Radiation from Moving Charges

Our ingredients are

1) ~~Relativistically~~ covariant Green's function Sec 14.1 (~~[illegible]~~)
$D(r,t;r',t')$

2) Rel. covariant source for point charge (~~[illegible]~~) eq. 12.139, derivation follows

$$J^{\mu}(x';t')$$

Then

$$A^{\mu}(x) = \frac{4\pi}{c}\int d^4x'\, D(x-x')\, J^{\mu}(x')$$

⇒ solve for potentials of pt charge
≡ Lienard-Wiechert potentials

2) check they agree w/ Lorentz transform

3) ~~Larmor formula~~ general result
(arbitrary motion of charge)

4) NR Larmor formula

5) applications

~~Skip to RM-5~~

Recall Lorentz-gauge Green's function

$$D(r,t) = \frac{\Theta(t-t')}{4\pi R}\,\delta\left(t-t'-\frac{R}{c}\right)$$

Θ: causality, $\vec{R} = \vec{x}-\vec{x}'$

Useful to write this in a way that Lorentz invariance is obvious.

define
$$z_0 \equiv t-t'$$
$$z^\mu \equiv (z_0, \vec{R})$$

$$\delta(z_\mu z^\mu) = \delta(z_0^2 - R^2) = \delta[(z_0-R)(z_0+R)]$$
$$= \frac{\delta(z_0-R)+\delta(z_0+R)}{2R}$$

using $\delta(f(x)) = \sum \frac{\delta(x-x_0)}{f'(x_0)}$

$$\therefore\quad D(z) = \Theta(z_0)\,\frac{\delta(z_\mu z^\mu)}{2\pi}$$

Now define $\eta^\alpha = (1,0,0,0)$ in our frame

$z^0 = z^\alpha \eta_\alpha$ in any frame

$$D(z) = \Theta(z^\alpha \eta_\alpha)\,\frac{\delta(z_\mu z^\mu)}{2\pi} \equiv \text{retarded Green's fn}$$

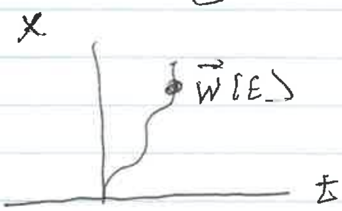
$$A^\mu(x) = \frac{4\pi}{c}\int d^4x'\, D(x-x')\, J^\mu(x')$$

Start with charge density moving on prescribed trajectory

J^μ:

$$\rho(x) = q\,\delta^3(\vec{x} - \vec{W}(t))$$

(always the case in what follows)



$$= q\int dt'\,\underbrace{\delta(t-t')\,\delta^3(\vec{x}-\vec{W}(t'))}$$

$\vec{W}$ = 3 vector

$W_0 = ct'$

$$\int c\,\delta^4(x - W(\tau))\cdot$$

express in terms of proper time τ

$$= c\,\delta(ct' - ct)\,\delta^3(\vec{x} - \vec{w}(t'))$$

convert $dt' = \gamma(\tau)\,d\tau$ to proper time $\int$

$$\rho(x,t) = q\int d\tau\,[c\gamma(\tau)]\,\delta^4(x - W(\tau))$$

Now recall 4-velocity

$$u^\mu = (\gamma c, \gamma\vec{v})$$

$$J^\mu = (c\rho, \vec{J})$$

$$J^\mu_{pt}(x) = qc\int d\tau\, u^\mu(\tau)\,\delta^4(x - W(\tau))$$

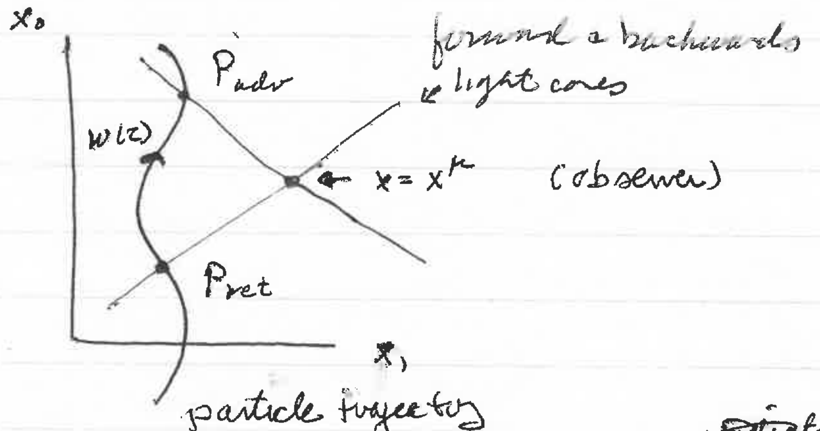
in terms of 4-velocity and 4-coordinate of particle

$$J^\mu = qc \int d\tau \; u^\mu(\tau)\, \delta^4(x - w(\tau))$$

$$A^\mu(x) = \frac{4\pi}{c}\int d^4x' \, D(x-x')\, J^\mu(x')$$

$$D(x-x') = \frac{\theta(x_0 - x_0')\,\delta\left((x-x')^2\right)}{2\pi}$$

$$A^\mu(x) = 2q\int d\tau \; u^\mu(\tau)\, \theta(x_0 - w_0(\tau))\, \delta\left((x-w(\tau))^2\right)$$



The δ-function insures that the ~~particle's~~ radiating contribution to $A^\mu(x)$ comes only from the parts of the current's world line which are on a light cone centered at x. $\delta((x-w(\tau))^2)$ There are generally 2 solutions (corresponding to the advanced solution and the retarded solution) and the θ-function picks the retarded one.

Let's assume the zeros of $x - w(\tau)$ are linear (true if $v<c$)

$$\delta(f(x)) = \frac{\delta(x - x_0)}{\left.\frac{df}{dx}\right|_{x_0}}$$

$$\frac{d}{d\tau}(x - w(\tau))^2 = -2(x - w(\tau))_\mu \frac{dw^\mu}{d\tau} = -2(x-w(\tau))_\mu \, u^\mu(\tau) = -2u\cdot(x - w(\tau))$$

$$A^\mu(x) = \left.\frac{q\, u^\mu(\tau)}{u\cdot(x - w(\tau))}\right|_{\tau=\tau_0}$$

τ_0 on past LC

$$A^\mu(x) = 2g \int d\tau\, u^\mu(\tau)\, \Theta(x_0 - w_0(\tau))\, \delta((x-w(\tau))^2)$$

If $v<c$, δ-fn has a linear zero

$$\delta(f(x)) = \frac{\delta(x-x_0)}{\left|\left(\frac{\partial f}{\partial x}\right)_{x=x_0}\right|}$$

$$\frac{d}{d\tau}(x-w(\tau))^2 = -2\,(x-w(\tau))_\mu \frac{dw^\mu}{d\tau}$$

$$= -2\,(x-w(\tau))_\mu u^\mu(\tau)$$

$$= -2\,u\cdot(x-w(\tau))$$ (4 vector dot product)

$$A^\mu = \left.\frac{g\,u^\mu(\tau)}{u\cdot(x-w(\tau))}\right|_{\tau=\tau_0}$$ $\equiv$ Lienard Wiechert potential

A more useful form

$$x_0 - w_0(\tau_0) = |\vec{x}-\vec{w}(\tau_0)| = R \quad \equiv \text{“retarded time”}$$

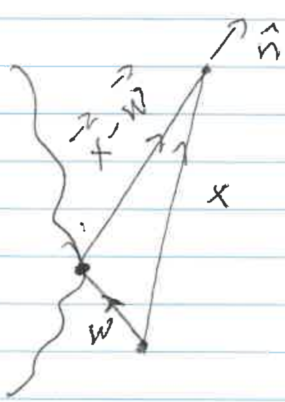
$u = (\gamma c, \gamma\vec{v})$

$$u\cdot(x-w) = u_0(x_0-w_0) - \vec{u}\cdot(\vec{x}-\vec{w})$$

$$= \gamma c R - \gamma \vec{v}\cdot\hat{n} R$$

$$= \gamma c R\left[1-\vec{\beta}\cdot\hat{n}\right] \quad ; \ \vec{\beta} = \frac{\vec{v}(\tau_0)}{c}$$

$\hat{n}$ unit vector along $\vec{x}-\vec{w}(\tau_0)$



$$\Phi(x,t) = \left.\frac{g}{(1-\vec{\beta}\cdot\hat{n})R}\right)_{ret}$$

$$\vec{A}(x,t) = \left.\frac{g\vec{\beta}}{(1-\vec{\beta}\cdot\hat{n})R}\right)_{ret}$$

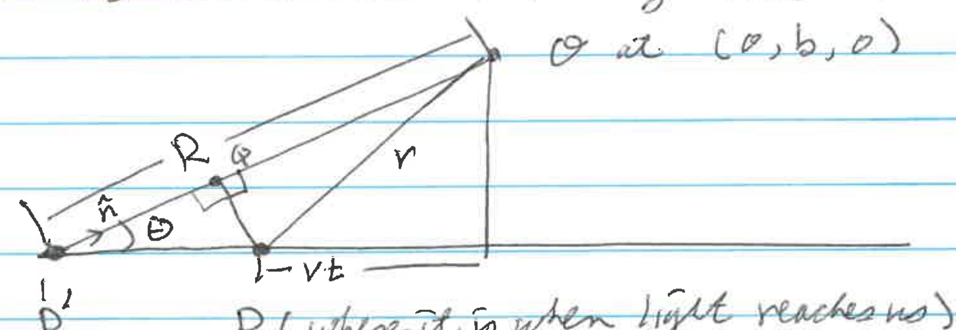
$w_0(\tau_0) = x_0 - R$
$\equiv$ ret

$$\Phi(x,t) = \frac{q}{(1-\vec{\beta}\cdot\hat{n})R}\bigg|_{ret} \qquad \vec{A}(x,t) = \frac{q\vec{\beta}}{(1-\beta\cdot n)R}\bigg|_{ret}$$

Ret: $w_0(\tau_0) = x_0 - R$

Contrast/compare Lorentz transforming potential of point charge –
then we expressed $\Phi, \vec{A}, \vec{E}, \vec{B}$ in terms of present position of charge. Here it is in terms of retarded position. First LW formula.

