} \cdot \frac{2\pi \hbar c^2}{\omega}$$

$$= \frac{e^2 \omega_{fi}^3}{2\pi \hbar c^3} \left| \hat{\epsilon} \cdot \langle f | \vec{x} | i \rangle \right|^2$$

Trans prob/unit time for atom to emit radiation of polarization $\hat{\epsilon}$ in ~~direction~~ $d\Omega$ about direction $\hat{n}$

Often we don't care about the polarization - so we sum over the 2 final pol's



choose $\hat{\epsilon}_2 \cdot x = 0$

$$\hat{\epsilon}_1 = \hat{\epsilon}_2 \times \hat{n}$$

$$\hat{\epsilon}_1 \cdot \vec{X} = \sin\theta$$

$$\left.\frac{d\Gamma}{d\Omega}\right)_{\text{pol sum}} = \frac{e^2 \omega^2}{2\pi\hbar c^3} |\langle f|\vec{x}|i\rangle|^2 \sin^2\theta$$

$$\Gamma = \frac{e^2 \omega^3}{2\pi\hbar c^3} |\langle f|\vec{x}|i\rangle|^2 \cdot 2\pi \int_{-1}^{1} d\cos\theta \, (1-\cos^2\theta)$$

$$2 - \frac{2}{3}$$

$$\boxed{\Gamma = \frac{4}{3} \frac{e^2}{\hbar c^3} \omega^3 |\langle f|\vec{x}|i\rangle|^2}$$

and $\hbar\omega = E_f - E_i$

$$|\langle f|\vec{x}|i\rangle|^2 = |\langle f|x|i\rangle|^2 + |\langle f|y|i\rangle|^2 + |\langle f|z|i\rangle|^2$$

The QM formula can also be written in terms of acceleration. Start with

$$\Gamma_{fi} = \frac{4}{3}\frac{e^2}{\hbar c^3}\omega_{fi}^3 |\langle f|\vec{x}|i\rangle|^2$$

use Heisenberg, twice

$$\langle f|\frac{d^2x_i}{dt^2}|i\rangle = \frac{1}{i\hbar}\langle f|H\frac{dx_i}{dt} - \frac{dx_i}{dt}H|i\rangle$$

$$= \frac{E_f - E_i}{i\hbar}\langle f|\frac{dx_i}{dt}|i\rangle = -\omega_{fi}^2\langle f|\vec{x}|i\rangle$$

so $\hbar\omega_{fi}\cdot\Gamma_{fi} = \frac{4}{3}\frac{e^2}{c^3}|\langle f|\frac{d^2\vec{x}}{dt^2}|i\rangle|^2$

Power is $\frac{\text{energy}}{\text{time}}$ so $\Gamma \Leftrightarrow \frac{1}{\text{time}}$, energy $\to \hbar\omega$.

Very similar to classical formula $P = \frac{2}{3}\frac{e^2}{c^3}|\vec{a}|^2$

Could time average for harmonic $x(t) \sim e^{i\omega t}$

$P_{ave} = \Gamma_{cl}\cdot\hbar\omega$, $\Gamma_{cl} = \frac{\omega^3}{3}\frac{e^2\langle x^2\rangle}{\hbar c^3}$

<u>But</u> 1) 2's?

2) in QM, it's ω_{fi}, in classical, it is $x(t) = e^{i\omega t}$.

3) $\langle f|x|i\rangle$ is <u>transition</u> matrix element, not expectation value

4) Classical $dP/d\Omega$ = antenna pattern, QM formula $d\Gamma/d\Omega$ = probability to detect photon at $\hat{n}$ – $d\Omega$ – it <u>should</u> be this, so far, it's <u>not</u>

Selection rules

$$B_{fi} = \frac{4\pi^2 e^2}{\hbar^2} |\langle f|\hat{\epsilon}\cdot\vec{r}|i\rangle|^2$$

1) $\langle f|\vec{r}|i\rangle = 0$; Forbidden transition – may not be really forbidden:

$$e^{ik\cdot r} = 1 + \underbrace{ik\cdot r}_{\substack{10^{-4} \text{ in atoms} \\ 10^{-2} \text{ in nuclei}}} - \frac{1}{2}(k\cdot r)^2 + \cdots$$

or $\langle f|\vec{r}\, e^{ik\cdot r}|i\rangle = 0$ really forbidden ($\ell = 0 \to \ell = 0$)

