Covariant version of stress tensor

Back story - we have a relativistic description of E&M beginning w/ Lagrangian. But something is missing - the Lorentz force law. Where does it come from? ~~How~~ What does relativity have to do with it? Complicated math (will skip parts), simpler physics.

To begin - notice $\mathcal{L}$ only depends on coordinates because fields do: $\mathcal{L}(x) = \mathcal{L}(\phi_j(x), \partial_\mu \phi_j(x))$ (for generic $\mathcal{L}$). Make a shift in x

$$x_\mu \to x_\mu + \epsilon_\mu$$

$$\delta\mathcal{L} = \epsilon_\mu \frac{\partial \mathcal{L}}{\partial x_\mu} = \epsilon_\mu \partial^\mu \mathcal{L} = \sum \frac{\partial \mathcal{L}}{\partial \varphi_j} \delta\varphi_j + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi_j)} \delta\partial^\mu\varphi_j$$

$$\delta\varphi = \epsilon_\mu \partial^\mu \phi, \quad \delta(\partial^\nu\phi) = \partial^\nu \delta\varphi = \epsilon_\mu \partial^\mu \partial^\nu \varphi$$

plus
$$\frac{\partial \mathcal{L}}{\partial \varphi_j} = \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi_j)}$$

$\to$ algebra
$$\epsilon_\mu \partial^\mu \mathcal{L} = \partial^\mu \left\{ \sum_j \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi_j)} \epsilon_\mu \partial^\nu \varphi_j \right\}$$

or
$$0 = \epsilon_\nu \partial_\mu T^{\mu\nu}$$

$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \sum_j \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi_j)} \partial^\nu \varphi_j$$

$\equiv$ stress tensor, energy-momentum tensor

Integrals of $T^{\mu\nu}$ are conserved quantities, in analogy to $J^\mu \to Q$.

What is the physical meaning of $T^{\mu\nu}$?

Back up to something simpler, N particles in a box, box has volume V_* in its rest frame, at-rest number density is $n_0 = \frac{N}{V_*}$

In frame where box moves w/ velocity ~~v~~ v, length contractions says $V = \frac{V_*}{\gamma}$, number density is $n = \frac{N}{V} = \gamma n_0$

Recall charge conservation: $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 = \partial_\mu J^\mu$

define N^μ ~~= or~~ $n v^\mu$, v^μ = 4-velocity

$$v^\mu = [\gamma c, \gamma \vec{v}]$$

N^i = number current $\propto \gamma n \vec{v}$.

express conservation of number as $\partial_\mu N^\mu = 0$.

Note density is a scalar associated w/ 3-volume V. 3-volume in 4-d space needs ~~to~~ a normal vector to be specified [analog of $\hat{n}\, dA$ as 2-d surface in 3-d) –

$n_\perp$ = 4-vector "normal to" 3-d region

# of particles in 3 vol $n_\alpha \Delta V = N^\alpha [n_\alpha \Delta V]$
to contract indices (*)

physics: conservation of a scalar quantity (particle number) needs a vector quantity to dot into normal direction.

Energy & momentum - these are part of a 4-vector.

Call Δp^μ the amount of p^μ in a 3-volume ΔV. Analog of (*)

$$\Delta p^\alpha = [n_\beta \Delta V] \, T^{\alpha\beta}$$

point at *

$T^{\alpha\beta}$ has one more index - a rank 2 tensor.

Examples. Consider ΔV at rest, $n_\beta = (1, 0, 0, 0)$
purely timelike $\qquad \uparrow t$

$$\Delta p^\alpha = T^{\alpha t} \Delta V$$

- $\alpha = 0$ $\quad T^{tt}$ = energy density (T^{00})
- $\alpha = i$ $\quad \dfrac{\Delta p^i}{\Delta V} = T^{it} \; (T^{i0}) \equiv \pi^i$

~~[illegible]~~ T^{0i}, T^{ij} ?