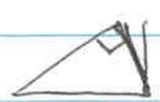
Consider charge moving at constant velocity in x, observer at b on y axis, at time $t=0$ closest approach



(retarded position)

$v(t_r - t)$
$|v(t-t_r)| = \beta R$

In time light moves a distance R, particle goes from P' to P, a distance $v\cdot\left(\frac{R}{c}\right) = \beta R$



$(\vec{\beta}\cdot\hat{n})R = \beta\cdot R = \beta R\cos\theta = \underline{P'Q}$

$OQ = \vec{R} - \hat{n}\beta\cdot R = \vec{R}(1-\beta\cdot n)$

$QP = \beta R\sin\theta$

$R\sin\theta = b$

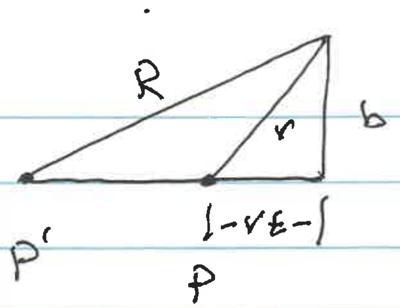
KM8

particle moves at v = constant in x

At time $t=0$ particle is at $x=0$

at time t it is at P

(vt away — for radiation seen at $t=0$, $t<0$)



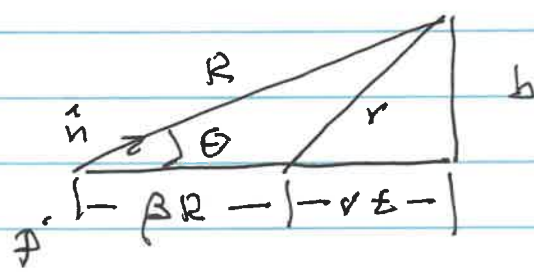
At time t observer sees light from when particle is at P' where time is t_r,

$$t - t_r = \frac{R}{c}$$

In a time $t - t_r$, particle moves a distance

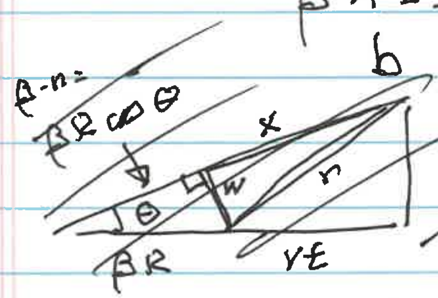
$$v(t - t_r) = \beta R$$

i.e. $v t_r = vt - \beta R$ an earlier time



Now it's geometry

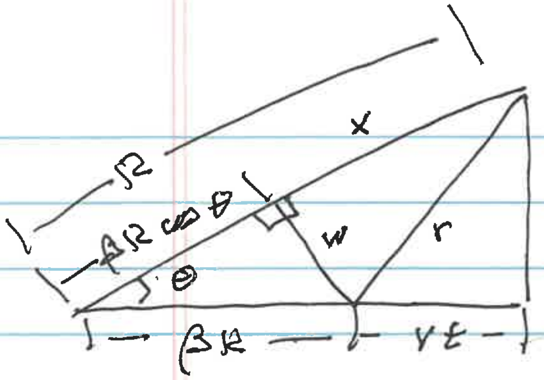
$$\left.\begin{array}{l} R \\ \beta \cdot n = \beta \cos\theta \end{array}\right\}\left\{ r,\ t,\ v,\ b \right.$$



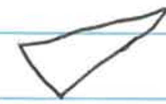
$\cancel{w = \beta R \sin\theta}$

$\cancel{x = R - \beta n \cdot R = R(1 - \beta \cdot n)}$

$\cancel{r^2 = x^2 + w^2}$



$$x = R(1-\beta\cdot\hat{n})$$
$$w = \beta R \sin\theta$$
$$b = R\sin\theta$$
$$\Big\}\ w = \beta b$$



a) $r^2 = w^2 + x^2$
$$= \beta^2 b^2 + R^2(1-\beta\cdot n)^2$$

b) $r^2 = b^2 + (vt)^2$

$$\therefore\ R^2(1-\beta\cdot n)^2 = b^2 + (vt)^2 - \beta^2 b^2$$

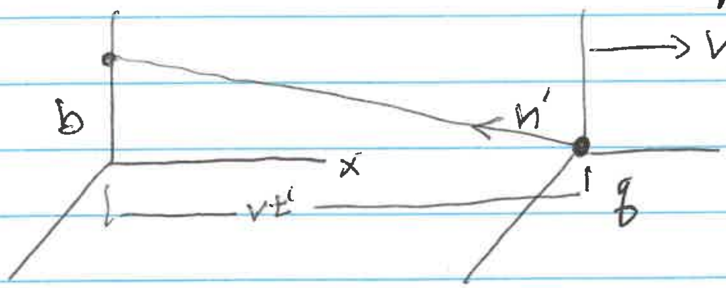
$$\Phi = \frac{q}{(1-\beta\cdot n)R}\Bigg)_{ret} = \frac{q}{\left[b^2 + (vt)^2 - \beta^2 b^2\right]^{1/2}}$$

$$= \frac{q}{\left[(1-\beta^2)\left(b^2 + \left(\frac{vt}{\sqrt{1-\beta^2}}\right)^2\right)\right]^{1/2}}$$

$$= \frac{\gamma q}{\left[b^2 + (\gamma v t)^2\right]^{1/2}}$$

$$A_x = \frac{\gamma q \vec{\beta}}{\left[b^2 + (\gamma v t)^2\right]^{1/2}}$$

Lorentz transformation instead
K' = frame of charge



in K', observer is at $x' = -vt'$, $y' = b$, charge is static

$$\Phi' = \frac{q}{(b^2 + (vt')^2)^{1/2}} \qquad A'_x = 0$$

$$\begin{pmatrix} \Phi \\ A_x \end{pmatrix} = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} \begin{pmatrix} \Phi' \\ A' \end{pmatrix} = \begin{bmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} \Phi' \\ 0 \end{bmatrix}$$

$$\Phi = \frac{\gamma q}{(b^2 + (vt')^2)^{1/2}} \quad \text{and } t' = \gamma t$$

$$\Phi = \frac{\gamma q}{(b^2 + (\gamma v t)^2)^{1/2}}$$

$$A_x = \beta\gamma\Phi' = \frac{\beta\gamma q}{[b^2 + (\gamma v t)^2]^{1/2}}$$

Lorentz transformation agrees with Liénard-Wiechert potentials.

Constant velocity part of $\vec{E}$ is

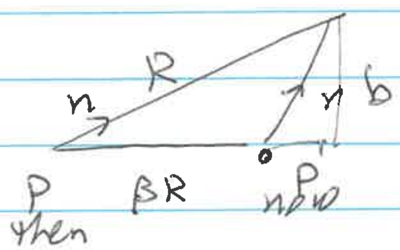
skip

$$\vec{E} = q\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3R^2}\right]_{ret}$$

Recall geometry of potential

$\hat{n}-\vec{\beta}$ from $\vec{r}+\vec{\beta}R = \vec{R}$ ∴ $\vec{r} = \vec{R}-\beta\vec{R}$

$= R(\hat{n}-\vec{\beta})$



$$R^2(1-\beta\cdot n)^2 = r^2-\beta^2b^2 = b^2+v^2t^2-\beta^2b^2$$
$$= \frac{1}{\gamma^2}\left(b^2+\gamma^2v^2t^2\right)$$

$$\vec{E} = q\,\frac{\frac{\vec{r}}{R}\frac{1}{\gamma^2}\frac{1}{R^2}}{\left[\frac{1}{\gamma^3R^3}\left(b^2+\gamma^2v^2t^2\right)^{3/2}\right]}$$

$$= \frac{q\gamma\vec{r}}{\left[b^2+(\gamma vt)^2\right]^{3/2}}$$

— note how it points along present position of charge : i.e. $\vec{r} = -vt\,\hat{z}$

$r\cdot x = -\cancel{\gamma} vt$

$$E_1 = -\frac{\gamma qvt}{\left(b^2+(\gamma vt)^2\right)^{3/2}},\quad E_2 = \frac{\gamma qb}{\left(b^2+(\gamma vt)^2\right)^{3/2}}$$

again, agrees w/ LT formula

RMK

$$\Phi = \frac{q}{[1-\vec\beta\cdot\hat n]_{ret}} \qquad \vec A = \frac{q\vec\beta}{[1-\vec\beta\cdot\hat n]_{ret}}$$

Fields: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$A^\mu(x) = 2q\int d\tau\, \Theta(x_0 - w_0(\tau))\, u^\mu(\tau)\, \delta((x-w(\tau))^2)$$
$$= \frac{q\, u^\mu(\tau)}{u\cdot(x-w(\tau))}\bigg|_{ret}$$

Horribly long, un-illuminating derivation ...

$$\vec E(x,t) = q\left[\frac{\hat n - \vec\beta}{\gamma^2(1-\vec\beta\cdot\hat n)^3 R^2}\right]_{ret}$$

problem 14.1 ← to show this agrees w/ LT

no higher deriv because $A^\mu \sim u^\mu$

$$+ \frac{q}{c}\left[\frac{\hat n \times \{(\hat n - \vec\beta)\times\dot{\vec\beta}\}}{(1-\vec\beta\cdot\hat n)^3 R}\right]_{ret}$$

$$\vec B(x,t) = (\hat n \times \vec E)_{ret}$$

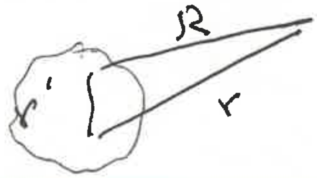
Note first term falls off like $\frac{1}{R^2}$, 2nd term as $\frac{1}{R}$

2nd term gives radiation — only accelerated charges radiate

- well, yes, this makes sense Lorentz transf. gives answer if particle velocity is constant (and gives first term), go to frame moving w/ particle, no radiation there!

NR limit is pretty easy to get and most useful!, let's do it explicitly

$$\Phi(r) = \int d^3r' \frac{\rho(r', t-\frac{R}{c})}{R}$$



$$R = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'} \simeq r - \hat{n}\cdot r' \quad \text{far field}$$

$$\frac{1}{R} \simeq \frac{1}{r}\left(1 + \frac{\hat{n}\cdot\vec{r}'}{r}\right)$$

$$\rho(r', t - R/c) = \rho(r', \overbrace{t - \frac{r}{c}}^{t_0} + \frac{\hat{n}\cdot\vec{r}'}{c})$$

$$\equiv \rho\left(r', t_0 + \frac{n\cdot r'}{c}\right) \quad \text{which we Taylor expand about } t_0$$

$$= \rho(r', t_0) + \frac{d\rho}{dt}(r', t_0)\cdot\left(\frac{\hat{n}\cdot\vec{r}'}{c}\right) + \cdots$$

expand $\frac{1}{R}$

$$\Phi(r,t) = \frac{1}{r}\int d^3r'\,\rho(r') + \frac{\hat{n}\cdot}{r^2}\int \hat{r}'\rho(r', t_0)\,d^3r'$$

$$+ \frac{1}{rc}\,\hat{n}\cdot\frac{d}{dt}\int \vec{r}'\rho(r', t_0)\,d^3r'$$

expand ρ

$$= \frac{Q}{r} + \frac{\hat{n}\cdot\vec{p}(t_0)}{r^2} + \frac{1}{c}\frac{n\cdot}{r}\frac{d}{dt}\vec{p}(t_0) + \cdots$$

introducing dipole moment and its derivative. For a radiation field we want a $\frac{1}{r}$ term in $\nabla\Phi$, which can only come from the last term

$$\vec{\nabla}\Phi \sim \frac{1}{rc}\vec{\nabla}\left[\hat{n}\cdot\dot{\vec{p}}(t_0 = t - r/c)\right]$$