2) Dipole selection rules

$\vec{r}$ is a vector operator

$$\vec{r} = \left(-\left(\frac{\vec{x} - i\vec{y}}{\sqrt{2}}\right), \vec{z}, \left(\frac{x + iy}{\sqrt{2}}\right)\right) \propto Y_1^m$$

suppose $|f\rangle = |n_f\, \ell_f\, m_f\rangle$
$|i\rangle = |n_i\, \ell_i\, m_i\rangle$

$$\langle f|r_m|i\rangle \propto \langle \ell_f m_f|r_m|\ell_i m_i\rangle$$

$$\propto \langle \ell_i, 1; m_i, m|\ell_f m_f\rangle$$

a CG coeff

$\therefore$ $\Delta m = m_f - m_i = \pm 1, 0$
$\Delta \ell = \ell_f - \ell_i = \pm 1$

and $\Delta S = 0$ (in $|\ell m_\ell s m_s\rangle$ basis –
$\Delta m_s = 0$ H_I is spin indep.

QM of EM field – just an SHO w/ many indices

One Classical SHO $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

$$\dot{p} = \{H, p\} = -\frac{\partial H}{\partial q} = -m\omega^2 q$$

$$\dot{q} = \{H, q\} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\frac{d}{dt}\left\{ q \pm \frac{ip}{m\omega} \right\} = \frac{p}{m} \mp i\omega q = \mp i\omega \left\{ q \pm \frac{ip}{m\omega} \right\}$$

define $A(t) = q(t) + i\frac{p(t)}{m\omega}$

$$\boxed{\frac{dA}{dt} = -i\omega A \,, \quad \frac{dA^*}{dt} = +i\omega A^*} \; *$$

$$A^*A + AA^* = 2\left(q^2 + \frac{p^2}{m^2\omega^2} \right)$$

$$\boxed{H = \frac{m\omega^2}{4}\left[A^*A + AA^* \right]} \; *$$

these define a ~~SHO~~ classical SHO

Many SHO: find normal modes

$$\cancel{\frac{dA_j}{dt} = -i\omega_j A_j \,, \quad \frac{dA_j^*}{dt} = i\omega_j A_j}$$

$$\cancel{H = \sum_j \frac{m\omega_j^2}{4}\left[A_j^* A_j + A_j A_j^* \right]}$$

Finding normal modes - by example

$$H = \sum_j \frac{1}{2} p_j^2 + \frac{1}{2} \sum_{jk} x_j V_{jk} x_k \, , \quad V_{jk} = V_{kj}$$

$$\dot{p}_j = -\frac{\partial H}{\partial x_j} = -V_{jk} x_k \quad (1)$$

$$\dot{x}_j = \frac{\partial H}{\partial p_j} = p_j \quad (2)$$

Define $P_n = R_{nj} p_j \qquad X_n = R_{nj} x_j$

or $p_j = R^{-1}_{jn} P_n \, , \quad x_j = R^{-1}_{jn} X_n$

(and for convenience) $\sum_j p_j^2 = \sum_{jne} R^{-1}_{jn} R^{-1}_{je} P_n P_e$

$$= \sum P_n^2 \qquad (R^{-1}_{jn} = R_{nj} \text{ - orthogonal transf.})$$

Then $\dot{P}_n = R_{nj} \dot{p}_j = -R_{nj} V_{jk} x_k$

$$= -R_{nj} V_{jk} R^{-1}_{ke} X_e$$

choose R so $R_{nj} V_{jk} R^{-1}_{ke} = \delta_{ne} \omega_n^2$

$$\dot{P}_n = -\omega_n^2 X_n$$

$$\dot{X}_n = P_n \quad (2)$$

$$H = \sum_k \frac{1}{2} P_k^2 + \frac{1}{2} \omega_k^2 X_k^2 \quad *$$

or $$A_k(t) = X_k(t) + i \frac{P_k(t)}{\omega}$$

$$\frac{dA_k}{dt} = -i\omega_k A_k$$

$$\frac{dA_k^*}{dt} = i\omega_k A_k^*$$

$$H = \sum_k \frac{m\omega_0^2}{4} \left[A_k A_k^* + A_k^* A_k \right]$$

$$X_k = R_{kj} x_j$$

Note normal modes are superpositions of original DoF's - generally <u>not</u> localized in space.