Consider a box in $\Delta y \Delta z \Delta t$ –

$n_\beta = \hat{x}$

$$\Delta p^\alpha = T^{\alpha x} \Delta y \Delta z \Delta t$$

$$T^{tx} = \frac{\Delta p^t}{\Delta t \Delta y \Delta z} = \frac{\Delta E}{\Delta t} \frac{1}{\Delta A}$$

= flux of energy in x-direction

$$T^{ix} = \frac{\Delta p^i}{\Delta t} \frac{1}{\Delta A} = \text{force/unit area} = \text{pressure}$$

T^{ij} = ith component of force/unit area across surface w/ normal j

Integrals of T ?

$$p^\nu = \int d^3x \, T^{0\nu} = \int d^3x \Bigg[\sum_j \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}_j}}_{\pi_j} \frac{\partial \phi_j}{\partial x_\nu} - g^{0\nu} \mathcal{L} \Bigg]$$

set $\nu = 0$: $T^{00} = \sum_j \pi_j \dot{\phi}_j - \mathcal{L} \sim p\dot{q} - L$

this is like the Hamiltonian – T^{00} = energy density

$$P^0 = \int d^3x \, T^{00} = \text{total energy}$$

Back to electrodynamics

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \underbrace{\frac{1}{c} J_\mu A^\mu}_{\text{drop for a while}}$$

take derivatives – $\phi_j \to A_\mu$

$$T^{\alpha\beta} = -g^{\alpha\beta}\mathcal{L} - \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \partial^\beta A^\lambda$$

2 issues

a) people like $T^{\alpha\beta}$ to be symmetric, this isn't

b) obviously not gauge invariant

Long story -- solution to a), b) involves

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right]$$

If $J = 0$ $\partial_\alpha \Theta^{\alpha\beta} = 0$. Components are

$$\Theta^{00} = \frac{1}{8\pi}\left[E^2 + B^2 \right] \equiv u$$

$$\Theta^{0i} = \Theta^{i0} = \frac{1}{4\pi}\left(\vec{E} \times \vec{B} \right) = c \times \text{momentum flux } g_i$$

$$c^2 \vec{g} = \vec{S}$$

$$\Theta^{ij} = -\frac{1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(E^2 + B^2) \right]$$

minus the Maxwell stress tensor

Including J_ν conservation laws we

$$\partial_\alpha \Theta^{\alpha\beta} = -\frac{1}{c} F^{\beta\lambda} J_\lambda$$

or $\beta = 0$: energy conservation

$$\frac{1}{c}\left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S}\right] = -\frac{1}{c}\vec{J}\cdot\vec{E}$$

$\beta = i$ momentum conservation

$$\underbrace{\frac{d g_i}{dt}}_{\text{change}} - \sum_j \frac{\partial T_{ij}}{\partial x_j} = -\left[\rho E_i + \frac{1}{c}(\vec{J}\times\vec{B})\right]$$

field momentum density = − change of particle momentum density

$[\, J:\ F^{i0} J_0 - F^{ij} J_j$

$$\frac{d P_{\text{field}}}{dt} = -\int d^3x \left[\rho E_i + \frac{1}{c}(\vec{J}\times\vec{B})\right]$$

Point particles in E+M fields

Best way to proceed - start with a Lagrangian!

And best to start with a free particle, first!

We expect to get $\frac{d\vec{p}}{dt} = 0$, $\vec{p} = \gamma m \vec{v}$.

What L gives that as an eqn of motion?

Ans #1 $\quad L = -mc^2\sqrt{1-\frac{v^2}{c^2}} = -\frac{mc^2}{\gamma}$

just a guess... $\quad L = -mc^2\sqrt{1-\frac{\dot{q}^2}{c^2}}$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

$$= \frac{d}{dt}\left[\frac{-mc^2}{\sqrt{1-\frac{\dot{q}_i^2}{c^2}}}\left[-\frac{\dot{q}_i}{c^2}\right]\right] = \frac{d}{dt}\gamma m v_i$$

Note $\quad L = -mc^2\left[1 - \frac{1}{2}\frac{v^2}{c^2} + \cdots\right] = \frac{1}{2}mv^2 + \text{const}$

Why is this reasonable? Ask for the action to be a scalar

$$S = \int d^4x\, \mathcal{L} = \int dt\, L = \int \overset{\text{proper time}}{[\gamma d\tau]} L$$

i.e. $\gamma L = \text{scalar} = -mc^2$

Interaction term? recall

$$S = \int d^4x\, \mathcal{L} = \int dt\, d^3x\, \mathcal{L} = \int dt\, L$$

$$\mathcal{L}_I = -J_\mu A^\mu.$$

What's a good J_μ for a point particle?