$$= -\frac{1}{rc^2}\left(\hat{n}\cdot\ddot{\vec{p}}(t)\right)\hat{n}$$

chain rule + $\vec{\nabla} r = \hat{n}$

$$\vec{p} = q\vec{r} \quad \text{so} \quad \frac{\dot{\vec{p}}}{c} = q\vec{\beta}$$

$$\nabla(n\cdot\dot{p}[t - \tfrac{r}{c}])$$

$$-\nabla\Phi = \frac{q}{rc}\,\hat{n}(\hat{n}\cdot\dot{\vec{\beta}})$$

$$= -\frac{1}{c}\,n\cdot\frac{d}{dt}\dot{p}$$

$$\nabla \Phi = \frac{1}{rc} \vec{\nabla}\left[\hat{n} \cdot \vec{\dot{p}}\left(t_0 = t - \frac{r}{c}\right)\right]$$

call $\dot{p}_i(t_0) \equiv K_i(t_0)$

$$\nabla \Phi = \frac{1}{rc} \, n_i \, \vec{\nabla} K_i(t_0)$$

$$\vec{\nabla} K_i(t_0) = \frac{dK_i}{dt_0} \vec{\nabla}(t_0)$$

$$= \frac{dK_i}{dt} \vec{\nabla}\left(t - \frac{r}{c}\right)$$

and $\nabla \vec{r} = \hat{n}$ also a $-\frac{1}{c}$ from $-\frac{r}{c}$

$$\vec{\nabla} \Phi = -\frac{1}{rc^2}\left(\hat{n} \cdot \frac{d^2\vec{p}}{dt^2}\right)\hat{n}$$

$$\vec{p} = q\vec{r} \quad \text{so} \quad \frac{1}{c}\frac{d^2 \vec{p}}{dt^2} = q\dot{\vec{\beta}}$$

$$-\nabla \Phi = \frac{q}{rc}\, \hat{n}\,(\hat{n} \cdot \dot{\vec{\beta}})$$

$$\vec{A} = \int \frac{J(r', t - R/c)}{R} \, d^3r'$$

$$\simeq \frac{1}{r} \int J(r', t - R/c) \, d^3r'$$

Now recall Ch 9: $\int \vec{J} \, d^3r = \frac{d}{dt} \int \vec{r}' \rho \, d^3r$
after much fiddling

$$= \dot{p}(t_0)$$

$$\vec{A} = \frac{1}{r} \dot{p}(t_0)$$

$$\frac{1}{c}\frac{\partial A}{\partial t} = \frac{1}{rc} \ddot{p}(t_0) = \frac{q}{r} \dot{\beta}(t_0), \text{ as } \dot{p} = q\dot{r}', \; \ddot{p} = q\ddot{r}$$

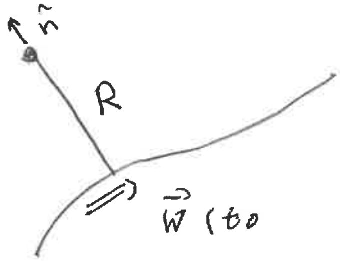
$$E = -\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t} = \frac{q}{cr}\left[\hat{n}(\dot{\vec{\beta}} \cdot \hat{n}) - \dot{\vec{\beta}}\right]$$

$$= \frac{q}{cr}\left[\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})\right]_{ret}$$

NR Larmor formula! ~~("ret" not necessary here)~~

but this isn't important for NR motion

Larmor formula : ~~[illegible]~~



$$\vec{E} = \frac{e}{Rc}\, \hat{n} \times (\hat{n} \times \dot{\vec{\beta}})\Big]_{ret}$$

if $\beta \ll 1$, negligible retardation,

$\hat{n}$ points at present position.

alternatively for instantaneous acceleration

$\vec{a}(t)$, $\quad \vec{E} = \frac{e}{Rc^2}\left[\hat{n} \times (\hat{n} \times \vec{a})\right]$

Antenna pattern? Recall radiation from

dipole, ~~$\vec{p}\,e^{i\omega t}$~~ $\vec{p}(t) = \mathrm{Re}\, \vec{p}\, e^{i\omega t}$

$$\vec{E} = -k^2 \frac{e^{ikR}}{R}\, \hat{n} \times (\hat{n} \times \vec{p}) \quad ; \quad \vec{p} \leftrightarrow \vec{a}$$

identical vector algebra - but

Now we compute an instanteous energy flux

(Then - we time averaged)

$\vec{E}$ & $\vec{B}$ are real; no time average, ~~[illegible]~~ :

* no extra $\frac{1}{2}$'s

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} \hat{n} E^2 \quad (\vec{B} = \hat{n} \times \vec{E})_{ret}$$

$$\frac{dP(t)}{d\Omega} = R^2 \hat{n} \cdot \vec{S} = \frac{e^2}{4\pi c^3} \left[\hat{n} \times (\hat{n} \times \vec{a})\right]^2$$

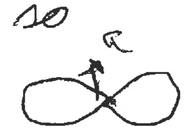
$$= \frac{e^2}{4\pi c^2} \left[\vec{a} \cdot \vec{a} - (\hat{n} \cdot \vec{a})^2\right]$$

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c^3}\left[\vec{a}\cdot\vec{a} - (\hat{n}\cdot\vec{a})^2\right]$$

the instantaneous angular distributed power emitted.

Total instantaneous power is easy: let $\vec{a}(t)$ define the $\hat{z}$ axis + $[\ ] = a^2[1-\cos^2\theta]$ so

general result: emission $\perp$ acceleration!



$$P = \frac{e^2 a^2}{4\pi c^3}\int [1-\cos^2\theta]\, d\cos\theta\, d\varphi$$

$$\left(2-\frac{2}{3}\right) \qquad \underbrace{\qquad}_{2\pi}$$

(not true for time ave!)

$$\frac{1}{4\pi}\cdot\frac{4}{3}\cdot 2\pi$$

$$\boxed{P(t) = \frac{2}{3}\frac{e^2}{c^3}a^2(t)}$$

instantaneous power emitted from arbitrary NR $\vec{a}(t)$.

Compare & contrast w/ linear dipole

$$x(t) = l\cos\omega t$$

$$a(t) = -\omega^2 l\cos\omega t$$

$$P(t) = \frac{2}{3}\frac{e^2\omega^4 l^2}{c^3}\cos^2\omega t$$

Time average: $\langle P\rangle = \frac{1}{3}\frac{\omega^4 e^2 l^2}{c^3}$

Dipole formula $\langle P\rangle = \frac{c}{8\pi}k^4|\hat{n}\times(\vec{p}\times\hat{n})|^2$

$\longrightarrow$ linear dipoles $p = el$, $\langle P\rangle = \frac{ck^4}{3}(el)^2$

$\longrightarrow$ $\omega = ck$ $\langle P\rangle = \frac{1}{3}\frac{\omega^4}{c^3}e^2l^2$

Example: Thompson scattering
AKA scattering on a free electron

NR power formula is

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\left[n\times(n\times\vec{a})\right]^2$$

measure (linear) polarization of outgoing radiation:

$$\frac{dP(\epsilon)}{d\Omega} = \frac{e^2}{4\pi c^3}\left[\hat{\epsilon}\cdot(\hat{n}\times(\hat{n}\times\vec{a}))\right]^2$$

$$= \frac{e^2}{4\pi c^3}\left|\hat{\epsilon}\cdot\vec{a}\right|^2 \quad \text{since } \hat{\epsilon}\cdot\hat{n}=0$$

Source of acceleration is an incident $\vec{E}$ field

$$\vec{a}(t) = \hat{\epsilon}_0 \frac{e}{m} E_0 \exp i(k_0\cdot x - \omega t)$$

Time average power formula

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{c}{8\pi}|E_0|^2 \cdot \underbrace{\left(\frac{e^2}{mc^2}\right)^2 |\hat{\epsilon}^*\cdot\hat{\epsilon}_0|^2}_{\frac{d\sigma}{d\Omega}}$$

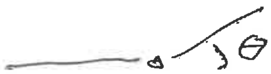
r_0 = classical electron radius

$$mc^2 = \frac{e^2}{r_0}, \quad r_0 = 2.8\times 10^{-13}\,\text{cm} = \frac{e^2}{\hbar c}\left(\frac{\hbar c}{mc^2}\right) = \alpha\,\frac{(197\text{ MeV}\times 10^{-13}\text{ cm})}{\frac{1}{2}\text{ MeV}}$$

$$\frac{d\sigma}{d\Omega} = r_0^2\,|\vec{\epsilon}^*\cdot\vec{\epsilon}_0|^2$$

This is for a particular initial & final polarization states. Notice $|\hat{\epsilon}^*\cdot\hat{\epsilon}_0|^2$ is the same as for E1 scattering - so copy the answer!

$$\frac{1}{2}\sum_{\epsilon_0}\sum_{\epsilon}\frac{d\sigma}{d\Omega} \equiv \frac{d\bar{\sigma}}{d\Omega} = \frac{1}{2} r_0^2 \left[1+\cos^2\theta\right]$$



$$\bar{\sigma} = \frac{8\pi}{3} r_0^2 = 0.66\times 10^{-24}\ \mathrm{cm}^2$$
$$= 0.66 \text{ barn.}$$

This is called the Thompson cross section.
The formula is valid in QM as well:

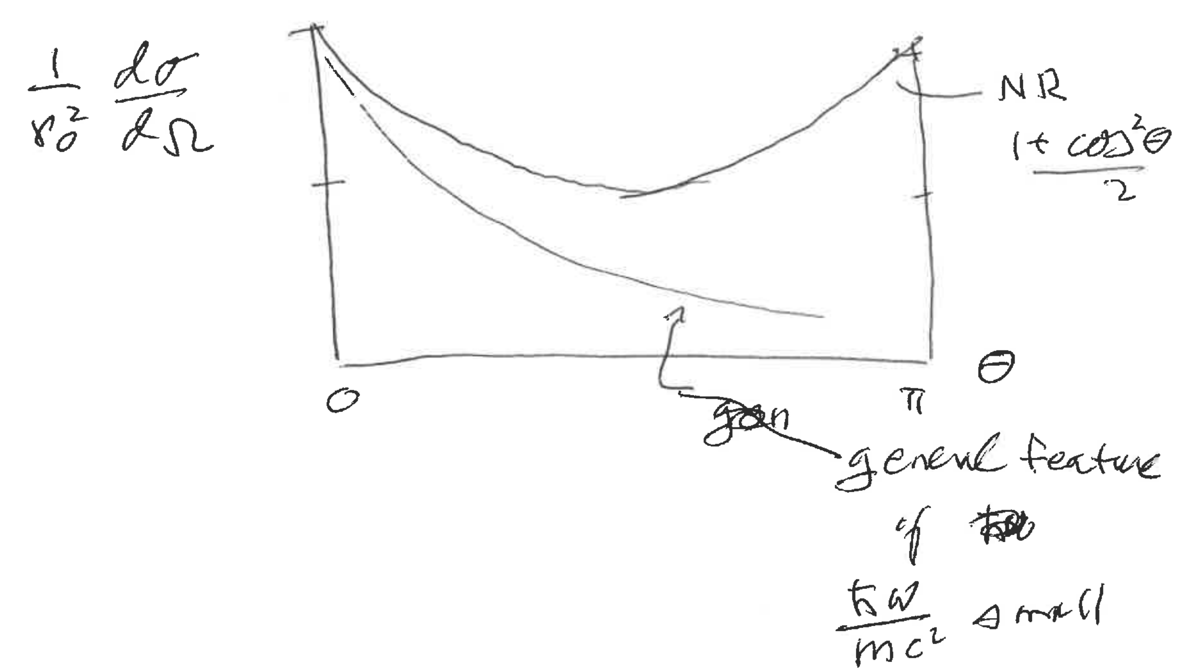
$$H = \frac{1}{2m}\left(\vec{p} - e\frac{\vec{A}}{c}\right)^2 \Rightarrow H_I = \frac{e^2}{2mc^2}\vec{A}\cdot\vec{A}$$

($p\cdot A$ is $\frac{v}{c}$ correction)

Valid if dynamics are NR;

$\hbar\omega \ll m_e c^2$ ($10^{-2} m_e c^2 = 5.1$ keV = x-rays)

Generalization to high energy (w/ Dirac equation) = "Klein-Nishina formula."