~~* for real H, useful to define R as orthogonal transf~~

$$\cancel{\sum_k P_k^2 = \sum_{knl} R_{kn} R^{t}_{kl} P_n P_l}$$

$$\cancel{R_{nk} R^{-1}_{kl} P_k P_l}$$

$$=$$

One quantum SHO

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x + \frac{i p}{\sqrt{2m\omega\hbar}} \quad \text{and} \quad a^+ = \sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{i p}{\sqrt{2m\omega\hbar}}$$

$$\boxed{[x, p] = i\hbar} \rightarrow [a, a^+] = 1 \quad ** \left\{ \sqrt{\frac{m\omega}{2\hbar}} \text{ etc for convenience!} \right\}$$

$$** \quad \boxed{H = \frac{\hbar\omega}{2}[a a^+ + a^+ a] = \hbar\omega[a^+ a + \tfrac{1}{2}]} \quad **$$

$$\left. \begin{array}{l} a^+ a = \frac{m\omega}{2\hbar} x^2 \\ \frac{m x^2 \omega^2}{2} = \hbar\omega a^+ a \end{array} \right\}$$

Heisenberg EoM

$$i\hbar \dot{a} = [H, a] = \hbar\omega a$$

$$\boxed{\begin{array}{l} \dot{a} = -i\omega a \\ \dot{a}^+ = i\omega a^+ \end{array}} \quad ** \text{ see page Q-1}$$

** define a quantum SHO

And so, a familiar solution $\quad a^+ a|n\rangle = n|n\rangle$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$H|n\rangle = E_n|n\rangle = \hbar\omega(n + \tfrac{1}{2})$$

$$n = 0, 1, 2 \ldots$$

Many oscillators – add an index!

$$H = \sum_{ij} c_{ij} p_i p_j + d_{ij} x_i x_j + \ldots$$

$$\rightarrow \sum_k A_k p_k^2 + B_k x_k^2, \quad p_k = \sum_i \alpha_i p_i, \quad x_k = \sum \beta_i x_i$$

$$H = \sum_k \frac{\hbar\omega_k}{2}[a_k^+ a_k + a_k a_k^+] \quad n_k = 0, 1, 2 \ldots$$

$$H|\Psi\rangle = E|\Psi\rangle, \quad |\Psi\rangle = |\{n_k\}\rangle, \quad E = \sum_k \hbar\omega_k(n_k + \tfrac{1}{2})$$

Many quantum oscillators - again, decompose into normal modes! (many = one with an index)

$$H = \sum_j \frac{1}{2} P_j^2 + \frac{1}{2} \sum x_j V_{kj} x_k$$

$$X_j = R_{je} x_e \quad \text{or} \quad x_g = R_{je}^{-1} X_e$$

$$\underline{P}_k = R_{kn} P_n$$

~~discover~~ $H = \sum \frac{1}{2} \underline{P}_k^2 + \frac{1}{2} w_k^2 X_k^2$

discover that if $[x_i, P_j] = \delta_{ij} \times i\hbar$

$$[X_i, P_j] = \delta_{ij} \cdot i\hbar$$

$$a_k = \sqrt{\frac{w_k}{2\hbar}}\, X_k + \frac{i \underline{P}_k}{\sqrt{2\hbar w_k}}$$

$$H = \sum_k \frac{\hbar w_k}{2} \left[a_k a_k^\dagger + a_k^\dagger a_k \right]$$

$$= \sum_k \hbar w_k \left[a_k^\dagger a_k + \frac{1}{2} \right]$$

and $\boxed{\begin{aligned} \dot{a}_k &= -i w_k a_k \\ \dot{a}_k^\dagger &= i w_k a_k^\dagger \end{aligned}}$

$[a_k, a_{k'}^\dagger] = \delta_{kk'}$, $[a_k, a_{k'}] = 0$, $[a_k^\dagger, a_{k'}^\dagger] = 0$