Many electrons in an atom? Blue sky, all over again

$$\frac{d\sigma}{d\Omega}(\text{many}) = \frac{d\sigma}{d\Omega}(\text{one}) \times \left| \sum_{i=1}^{Z} e^{i\vec{q}\cdot\vec{x}_i} \right|^2$$

with $\vec{x}_i$ = location of ith e^-, $q = \vec{k}_0 - \vec{k}$.

Problem: electrons aren't free!

Solution ①: if $\hbar\omega \gg$ binding energy, ignore binding: "quasi free scattering"

and if $qa \ll 1$ $e^{i\vec{q}\cdot\vec{x}_i} \sim 1$ $\quad a = \langle |\vec{x}_i - \vec{x}_j| \rangle$

$$\left| \sum_{i=1}^{Z} 1 \right|^2 = Z^2$$

$\Rightarrow$ coherent scattering at small angle

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{mc^2}\right)^2 |\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_0|^2$$

If $qa \gg 1$, cross terms average to zero

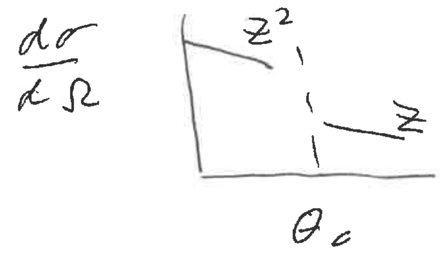
$$|\Sigma|^2 = Z, \quad \frac{d\sigma}{d\Omega} = Z\left(\frac{e^2}{mc^2}\right)^2 |\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_0|^2$$

$\Rightarrow$ incoherent scattering at large angles

$Z^2 \to Z$ at $qa \sim 1$, $\quad 2ka\sin\frac{\theta_c}{2} \sim 1$

$ka\,\theta_c \sim 1 \Rightarrow \theta_c = \frac{1}{ka}$

se Thomas-Fermi $\Rightarrow a \sim 1.4\, Z^{1/3} a_0$



Solution ② : form factor

$$e\sum e^{iq\cdot x} = \int d^3x'\, e^{iq\cdot x'}\rho(x') = \rho(q)$$

Note $\rho(0) = \int d^3x'\, \rho(x') = Ze$

$$F(q) = \frac{\rho(q)}{Ze}\,, \quad F(0) = 1$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{mc^2}\right)^2 |\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_0|^2\, |F(q)|^2$$

Solution ③

real QM calc : 2nd order PT

or

"mass on spring" model

$$a = \frac{e}{mc}\omega^2 \sum_j \frac{1}{\omega_{0j}^2 - \omega^2 - i\omega\Gamma_j}$$

Analogy with dipole radiation: assume harmonic motion

$p = ql$, $\quad \underline{P} = \frac{ck^4}{3} p^2 = \frac{1}{3c^3} q^2 l^2 \omega^4$

For up, if $\quad v = \omega l \sin \omega t \quad a = -\omega^2 l \cos \omega t$

$\langle a^2 \rangle = \frac{1}{2} \omega^4 l^2 \quad$ time average gives $\frac{1}{2}$

$$\underline{P} = \frac{2}{3} \frac{q^2}{c^3} \langle a^2 \rangle = \frac{1}{3} \frac{q^2 l^2 \omega^4}{c^3}$$

We can generalize the Larmor formula to arbitrary velocity motion (relativistic motion)

Recall energy & time are 4th components of 4 vectors

power $= \frac{\text{energy}}{\text{time}}$ is invariant

Rewrite $\quad P = \frac{2}{3} \frac{q^2}{m^2 c^3} (m\dot{\vec{v}}) \cdot (m\dot{\vec{v}})$

$$\rightarrow -\frac{2}{3} \frac{q^2}{m^2 c^3} \frac{dp^\mu}{d\tau} \cdot \frac{dp_\mu}{d\tau}$$

(this is the unique invariant built of $\frac{dp_\mu}{d\tau}$ and p_μ, which has the correct NR limit)

Component formula is useful: recall $p^\mu = mu^\mu$ and

4-acceleration! $\quad \frac{du^\mu}{d\tau} = c\gamma^2 \left[\gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}}), \ \dot{\vec{\beta}} + \gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}}) \vec{\beta} \right]$

$$\left[\frac{du}{d\tau}\right]^2 = c^2 \gamma^4 \left[\gamma^4 (\beta \cdot \dot{\beta})^2 - (\dot{\beta})^2 - 2(\beta \cdot \dot{\beta}) \gamma^2 (\beta \cdot \dot{\beta}) - \gamma^4 \beta^2 (\beta \cdot \dot{\beta})^2 \right]$$

$$= c^2\gamma^4\Big[(\beta\cdot\dot\beta)^2\big[\underbrace{\gamma^4 - 2\gamma^2 - \gamma^4\beta^2}\big] - (\dot\beta)^2\Big]$$

$$\gamma^4(1-\beta^2) - 2\gamma^2 = \frac{\gamma^4}{\gamma^2} - 2\gamma^2 = -\gamma^2$$

$$\left(\frac{du}{d\tau}\right)^2 = -c^2\gamma^4\left[(\dot\beta)^2 + \underline{\gamma^2(\beta\cdot\dot\beta)^2}\right]$$

Note $\gamma^2(\vec\beta\times\dot{\vec\beta})^2 = \gamma^2\left(\beta^2\dot\beta^2 - (\beta\cdot\dot\beta)^2\right)$

so $\gamma^2(\beta\cdot\dot\beta)^2 = \gamma^2(\beta^2\dot\beta^2) - \gamma^2(\beta\times\dot\beta)^2$

$$\left(\frac{du}{d\tau}\right)^2 = -c^2\gamma^4\left[(1+\gamma^2\beta^2)\dot\beta^2 - \gamma^2(\beta\times\dot\beta)^2\right]$$

$$1+\gamma^2\beta^2 = 1 + \frac{\beta^2}{1-\beta^2} = \frac{1-\beta^2+\beta^2}{1-\beta^2} = \gamma^2$$

$$= -c\gamma^6\left[(\dot\beta)^2 - (\beta\times\dot\beta)^2\right]$$

NR: $\gamma \to 1$

$\beta\times\dot\beta \ll \dot\beta$

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot{\vec\beta})^2 - (\vec\beta\times\dot{\vec\beta})^2\right]$$

Lienard (1898) result. You can grind it out of the L-W potential (see Griffiths!)

Note γ^6 behavior of radiated power!

Examples: pure linear, pure circular motion $\to$

1) Linear accelerator – 1-d motion, $\vec\beta \times \dot{\vec\beta} = 0$

easiest to back up: $P = -\frac{2}{3}\frac{e^2}{m^2c^3}\frac{dP_\mu}{d\tau}\frac{dP^\mu}{d\tau}$ $\qquad dt = \gamma d\tau$

$$\left(\frac{dP^\mu}{d\tau}\right)^2 = \gamma^2\left[\frac{1}{c^2}\left(\frac{\partial E}{\partial t}\right)^2 - \left(\frac{\partial \vec p}{\partial t}\right)^2\right]$$

First term: $E^2 - c^2p^2 = (mc^2)^2$

$2E\,dE - 2c^2 p\,dp = 0$ $\qquad \left(\frac{cp}{E} = \beta\right)$

$dE = c^2 \frac{p}{E} dp = c\,\beta\,dp$

so $\frac{dE}{dt} = c\beta\frac{dp}{dt}$

$$\left(\frac{dp}{d\tau}\right)^2 = \gamma^2(\beta^2 - 1)\left(\frac{d\vec p}{dt}\right)^2 = \left(\frac{d\vec p}{dt}\right)^2$$

$$\frac{\partial p}{\partial t} = \text{force} = -\frac{\partial E}{\partial x} \Rightarrow P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2$$

Compare to ~~pow~~ power supplied by external force

$$P_{supp} = \frac{dE}{dt} = \frac{dE}{dx}\cdot v$$

$$r = \frac{P_{radiated}}{P_{supplied}} = \frac{2}{3}\frac{e^2}{m^2c^3}\frac{1}{v}\frac{\partial E}{\partial x} \underset{\substack{v=c\\ e=e}}{\Rightarrow} \frac{2}{3}\frac{e^2}{mc^2}\frac{1}{mc^2}\frac{\partial E}{\partial x}$$

e^2/mc^2 = classical electron radius = 3×10^{-13} cm = 3 fm

$r = \frac{2}{3} \times \frac{3\text{ fm}}{\frac{1}{2}\text{MeV}}\frac{dE}{dx} \Rightarrow$ to be $O(1)$ $\frac{dE}{dx} = \frac{\text{MeV}}{\text{fm}} = 10^{18}\frac{\text{eV}}{\text{cm}}$!

Negligible effect of radiation on macroscopic "linear acceleration" (but can be important in high energy process)

Circulating accelerators are another story!