~~States~~ Energy eigenstates are product states

$$|\psi\rangle = |n_1, n_2, n_3 \cdots n_k \cdots\rangle$$

each n_k = integer $n_k = 0, 1, 2 \cdots$

$$a_j |n_1 n_2 \cdots n_j \cdots\rangle = \sqrt{n_j}\, |n_1 n_2 \cdots n_j - 1, \cdots\rangle$$

$$a_j^+ |n_1 n_2 \cdots n_j \cdots\rangle = \sqrt{n_j + 1}\, |n_1 n_2 \cdots n_j + 1, \cdots\rangle$$

also note $X_e \sim a_e + a_e^+$ for eth normal mode

$$\underset{\sim}{x_k} = \sum_e R^{-1}_{ke} (a_e + a_e^+)$$

operator at a site in original system is a superposition of creation + annihilation ops for all normal modes.

Classical ~~[illegible]~~ Electromagnetism for a solution of Maxwell's eqns in free space -

$$\vec{A}(x,t) = \sum_{\vec{k}} \sum_{\sigma=1,2} \left(\frac{2\pi\hbar c}{V\omega_k}\right)^{1/2} \hat{\epsilon}_{k\sigma}$$

$$\times \left[a_{k\sigma}(t)\, e^{i\vec{k}\cdot\vec{x}} + a_{k\sigma}(t)^* e^{-i\vec{k}\cdot\vec{x}}\right] \quad (*)$$

2 polarizations, $\hat{\epsilon}_{k\sigma} \cdot \vec{k} = 0$

$a_{k\sigma}(t)$ = classical Fourier amplitude

"photon normalization" strictly for convenience

Plug into wave eqn $\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 A(x,t)}{\partial t^2} = 0$

get $\sum_{k\sigma} \cdots e^{ik\cdot x}\left[-\omega_k^2 + \frac{d^2}{dt^2}\right] a_{k\sigma}(t) \; -- = 0$

use orthogonality: $\int d^3x\; e^{-ik'\cdot x} [\;] \Rightarrow$

separate, discover mode by mode

$$\frac{d^2 a_{k\sigma}(t)}{dt^2} + \omega_k^2\, a_{k\sigma}(t) = 0$$

solution $a_{k\sigma}(t) = \underline{a^{(1)}_{k\sigma}(0)\, e^{-i\omega_k t}} + a^{(2)}_{k\sigma}\, e^{i\omega_k t}$

An annoying technical factor of 2 -- a & a^* overcount the # of fourier modes in $(*)$ - bottom line - keep only one term in $*$, $a_{k\sigma} = a^{(1)}_{k\sigma}$

$$\boxed{\frac{d a_{k\sigma}}{dt} = -i\omega\, a_{k\sigma}} \quad **$$

$$U = \frac{1}{8\pi}\int_V d^3x \left[E^2 + B^2\right] = \frac{1}{8\pi}\int d^3x \left[\left(\frac{\partial A}{\partial t}\right)^2 + (\vec{\nabla}\times\vec{A})^2\right]$$

$$A = \sum_k \cdots \left(e^{ik\cdot x} a + e^{-ikx} a^*\right)$$

square, $\int d^3x \sum_{kk'} e^{ik\cdot x} e^{-ik'\cdot x} \cdots = \sum_{kk'} \delta_{kk'}$

$$\boxed{U = \frac{1}{2}\sum_{k\sigma} \hbar\omega_k \left[a_{k\sigma} a^*_{k\sigma} + a^*_{k\sigma} a_{k\sigma}\right]} \quad **$$

**: classical E+M in free space is a set of uncoupled classical harmonic oscillators!

Classical to quantum

Classical		Quantum
$a_{k\sigma}$	$\longrightarrow$	$\hat{a}_{k\sigma}$ where $[\hat{a}_{k\sigma}, \hat{a}^+_{k'\sigma'}] = \delta_{kk'}\delta_{\sigma\sigma'}$
$a^*_{k\sigma}$		$\hat{a}^+_{k\sigma}$

$$[\hat{a}_{k\sigma}, \hat{a}_{k'\sigma'}] = [\hat{a}^+_{k\sigma}, \hat{a}^+_{k'\sigma'}] = 0$$

$$H_{rad} = \sum_{k\sigma} \hbar\omega_k \left[a^+_{k\sigma} a_{k\sigma} + \frac{1}{2}\right]$$