Change in energy per orbit is small (few keV)

$$\left|\frac{d\vec{p}}{d\tau}\right| = \gamma\left|\frac{d\vec{p}}{dt}\right| = \gamma\omega p \gg \frac{1}{c}\frac{dE}{dt} \qquad p \sim e^{i\omega t}$$

$$\underline{P} \simeq \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\omega^2|\vec{p}|^2$$

$p = \gamma m v = \gamma m \omega \rho$ for bend radius ρ, freq. ω

$\omega\rho = v = c\beta$

$$\underline{P} = \frac{2}{3}\frac{e^2}{m^2c^3}\left[\begin{array}{l} p = \cancel{\gamma m \rho \omega} \\ \quad = \gamma m \rho \cdot \frac{c\beta}{\rho}\end{array}\right]^2 \cdot \left[\omega = \frac{c\beta}{\rho}\right]^2 \gamma^2$$

$$= \frac{2}{3}e^2 c\frac{\gamma^4\beta^4}{\rho^2}$$

The ~~loss per rev~~ energy loss per revolution

$\delta E = \underline{P} \cdot T$ where $T = \frac{2\pi\rho}{\omega} = \frac{2\pi\rho}{c\beta}$

$$\delta E = \frac{4\pi}{3}\frac{e^2}{\rho}\beta^3\gamma^4$$

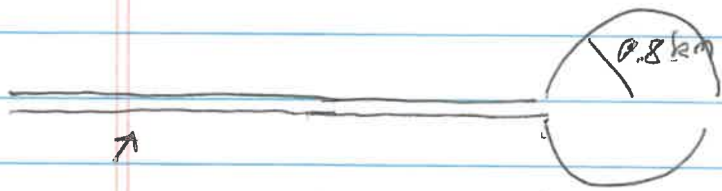
For electrons $\quad 0.0885 \frac{[E(\text{GeV})]^4}{\rho \text{ in meters}}$ MeV

Bad news for circular electron accelerators!

LEP (1990's) $\quad 0.0885 \frac{[\sim 100\ \text{GeV}]^4}{\left[\begin{array}{l}\text{circumf.} = 27\text{ km} \\ \Rightarrow \rho \sim 10\text{ km}\end{array}\right]}$

$$\delta E \sim 0.1 \times \frac{(10^8)^{\text{MeV}}}{10^4_{\,1000}} \sim 10^3 \text{ MeV} = 1 \text{ GeV/turn}$$

Need to add 1 GeV per turn to maintain beam
or - lose 1% of energy per turn
Stanford Linear Collider (1990's)



SLAC : $\frac{50 \text{ GeV}}{2 \text{ mi} = 3 \text{ km}} \cdot \frac{15 \text{ GeV}}{\text{km}}$

$$\frac{0.08}{800 \text{ m}} (50 \text{ GeV})^4 = 10^{-4} \times 5^4 \times 10^4 \text{ MeV}$$

$$= 0.625 \text{ GeV}$$

in energy loss
per beam

All future (if any!) e^+e^- machines will
be colliding linacs! (E^4/m)

$$\gamma = \frac{E}{mc^2}, \quad \frac{m_p}{m_e} = 2000, \quad \gamma^4$$

so no issue for proton rings
at any imaginable present day energy

Next, want antenna pattern, then want to Fourier transform – frequency spectrum

Antenna pattern – $\vec{E} = \frac{q}{c}\left[\frac{n\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{R(1-\beta\cdot n)^3}\right]\Bigg|_{ret}$

$\vec{B} = \hat{n}\times\vec{E}$

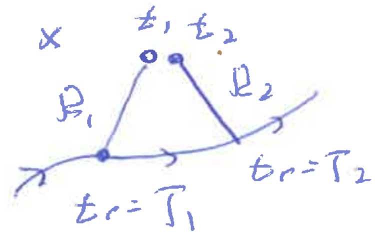
$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}\hat{n}E^2$, $\frac{dP}{d\Omega} = R^2(t_0)\,\hat{n}\cdot\vec{S}$ (*)

Path of radiator is $\vec{w}(t)$. R = distance between observer and retarded position $\vec{R} = \vec{x} - \vec{w}(t_0)$

(*) gives energy flux seen by observer. Jackson's result, 14.38, is for angular distribution of flux as it leaves the ~~observer~~ radiator, not as it arrives at the observer.

This is the rate at which the particle loses energy.

To get it:



emission at $t_r = T_1$ arrives at observer at $t_1 = T_1 + R_1/c$
at $t_r = T_2$ " " " " $t_2 = T_2 + \frac{R_2}{c}$

Energy / solid angle seen at x, over t_1 to t_2 is

$$E = \int_{t_1(obs) = T_1 + R_1/c}^{t_2(obs) = T_2 + R_2/c} \hat{n}\cdot\vec{S}(t_{obs})\,dt_{obs}$$

Convert to the particle's own time in the $\int$:

$$E = \int_{T_1}^{T_2} \Big[\hat{n}\cdot \vec{S}(t)\, \underline{\frac{dt_{obs}}{dt_r}} \Big] dt_r \equiv \int_{T_1}^{T_2} n\cdot \vec{S}'_{part}(t_r)\, dt_r$$

$$\frac{dP}{d\Omega} = R^2(t_0)\, \hat{n}\cdot\vec{S}\, \frac{dt_{obs}}{dt_r} \equiv \cancel{\hat{n}\,\cdot} \; R^2(t_0)\, \hat{n}\cdot \vec{S}'_{part}(t_r)$$

Observer measures dE/dt_{obs}, loss ~~p~~ rate in particle's frame is $\frac{dP_{part}}{d\Omega} = \frac{d^2E}{dt_r\, d\Omega} = \frac{dt_{obs}}{dt_r}\frac{d^2E}{dt_{obs}\, d\Omega}$

= the underlined term above.

To get it: $t_{obs} = t_r + \frac{R}{c} = t_r + \frac{\sqrt{(\vec{x}-\vec{w}(t))^2}}{c}$

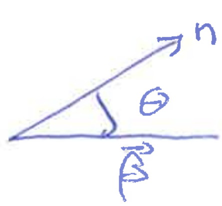
$$\frac{dt_{obs}}{dt_r} = 1 + \frac{1}{2Rc}\frac{d}{dt}(\vec{x}-\vec{w}(t))^2$$

$$= 1 + \frac{1}{2Rc}\left[-2\vec{v}\cdot(\vec{x}-\vec{w})\right] \quad ; \quad \vec{x}-\vec{w} = \vec{R}$$

$$= 1 - \frac{\vec{v}}{c}\cdot\frac{\vec{R}(t)}{R} = 1 - \vec{\beta}\cdot\hat{n}$$

(14.38) $$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left[\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}\right]^2}{(1-\vec{\beta}\cdot\hat{n})^5}$$

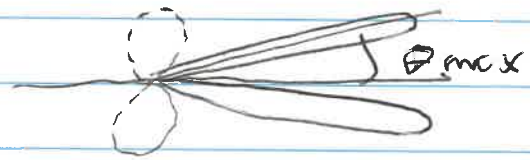
Applications 1) $\vec{\beta}$ and $\dot{\vec{\beta}}$ parallel



$$|\ \ |^2 = |\hat{n}\times(\hat{n}\times\dot{\beta})|^2 = \sin^2\theta\, |\dot{\beta}|^2$$

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3}\frac{\sin^2\theta}{[1-\beta\cos\theta]^5}$$

For $\beta << 1$ this is Larmor, $\sin^2\theta$ - but look at $\frac{1}{(1-\beta\cos\theta)^5}$ - Forward peaking



$$\frac{d}{d\theta}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} = 0 \text{ at } \theta_{max}$$

What is θ_{max}?

Useful to expand before differentiating: $\beta \sim 1$, $\sin\theta \sim \theta$, keep track of $1-\beta$

$$1-\beta\cos\theta = 1-\beta+\frac{\beta^2\theta^2}{2} = 1-\beta+\frac{\theta^2}{2}$$

$$\frac{1}{\gamma^2} = 1-\beta^2 = (1-\beta)(1+\beta) = 2(1-\beta)$$

$$\text{or } 1-\beta = \frac{1}{2\gamma^2}$$

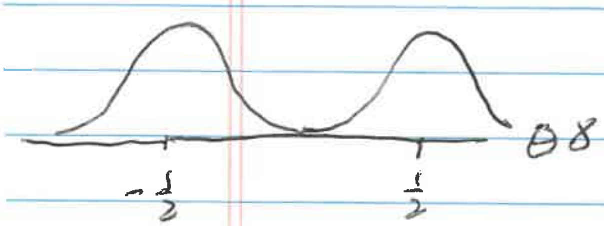
$$\boxed{1-\beta\cos\theta = \frac{1}{2\gamma^2}+\frac{1}{2}\theta^2}$$

$$\frac{dP}{d\Omega} = \frac{e^2a^2}{4\pi c^3}\frac{\theta^2}{\left[\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right]^5} = \frac{8e^2a^2\gamma^8[\gamma\theta]^2}{[1+\gamma^2\theta^2]^5}$$

Note $\gamma\theta$: $\theta_{max} \sim 1/\gamma$!

$$y = \gamma^2\theta^2 \quad \frac{d}{dy}\frac{y}{(1+y)^5} = 0 \Rightarrow \frac{1}{(1+y)^5} = \frac{5y}{(1+y)^6} \Rightarrow 1+y = 5y$$

$$\theta_{max} = \frac{1}{2\gamma}$$



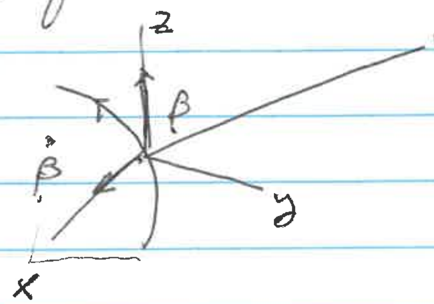
$$P \sim \gamma^8\int \underbrace{\theta d\theta}_{d\cos\theta}\frac{dP}{d\Omega} = \gamma^6\int \gamma\theta\, d\gamma\theta \times \frac{dP}{d\Omega}$$

$$= \frac{2}{3}\frac{e^2a^2}{c^3}\gamma^6.$$

Skip

or $\frac{dP}{d\Omega} = \frac{2}{3}\frac{e^2 a^2}{c^3}\gamma^6$ if you do the integral

2) Uniform circular motion in x-z plane — kinematics in notes



$\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$\vec{\dot\beta} = |\dot\beta|\hat{x} \qquad \vec{\beta} = |\beta|\hat{z}$

Skip!

$$\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{\dot\beta}] = |\dot\beta|\{\hat{n}\times(\hat{n}\times\hat{x}) - \beta\,\hat{n}\times(\hat{z}\times\hat{x})\}$$

$$= |\dot\beta|\{\hat{n}(\hat{n}\cdot\hat{x}) - \hat{x} - \beta\,\hat{n}\times\hat{y}\}$$

$$|\ \ |^2 = |\dot\beta|^2\{(n\cdot x)^2 - 2(n\cdot x)^2 + 1 + \beta^2(n\times y)^2$$

$$+2\beta x\cdot(\hat{n}\times\hat{y}) - 2\beta\,\underbrace{\hat{n}\cdot(\hat{n}\times\hat{y})}_{y\cdot(n\times n)=0}(n\cdot x)\}$$

$$\overset{\shortparallel}{-2\beta n\cdot(x\times y)}\ \ \overset{\shortparallel}{z}$$

$$= |\dot\beta|^2\{1 - (n\cdot x)^2 + \beta^2(\hat{n}\times\hat{y})^2 - 2\beta\,\hat{n}\cdot z)\}$$

$$(\hat{n}\times\hat{y})^2 = n^2y^2 - (n\cdot y)^2 = 1 - (n\cdot y)^2$$

$n\cdot z = \cos\theta \qquad n\cdot x = \sin\theta\cos\varphi \qquad n\cdot y = \sin\theta\sin\varphi$

$$|\ \ |^2 = |\dot\beta|^2\{1 - \sin^2\theta\cos^2\varphi + \beta^2(1-\sin^2\theta\sin^2\varphi) - 2\beta\cos\theta\}$$

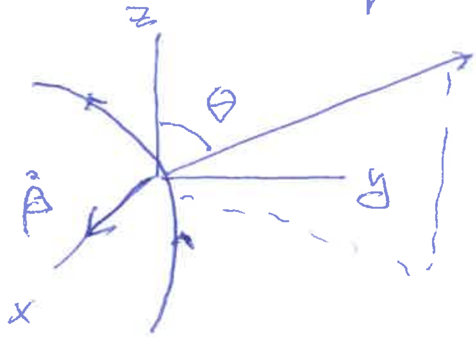
$$1-\sin^2\theta\sin^2\varphi = 1-(1-\cos^2\theta)(1-\cos^2\varphi) = \cos^2\theta + \sin^2\theta\cos^2\varphi$$

$$|\ \ |^2 = |\dot\beta|^2\{1 - \sin^2\theta\cos^2\varphi + \beta^2\cos^2\theta + \sin^2\theta\cos^2\varphi - 2\beta\cos\theta$$

$$|\ \ |^2 = |\dot\beta|^2\{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi$$

2) Uniform circular motion.

let particle move in x-z plane, center of circle in $\hat{x}$



$$\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\vec{\dot{\beta}} = |\dot{\beta}|\hat{x}$$

$$\vec{\beta} = |\beta|\hat{z}$$

skip algebra

$$1-\beta\cos\theta = 1-\beta+\frac{1}{2}\beta\theta^2$$

$$= \frac{1}{2\gamma^2}\left[1+\gamma^2\theta^2\right]$$

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^2} \cdot \frac{8\gamma^6}{[1+\gamma^2\theta^2]^3}\left[1 - \frac{4(\gamma\theta)^2\cos^2\varphi}{(1+\gamma^2\theta^2)^2}\right]$$

peaks at $\gamma\theta \sim 1$

$$P \sim \frac{2}{3}\frac{e^2a^2}{c^3}\gamma^4$$

~~skip~~

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \frac{1}{(1-\beta\cos\theta)^3}\left[1 - \frac{1}{\gamma^2}\frac{\sin^2\theta\cos^2\varphi}{(1-\beta\cos\theta)^2}\right]$$

14.44

Again use $1-\beta\cos\theta = 1-\beta+\frac{1}{2}\beta\theta^2$
$= \frac{1}{2\gamma^2}(1+\gamma^2\theta^2)$

skip to here

$$\frac{dP}{d\Omega} = \frac{q^2a^2}{4\pi c^2}\frac{8\gamma^6}{[1+\gamma^2\theta^2]^3}\left[1 - \frac{4(\gamma\theta)^2\cos^2\varphi}{(1+\gamma^2\theta^2)^2}\right]$$

Again, peaking at $\gamma\theta \sim 1$

$$P \sim \frac{2}{3}\frac{e^2a^2}{c^3}\gamma^4$$

Superficially, it appears that $\overset{a}{\Rightarrow}_{v}$ $P \sim \gamma^6$ is more efficient than $\uparrow^{a}\!\rightarrow_{v}$ $P\sim\gamma^4$ for fixed a — but in terms of applied force (pull γ's into $\frac{d\vec{p}}{dt}$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\left(\frac{d\vec{p}}{dt}\right)^2$$

for circular motion vs $\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{dt}\right)^2$ for linear.

That is, for given $\frac{d\vec{p}}{dt}$, $F\perp v$ produces more acceleration, hence more radiation. This makes sense — if $\gamma \gg 1$, $\dot{\beta}_{\parallel}$ has to be small because you are stuck at $c=1$.

In circular motion $\left|\frac{d\vec{p}}{dt}\right| = \omega|p| = \frac{v}{e} \cdot \gamma m v \sim \frac{\gamma m c^2}{e}$ if $v \sim c$, $\gamma >> 1$. Then

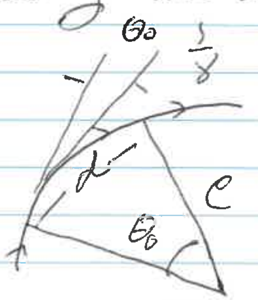
$$P_\perp \equiv \text{power from } \perp \text{ motion} \sim \gamma^2 \cdot \left(\frac{\gamma}{e}\right)^2 \sim \frac{\gamma^4}{e^2}.$$

Approximation - keep P only from $\perp$!

$\Rightarrow$ Radiation emitted by charged particle in arbitrary, extreme relativistic motion is like radiation emitted by a particle moving along an arc of a circular path, $\left|\frac{d\vec{p}}{dt}\right| = \frac{\gamma m c^2}{e} \Rightarrow e = \left(\frac{\gamma m c^2}{\left|\frac{d\vec{p}}{dt}\right|}\right) \sim \frac{c^2}{\dot{v}_\perp}$

This will give qualitative story for freq. dependence

The ~~radiation~~ radiation is a "cone" or searchlight along the direction of motion



Angular width is $\theta_0 \sim \frac{1}{\gamma}$

Distance travelled by charge while radiation is received by observer is $d = e\theta_0 = e/\gamma$

- time interval of observed radiation is $t_0 = \frac{d}{v} = \left(\frac{e}{\gamma}\right)\frac{1}{v}$ (for velocity of particle v) $t_0 = \frac{e}{\gamma v} = \frac{e}{c\gamma\beta}$

- Front of pulse travels $D = ct_0$ in time t_0, $D = \frac{e}{\gamma\beta}$

- Length of pulse is $L = D - d = \frac{e}{\gamma}\left(\frac{1}{\beta} - 1\right)$ as particle catches up w/ pulse (L = front - back)

$$L = \frac{e}{\gamma\beta}(1-\beta) = \frac{e}{\gamma}(1-\beta) \sim \frac{e}{2\gamma^3} \qquad 1-\beta = \frac{1}{2\gamma^2}$$

– Time duration of pulse seen by observer

$$\tau_0 = \frac{L}{c} = \frac{\rho}{2\gamma^3 c}$$

– "Characteristic frequency" of observed radiation

$$\omega_{rad} = \frac{1}{\tau_0} = 2\gamma^3 \frac{c}{\rho} = 2\gamma^3 \omega_{orbit}$$

ω_{orbit} = orbiting frequency of particle $= \frac{c}{\rho}$

This is a huge magnification – γ^3 !

For circular motion ω_{orb} = true ~~orbit~~ orbit frequency

For arbitrary motion ω_{orb} is just $\frac{c}{\rho}$ for tightest bend ~~radius~~ radius.

Cornell 10 GeV e^-

$$\gamma = \frac{E}{m} = \frac{10^4 \text{ MeV}}{\frac{1}{2}\text{ MeV}} = 2 \times 10^4$$

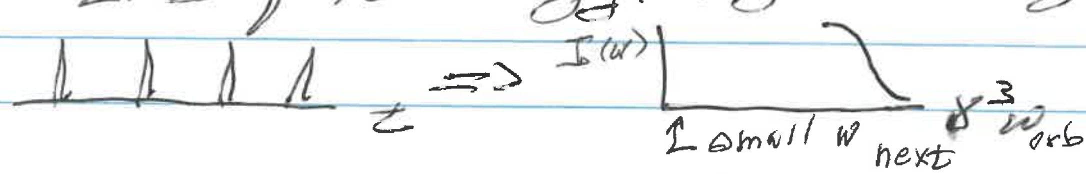
$$\omega_{orb} = 3 \text{ MHz} = 3 \cdot 10^6 \text{ sec}^{-1}$$

$(\gamma^3 \omega_0)$ $\quad \omega_{rad} = 2.4 \times 10^{19} \text{ sec}^{-1} \quad [10^{4\times 3+6}$

$$\lambda = \frac{c}{\omega} = \frac{3\times 10^{18} \text{ Å/sec}}{2.4 \cdot 10^{19} \text{ /sec}} \sim 10 \text{ Å}$$

= X rays – $E = \hbar\omega = 30$ keV

Lots of technology! (Synchrotron light sources)



A "better" formula gives the frequency dependence -

$$\vec{B}_{rad} = \hat{n} \times \vec{E}_{rad}, \quad E_{rad} = \frac{e}{cR}\left.\frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\beta\cdot n)^3}\right)_{ret}$$

$$\rightarrow \quad \left.\frac{dP(t)}{d\Omega}\right)_{observer} = \frac{c}{4\pi} R^2 \hat{n}\cdot(\vec{E}\times\vec{B}) = \frac{c}{4\pi}\left(R\vec{E}\right)^2$$

$$\equiv \left(\vec{A}(x,t)\right)^2 \quad \text{(Jackson's notation)}$$

$\vec{A}$ isn't vector potential, it is $\sqrt{\frac{c}{4\pi}}\, R\vec{E}$

$$\vec{R}(\tau_0) = \vec{x} - \vec{W}(\tau_0), \quad [x - W(\tau)]^2 = 0, \quad ct - W_0(\tau_0) > 0$$

This is in the observer's frame - we want the radiation seen by the observer

$$\frac{dW}{d\Omega} = \text{energy radiated per solid angle}$$

$$= \int dt \frac{dP}{d\Omega} = \int dt\, \vec{A}(x,t)\cdot\vec{A}^*(x,t)$$

A is real but we are going to pull a fast one . . .