conventional to drop the $\frac{1}{2}$ (= "zero point energy")

$$H|\psi\rangle = E|\psi\rangle$$

$$|\psi\rangle = |\{n_{k\sigma}\}\rangle \qquad n_{k\sigma} = 0, 1, 2 \cdots$$

i.e. for every mode ($\equiv k\sigma$ - think about modes in a cavity) assign an integer

$$|\Psi\rangle = |0_{k_1\sigma_1}, 0_{k_2\sigma_2}, 17_{k_3\sigma_3}, 3_{k_4\sigma_4} \cdots\rangle$$

$$H|\Psi\rangle = E|\Psi\rangle \qquad = |\{n_{k\sigma}\}\rangle$$

$$E = \sum_{k\sigma} \hbar\omega_k \, n_{k\sigma}$$

Interpretation of quantum $\vec{A}$ exactly the same as quantum x for SHO ($\hat{x}$)

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \; ; \quad \hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,|n+1\rangle + \sqrt{n}\,|n-1\rangle\right)$$

$\hat{A}$ is an operator which raises and lowers $n_{k\sigma}$'s

$$\vec{A}(x,t) = \sum_{k\sigma}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2} \epsilon_{k\sigma}\left[\hat{a}_{k\sigma} e^{ik\cdot x} + \hat{a}^\dagger_{k\sigma} e^{-ik\cdot x}\right]$$

"raises and lowers $n_{k\sigma}$'s" $\equiv$ creates and annihilates photons

$$\vec{A}(x,t)\,|n_{k_1\sigma_1} \, n_{k_2\sigma_2} \cdots\rangle$$

$$= \sum_{k\sigma} (\quad) \epsilon \left\{ e^{ik\cdot x}\sqrt{n_{k_1\sigma_1}+1}\,|n_{k_1\sigma_1}+1, \cdots\rangle + e^{-ik\cdot x}\sqrt{n_{k\sigma}}\,|n_{k_1\sigma_1} \cdots\rangle \right\}$$

complicated looking because system is diagonal in k-space.

Emission/absorption

$$H = H_{atom} + H_{rad} + H_I$$

$$\underbrace{\frac{p^2}{2m} + V + \sum \hbar\omega_k n_{k\sigma}}_{H_0} \quad \underbrace{- \frac{e}{mc} \hat{A}\cdot\hat{p}}_{H_I} \quad + \frac{e^2}{2m} A^2$$

Complete specification of eigenstate of H_0 is product

$$|a\rangle_{atom} \, |\{n_{k\sigma}\}\rangle_{rad}$$

$$E_i = E_a + \sum n_{k\sigma} \hbar\omega_k$$

Example for emission - suppose only have $n_{k\sigma}$ photons in one mode to start, at the end

$$n_{k\sigma} \rightarrow n_{k\sigma} + 1$$

$$|i\rangle = |a\rangle_{atom} \, | 0, 0, 0 \ldots, n_{k\sigma}, 0 \ldots \rangle, \quad E_i = E_a + n_{k\sigma}\hbar\omega_{k\sigma}$$

$$|f\rangle = |b\rangle_{atom} \, | 0, 0, 0 \ldots, n_{k\sigma} + 1, 0 \rangle \quad E_f = E_b + (n_{k\sigma}+1)\hbar\omega_{k\sigma}$$

emphasize <u>complete</u> again
specification

So much for H_0. What is operator $\hat{H}_I$?

$$\hat{H}_I = -\frac{e}{mc} \sum_{k'\sigma'} \left(\frac{2\pi\hbar c^2}{\omega_{k'} V} \right)^{\frac{1}{2}} \vec{p} \cdot \hat{\epsilon}_{k'\sigma'} \left[\hat{a}_{k'\sigma'} e^{i\vec{k}'\cdot x} + a^{+}_{k'\sigma'} e^{-i\vec{k}'\cdot x} \right]$$

$$\langle f | \hat{H}_I | i \rangle = ?$$

$$\langle f|\hat{H}_I|a\rangle = -\frac{e}{mc}\sum_{k'\sigma'}\langle n_{k\sigma}+1|a^+_{k'\sigma'}|n_{k\sigma}\rangle$$

$$\times\left(\frac{2\pi\hbar c}{\omega_k V}\right)^{1/2}\langle b|\hat{\epsilon}_{k'\sigma'}\cdot\vec{p}\,e^{-ik'\cdot x}|a\rangle$$

$$= -\frac{e}{mc}\left(\frac{2\pi\hbar c}{\omega_k V}\right)^{1/2}\sqrt{n_{k\sigma}+1}\,\langle b|\hat{\epsilon}_{k\sigma}\cdot\vec{p}\,e^{-ik\cdot x}|a\rangle$$

Energy difference is $E_b + \hbar\omega_k - E_a$.