Define Fourier transform of $\vec{A}(t)$

$$A(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(t)\,dt\, e^{i\omega t} \qquad A(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(\omega) e^{-i\omega t}\, d\omega$$

$$\frac{dW}{d\Omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\, e^{-i(\omega-\omega')t}\, \vec{A}^*(\omega')\,\vec{A}(\omega)$$

$$= \int_{-\infty}^{\infty} |A(\omega)|^2\, d\omega$$

$$\cong \int_0^{\infty \leftarrow 0 \text{ to } \infty!} \frac{d^2 I(\omega, \hat{n})}{d\omega\, d\Omega}\, d\omega$$

energy radiated per unit solid angle per frequency interval

$$\frac{d^2 I}{d\omega\, d\Omega} = |\vec{A}(\omega)|^2 + |\vec{A}(-\omega)|^2 = 2|A(\omega)|^2$$

because A is real, $A(-\omega) = A^*(\omega)$

$$\vec{A}(\omega) = \sqrt{\frac{e^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt\, e^{i\omega t} \left[\frac{n \times ((\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}})}{(1 - \beta \cdot n)^3} \right]_{ret}$$

ret: $t = t' + \frac{R(t')}{c}$ $\quad t' =$ retarded time

Exact - now for useful far-field approximations

$\vec{x} \gg \vec{w}$, $|x| = r$, $|\vec{x} - \vec{w}| = r - \frac{\vec{x} \cdot \vec{w}}{r}$

or $t' = t - \frac{r}{c} + \frac{\vec{w}(t_r) \cdot \hat{n}}{c}$, ~~$dt = dt'$~~

$\hat{n} = \frac{\vec{x}}{r}$

$$A(\vec{x}, \omega) = \sqrt{\frac{e^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt' \frac{dt}{dt'} \left(\frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \beta \cdot n)^3} \right)$$

$$\times \exp i\omega \left(t + \frac{r}{c} - \frac{\vec{w}(t) \cdot \hat{n}}{c} \right)$$

$$\frac{dt}{dt'} = (1 - \beta \cdot \hat{n})$$

↓ skip to Useful

$$= \sqrt{\frac{e^2}{8\pi c}} \int dt' \; \frac{n \times [(\hat{n}-\vec{\beta}) \times \dot{\vec{\beta}}]}{(1-\beta\cdot n)^2} e^{i\omega(t' - \vec{w}(t')\cdot\hat{n}/c)}$$

dropping an irrelevant exponential

$$\vec{\beta}(t') = \frac{1}{c}\frac{d\vec{w}(t')}{dt'} \;;\quad \dot{\vec{\beta}} = \frac{1}{c}\frac{d^2\vec{w}}{dt^2}$$

Now observe
$$\frac{d}{dt'}\left[\frac{n\times(n\times\beta)}{1-\beta\cdot n}\right] = \frac{n\times(n\times\dot{\beta})}{1-\beta\cdot n} + \frac{n\times(n\times\beta)}{(1-\beta\cdot n)^2}\,\dot{\beta}\cdot n$$

$$= \frac{n\times(n\times\dot{\beta})(1-\beta\cdot n) + (n\times(n\times\beta))\,\dot{\beta}\cdot n}{(1-\beta\cdot n)^2}$$

<u>Numerator</u> is
$$n\times(n\times\dot{\beta}) - (\beta\cdot n)(\dot{\beta} - n(n\cdot\dot{\beta})) + \dot{\beta}\cdot n\,(\beta - n(n\cdot\beta))$$

$$= n\times(n\times\dot\beta) - \dot{\beta}(\beta\cdot n) + \beta(\dot{\beta}\cdot n)$$

$$= n\times[(n-\beta)\times\dot{\beta}] \quad !$$

⇒ <u>Useful</u>
$$\vec{A}(x,\omega) = \sqrt{\frac{e^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt' \left[\frac{d}{dt'}\left(\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right)\right] \times \exp i\omega\left(t' - \frac{\vec{w}\cdot\hat{n}}{c}\right)$$

14.64
14.66 or 15.1
go to next page

Beginning of Bremsstrahlung uses this.

Synchrotron radiation, wigglers, undulators too –

Algebra is hellish

Do Bremsstrahlung as a simple example –

Can also do a parts $\int$

$$\vec{A}(x,\omega) = -\sqrt{\frac{e^2}{8\pi^2 c}}\int dt' \quad \text{skip this} \quad 1-\beta\cdot n]\, e^{i\omega(t' - \frac{w\cdot n}{c})}$$

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left| \int_{-\infty} (t)\, e^{i\omega(t - \frac{\hat{n}\cdot\vec{w}(t)}{c})} \right|^2$$

This hides all the time dependence in $\beta e^{i\cdots}$

And

$$e\vec{\beta}(t)e^{i\omega\cdots} \to \sum_i e_i \beta_i(t) e^{i\omega}$$

$$= \frac{1}{c}\int d^3x'\, \vec{J}(x', t)\, e^{i\omega(t - x'\cdot\hat{n}/c)}$$

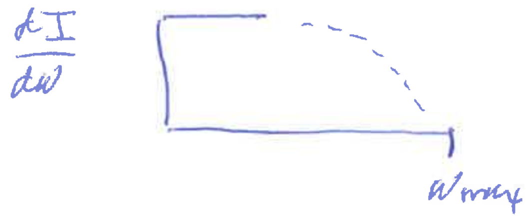
$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{\omega^2}{4\pi^2 c^3}\left| \hat{n}\times\left(\hat{n}\times \int_{-\infty}^{\infty} dt\, e^{i\omega t} \int d^3x\, J(x,t)\, e^{\frac{-i x\cdot\hat{n}}{c}}\right)\right|^2$$

again, energy radiated per unit solid angle per frequency interval.

Bremsstrahlung

Collision of 2 charged particles → change of velocity → acceleration (deceleration) → radiation.

A particular limit has (nearly) universal behavior:



small ω or $\omega \ll \omega_{max}$

$\frac{dI}{d\omega}$ = constant.

Not so at big ω! (examples follow)

Start w/ 15.1. Assume all the radiation comes from one charged particle, assume it has a velocity $\vec{\beta}(t)$, path $x(t)$

$$\frac{d^2I}{d\omega\, d\Omega_\gamma} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt \left[\frac{d}{dt}\left(\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right)e^{i\omega\left[t-\hat{n}\cdot\frac{x(t)}{c}\right]}\right.\right.$$

Assume collision time $0<t<\tau$, so $\frac{d}{dt}[\ ]=0$ outside that range → $\int_0^\tau dt \ldots$

Assume you are only interested in ω such that $\omega\tau \ll 1$ ($\omega_{max} \sim \frac{1}{\tau}$ by uncertainty principle). Then – a miracle – $e^{i\omega\ldots} \approx 1$, integral a perfect differential.

Pause to insert polarization $\hat{\epsilon}$ of outgoing radiation: i = initial, f = final

$$\lim_{\omega\to 0}\frac{d^2I}{d\omega\, d\Omega_\gamma} = \frac{q^2}{4\pi^2 c}\left|\hat{\epsilon}^*\cdot\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta}_f)}{1-\beta_f\cdot n} - \frac{n\times(n\times\beta_i)}{1-\beta_i\cdot n}\right]\right|^2$$

$\vec{\epsilon}\cdot\hat{n}=0$ so even simpler!

$$\frac{d^2 I(\epsilon)}{d\omega\, d\Omega_\gamma} = \frac{q^2}{4\pi^2 c}\left[\frac{\hat{\epsilon}^*\cdot\vec{\beta}_f}{1-\vec{\beta}_f\cdot\hat{n}} - \frac{\epsilon^*\cdot\beta_i}{1-\beta_i\cdot n}\right]^2 \quad (*)$$

- forward peaking
- ω-independent

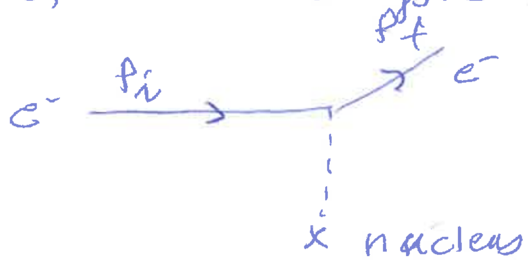
QM result for γ emission in a process reduces to this, for $\hbar\omega \ll$ other energy scales. Typically expressed as soft photon cross section,

$$N(\omega)\,\hbar\omega = I(\omega)$$

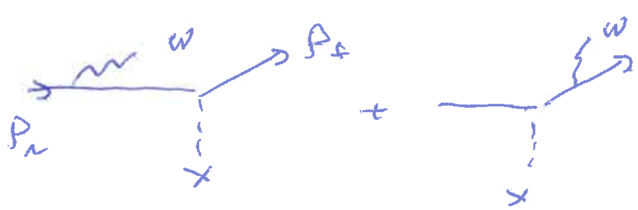
$$\frac{d^2 N}{d(\hbar\omega)\, d\Omega_\gamma} = \frac{1}{\hbar\omega}\frac{d^2 I}{d[\hbar\omega]\, d\Omega_\gamma}$$

= # of γ's emitted per energy interval per solid angle: note this diverges as $\omega \to 0$: many low energy photons

Quantum example as cartoon



$\rightarrow V(r) = \frac{Ze^2}{r}$, $\frac{d\sigma}{d\Omega_e}(i\to f)$

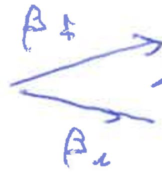


If $\omega \ll E_i, E_f$, neglect ω — photon does not affect other [illegible]

$$\frac{d^3\sigma(i\to f + N\gamma\text{'s})}{d\Omega_e\, d(\hbar\omega)\, d\Omega_\gamma} = \frac{d^2 N}{d(\hbar\omega)\, d\Omega_\gamma}\frac{d\sigma}{d\Omega}(i\to f)$$

(*) basis for many variations: Homework is $\beta_i = 0$ but coherent radiation from 2 charged particles.

Jackson's examples a bit different. First, $\beta_i, \beta_f \ll 1$,

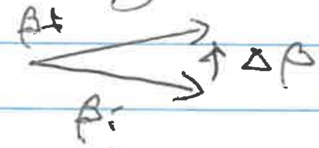


$\uparrow \Delta\beta$, neglect denominators

$$\sum_{pol} \frac{d^2 I_{NR}}{d\omega\, d\Omega_\gamma} = \sum_{pol} \frac{e^2}{4\pi^2 c} \left| \vec{\epsilon} \cdot \Delta\vec{\beta} \right|^2$$

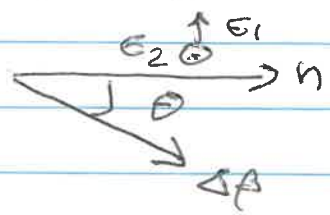
Special cases

1) $\beta_i, \beta_f << 1$: neglect denominators,



$$\sum_{pol} \frac{d^2 I_{NR}}{d\omega \, d\Omega_\gamma} = \sum_{pol} \frac{q^2}{4\pi^2 c} \left| \vec{\epsilon}^{\,*} \cdot \Delta\vec{\beta} \right|^2$$

Pick polarizations wrt n, $\Delta\beta$ plane



$$\epsilon_1 \cdot \Delta\beta = \sin\theta \, \Delta\beta$$
$$\epsilon_2 \cdot \Delta\beta = 0$$

$$\sum_{pol} \frac{d^2 I}{d\omega \, d\Omega_\gamma} = \frac{q^2 |\Delta\beta|^2 \sin^2\theta}{4\pi^2 c^2}$$

$$\frac{dI}{d\omega} = \int d\Omega_\gamma \frac{d^2 I}{d\omega \, d\Omega_\gamma} = \frac{q^2 |\Delta\beta|^2}{4\pi^2 c} \cdot 2\pi \cdot \underbrace{\int \frac{1-\cos^2\theta}{d\cos\theta}}_{2-\frac{2}{3}}$$

$$= \frac{2}{3} \frac{q^2 (\Delta\beta)^2}{\pi c}$$ (back to original formula: $\frac{dI}{d\omega}$ cutoff from $e^{i\omega t}$)

$$\frac{dN}{d\hbar\omega} = \frac{2}{3} \frac{q^2}{\hbar c \pi} \frac{(\Delta\beta)^2}{\hbar\omega} = \frac{2}{3} \frac{\alpha}{\pi} \frac{(\Delta\beta)^2}{\hbar\omega}$$

# of photons diverges (= "infrared divergence") lots of soft photons but add up to $O(\alpha)$ energy

Scratch for angle average

Scratch: $\beta = (0,0,1)$

$\Delta\beta = (\cos\varphi, \sin\varphi, 0)$

$n = (\sin\theta, 0, \cos\theta)$

$\epsilon_\perp = (0, 1, 0)$

$$\epsilon_\parallel = n \times \epsilon_\perp = -\begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix} = -\hat{i}\cos\theta + \hat{k}\sin\theta$$

$$\cancel{n \times (\beta \times \Delta\beta) = \beta(n\cdot\Delta\beta) - \Delta\beta(n\cdot\beta)}$$

$$= \cancel{\vec{\beta}(\sin\theta\cos\varphi)\Delta\beta - \Delta\beta}$$

$\epsilon_\perp \cdot \Delta\beta = \sin\varphi\,\Delta\beta \qquad \epsilon_\parallel \cdot \Delta\beta = -\Delta\beta\cos\theta\cos\varphi$

$$e\cdot(n \times (\beta \times \Delta\beta)) = \epsilon_{ijk}\, e_i\, n_j\, \epsilon_{klm}\, \beta_l\, \Delta\beta_m$$

$$\epsilon_{ijk}\epsilon_{lmk}$$

$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, e_i\, n_j\, \beta_l\, \Delta\beta_m$$

$$= (e\cdot\beta)(n\cdot\Delta\beta) - (e\cdot\Delta\beta)(n\cdot\beta)$$

so we have for $\epsilon_\perp$: $0 - \beta\Delta\beta\sin\varphi\cos\theta$

$\epsilon_\parallel$: $\beta\Delta\beta\left[\sin^2\theta\cos\varphi + \cos^2\theta\cos\varphi\right]$

$$= \beta\Delta\beta\cos\varphi$$

$$|\ \ |^2 \text{ for } e_\perp = \frac{\cancel{2}}{\cancel{b}}(\Delta\beta)^2 \frac{|\sin\varphi - \beta\sin\varphi\cos\theta|^2}{(1-\beta\cos\theta)^4}$$

PB

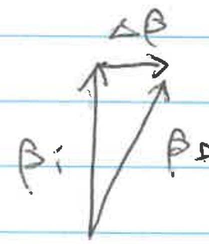
$$= \frac{1}{2}(\Delta\beta)^2 \frac{1}{(1-\beta\cos\theta)^2}$$

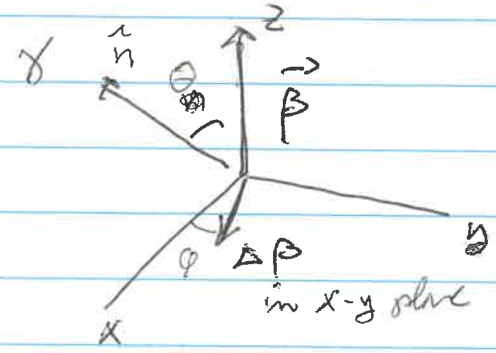
$$|\ \ |^2 \text{ for } e_\parallel = (\Delta\beta)^2 \frac{[-\cos\theta\cos\varphi + \beta\cos\varphi]^2}{(1-\beta\cos\theta)^4}$$

$$= \frac{1}{2}(\Delta\beta)^2 \frac{(\beta-\cos\theta)^2}{(1-\beta\cos\theta)^4}$$

upon angle average

2) Bremsstrahlung by a fast electron as it undergoes a collision, assume $\Delta\beta \ll \beta_i \approx \beta_f$, small deflection ⊥ original direction



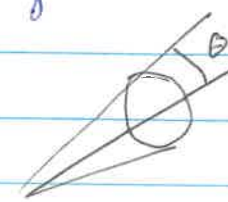


pick coordinates $\hat{n} = [\sin\theta, 0, \cos\theta]$ in x-z plane

$\vec{\Delta\beta} = \Delta\beta\,[\cos\varphi, \sin\varphi, 0]$ in x-y

Convenient to average over the direction of $\Delta\beta$, compute distribution along $\vec{\beta}$ (small deflection)

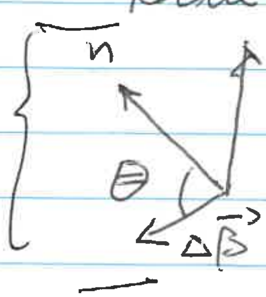
$$\frac{d^2 I_{ave}}{d\cos\theta\, d\omega} = \frac{1}{2\pi}\int d\varphi \frac{d^2 I}{d\omega\, d\Omega}$$



radiation at angle θ to initial velocity

There is a long way to do this or a clever way. The clever way is to boost the NR result along the $\vec{\beta}$ direction (the $\hat{z}$ direction here).

Recall



$$\frac{d^2 I_{NR}}{d\Omega_\gamma\, d\omega} = \frac{q^2|\Delta\beta|^2}{4\pi^2 c}\left(\sin^2\Theta\right)$$

$$\cos^2\Theta = \frac{(\hat{n}\cdot\vec{\Delta\beta})^2}{|\Delta\beta|^2} = (\sin\theta\cos\varphi)^2 \text{ here}$$

$$\frac{d^2 I_{NR}}{d\Omega_\gamma\, d\omega} = \frac{q^2|\Delta\beta|^2}{4\pi^2 C}\left[1 - \sin^2\theta\cos^2\varphi\right]$$

$\langle \cos^2\varphi \rangle = \frac{1}{2}$

$$\frac{d^2 I_{NR, ave}}{d\Omega_\gamma d\omega} = \frac{g^2 |\Delta\beta|^2}{4\pi^2 c}\left[1 - \underbrace{\frac{1}{2}\sin^2\theta}_{\frac{1+\cos^2\theta}{2}}\right]$$

$dI' = \hbar\omega' dN$

This is $\cancel{\frac{\hbar\omega' dN}{d\hbar\omega' d\Omega'_\gamma}} = \frac{d^2 I'}{d\Omega'_\gamma \, d\hbar\omega'} = \hbar\omega' \frac{dN}{d\hbar\omega' d\Omega'_\gamma}$

primes just to think about LT

Now to transform: useful facts

1) N is invariant, it's a number, $dI = \hbar\omega dN$

2) $$\frac{d^3 k}{2k_0} = d^3k \, dk_0 \, \delta(k_0^2 - k^2)$$
$$= d^4k \, \delta(k_\mu k^\mu)$$
is invariant

$\therefore \frac{dN}{\left(\frac{d^3k}{k_0}\right)}$ is invariant

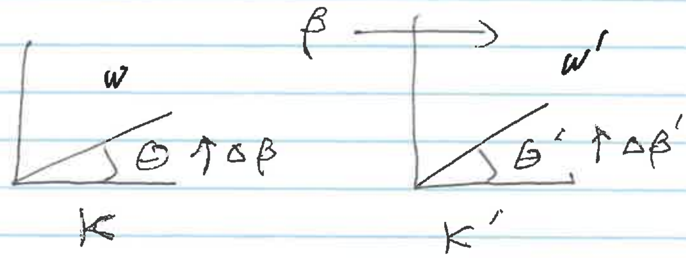
$$\frac{d^3k}{k_0} = \frac{d\Omega_\gamma}{\hbar\omega}\left[\frac{\hbar\omega}{c}\right]^2 d\frac{\hbar\omega}{c}$$
$$\propto \omega \, d\omega \, d\Omega_\gamma$$

so $\frac{dN}{d^3k/k_0} \propto \frac{dN}{\omega \, d\omega \, d\Omega_\gamma} \propto \frac{d^2 I}{\omega^2 d\omega \, d\Omega_\gamma}$ is invariant

$$\frac{d^2 I_{ave}}{d\Omega_\gamma\, d\omega} = \left(\frac{\omega}{\omega'}\right)^2 \frac{d^2 I'}{d\Omega'_\gamma\, d\omega'}$$

$$= \frac{q^2}{8\pi^2 c} |\Delta\beta'|^2 \left(\frac{\omega}{\omega'}\right)^2 \left[1+\cos^2\theta'\right]$$

And finally



$$\omega' = \gamma\omega(1-\beta\cos\theta) \qquad (1)$$
$$\omega = \gamma\omega'(1+\beta\cos\theta') \qquad (2)$$

Transverse doppler $\Delta\beta' = \gamma\Delta\beta$ $\quad$ $\Delta\beta \sim a$ (velocity!

(1) $$\frac{\omega}{\omega'} = \frac{1}{\gamma[1-\beta\cos\theta]}$$

(1+2) $$\cos\theta' = \frac{\cos\theta-\beta}{1-\beta\cos\theta}$$

$$\frac{d^2 I_{ave}}{d\Omega_\gamma\, d\omega} = \frac{q^2|\Delta\beta|^2}{8\pi^2 c}\left[\frac{1}{(1-\beta\cos\theta)^2} + \frac{(\cos\theta-\beta)^2}{(1-\beta\cos\theta)^4}\right]$$

Can get this directly from ~~beginning~~ beginning

$$1+\beta\cos\theta' = \frac{\omega}{\gamma\omega'} = \frac{1}{\gamma^2(1-\beta\cos\theta)}$$

($\epsilon\cdot\beta_f$, $\epsilon\cdot\beta$: awkward)

$$\beta\cos\theta' = \frac{1-\beta^2}{1-\beta\cos\theta} - 1 = \frac{-1+\beta\cos\theta+1-\beta^2}{1-\beta\cos\theta}$$
$$= \beta\left[\frac{\cos\theta-\beta}{1-\beta\cos\theta}\right]$$

At small angle

$$\cos\theta = 1 - \frac{1}{2}\theta^2$$

$$1 - \beta = \frac{1}{2\gamma^2} \quad \text{at high velocity}$$

$$1 - \beta\cos\theta = 1 - \left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{1}{2}\theta^2\right)$$

$$= \frac{1}{2\gamma^2}\left(1 + \gamma^2\theta^2\right)$$

$$\cos\theta - \beta = 1 - \beta - \frac{1}{2}\theta^2 = \frac{1}{2\gamma^2}\left(1 - \gamma^2\theta^2\right)$$

Everything again depends on $\gamma\theta$

$$\frac{d^2 I_{ave}}{d\omega\, d\Omega\, \gamma} = \frac{q^2 (\Delta\beta)^2}{\pi^2 c}\, \gamma^4 \, \frac{1 + \gamma^4\theta^4}{\left(1 + \gamma^2\theta^2\right)^4}$$

1) ~~peaks at~~ dies at $\gamma\theta \gg 1$

2) independent of ω

$$\frac{dN}{d(\hbar\omega)} \sim \alpha \cdot \frac{1}{\hbar\omega} \quad \text{again}$$