Note no $e^{\pm i\omega t}$ is in H_I – this is a time independent perturbation, so Golden Rule is

$$d\Gamma = \frac{2\pi}{\hbar}|\langle f|H_I|a\rangle|^2\,\delta(E_f - E_i)\cdot\underbrace{V\frac{d^3k}{(2\pi)^3}}_{\text{many possible } |k|\text{'s}}$$

$$= \frac{2\pi}{\hbar}\left(\frac{2\pi\hbar c}{\omega_k V}\right)(n_{k\sigma}+1)\cdot|\langle b|\hat{\epsilon}\cdot\vec{p}\,e^{-ik\cdot r}|a\rangle|^2$$

$$\times\,\delta(E_b + \hbar\omega - E_a)\cdot V\frac{d^3p_\gamma}{(2\pi\hbar)^3}$$

Apart from the $n_{k\sigma}+1$) exactly what we had before. Now can discuss 2 kinds of emission: in particular, lifetime, selection rules, --- intensities of spectral lines

spontaneous emission $n_{k\sigma}=0$

<u>stimulated</u> emission $\Gamma \sim n_{k\sigma}+1$

(light amplification by stimulated emission of radiation ???)

absorption

$$|\langle n_{k\sigma}-1|\,a_{k\sigma}\,|n_{k\sigma}\rangle|^2 = n_{k\sigma}$$

$$d\Gamma \sim \frac{2\pi}{\hbar}\underbrace{\left(\frac{2\pi\hbar c}{\omega V}\right) n_{k\sigma}}_{\text{we'd call this } \frac{2\pi c^2}{\omega^2} I(\omega)\Delta\omega} |\langle b|p\cdot\epsilon|a\rangle|^2 \times \delta(\Delta E-\hbar\omega)$$

or $$n_{k\sigma}\left(\frac{2\pi\hbar c}{\omega V}\right) = \frac{2\pi c^2}{\omega^2} I(\omega)\Delta\omega$$

$$n_{k\sigma}\hbar\omega = V I(\omega)\Delta\omega$$

$I(\omega)$ = energy density per freq interval

Can I summarize the ~~course~~ course?

The unifying physics theme is radiation

$$\vec{A}(x,t) = \frac{4\pi}{c}\int d^3x'dt' \, J(x',t')\, D(x-x', t-t')$$

realized in many "special cases" – and each special case had a long back story which (to be honest) was not fully explained or followed up.

1) No reference to relativity, $J(t) \sim e^{i\omega t}$

antennas – mostly in terms of multipoles

scattering – nearly all, variations ~~on~~ on dipole formula

diffraction

2) Relativity for itself – beginning almost at sophomore level, extending to (almost) what you need for GR

Lagrangian for E&M – plus – can you motivate it?

7320 Answer: partially: symmetries constrain $\mathcal{L}$. but this is incomplete!

3) Radiation & relativity combined – just a "fingertip feel" for this very extended subject – with ~~practical~~ applications ~~&~~ ranging from practical to astrophysical.

And of course, everything was done in approximation, nothing was exact.

Okay, I tried to come up with something inspirational to say at the end, but everything I thought of sounded pretentious.

So I'll just say a couple of practical things: I will try to be around my office for questions

but Thursday I have comps exams from 10 to 2 and a meeting at 3 - 4 or 5 (?)

Friday I have a comps exam 11-1

If you get extra time Monday, check with me today, show up at my office at appropriate time

Final 430-7 here - covers entire semester (~~[illegible]~~ slight bias toward end)

I'll get grades out as quickly as I can, no promises

pick me: I hope the class was { fun, interesting or not the worst experience } of your life