We only have one available 4-vector, the 4-velocity $U^\mu = (\gamma c, \gamma \vec{u})$

so $J_\mu A^\mu \propto q\, U_\mu A^\mu$

Again, treat S as invariant

$$S = \int \gamma\, d\tau\, L_I \qquad (\gamma L_I = \text{scalar})$$

$$L_I = -\frac{e}{c}\frac{1}{\gamma} U_\mu A^\mu = -e\left[A_0 - \frac{\vec{u}\cdot\vec{A}}{c}\right]$$

or the total Lagrangian is

$$L = L_0 + L_I, \quad L_0 = -mc^2\sqrt{1-\frac{u^2}{c^2}}$$

so Eqn of motion is

$$\frac{d}{dt}\frac{\partial L_0}{\partial \dot{x}_i} - \frac{\partial L_0}{\partial x_i} = -\frac{d}{dt}\left[\frac{\partial L_I}{\partial \dot{x}_i} - \frac{\partial L_I}{\partial x_i}\right]$$

or

$$\frac{dp_i}{dt} = -\left[\frac{d}{dt}\frac{\partial L_I}{\partial \dot{x}_i} - \frac{\partial L_I}{\partial x_i}\right]$$

2 terms. $$\frac{\partial L_I}{\partial x_i} = -e\left[\frac{\partial A_0}{\partial x_i} - \frac{u_j}{c}\frac{\partial A_j}{\partial x_i}\right]$$

$\frac{d}{dt}$ is a total derivative — need to include direct and indirect time variation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_i \frac{\partial x_i}{\partial t}\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}$$

so $$\frac{d}{dt}\frac{\partial L_I}{\partial u_i} = \left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\frac{\partial}{\partial u_i}\left(A_0 - \frac{\vec{u}\cdot\vec{A}}{c}\right)$$

$$= \left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\left(-\frac{A_i}{c}\right)$$

$$= -\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{u_j}{c}\partial_j A_i$$

so

$$\frac{dp_i}{dt} = e\left[-\partial_i A_0 + \frac{u_j}{c}\partial_i A_j\right]$$

$$+ e\left[\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{u_j}{c}\partial_j A_i\right]$$

$$= e\left[-\partial_i A_0 + \frac{1}{c}\frac{\partial A_i}{\partial t}\right]$$

$$+ \frac{e}{c}u_j\left[\partial_i A_j - \partial_j A_i\right]$$

First term is $-\nabla\Phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{E}$

2nd term uses $\partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k$

$\epsilon_{ijk} u_j B_k = (\vec{u} \times \vec{B})_i$

Lorentz Force Law!

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m \vec{v}) = e\left[E + \frac{\vec{v} \times \vec{B}}{c}\right]$$

Alternative covariant form
is often useful

$$m\frac{du^\mu}{d\tau} = \frac{e}{c}F^{\mu\nu}u_\nu$$

Check: $u^\mu = (\gamma c, \gamma \vec{u})$, ~~___~~ $d\tau = \gamma dt$

$\underline{\mu = i}$: $F^{\mu\nu}u_\nu = F^{i0}u_0 - F^{ij}u_j$

$$\left(\gamma\frac{d}{dt}\right)\left[\gamma m\vec{u}\right] = \frac{\gamma e}{c}\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

$\underline{\mu = 0}$

$$m\frac{du^0}{d\tau} = \frac{e}{c}F^{0\nu}u_\nu$$

$$\gamma\frac{d}{dt}\gamma mc = \frac{e}{c}F^{0j}\gamma u_j$$

$$\frac{d}{dt}\gamma mc^2 = e\vec{v}\cdot\vec{E}$$

(power formula : RHS is $J\cdot E$)
LHS is $\frac{d\,\text{Energy}}{dt}$

A more reputable derivation –

⇒ ~~and~~ free particle moves to extremize its proper time.

$$J = \int ds = \cancel{\int d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}}$$

Analog of "geodesic equation" in GR

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$= g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau^2$$

$$J = \int d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \equiv \int d\tau \sqrt{f(\tau)}$$

Not what you see in books – but ask for

$\delta J = 0$ if $x \to x + \delta x$

$$\delta J = \int d\tau \frac{\delta f^{1/2}}{\delta x} \cdot \delta x = \int d\tau \frac{1}{2\sqrt{f}} \frac{\delta f}{\delta x} \delta x = 0$$

Easier to minimize $\int d\tau\, f(\tau)$ – and the extremum condition is the same. In GR the "geodesic eqn" ~~is~~ minimizes

$$I = \int d\tau\, g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

Now shift $x \to x + \delta x$, ask $\delta I = 0$.

For us, $g_{\mu\nu} = \eta_{\mu\nu}$ has <u>no</u> x dependence

$$0 = \delta I = \int d\tau \, \eta_{\mu\nu} \cdot \left[2 \frac{dx^\mu}{d\tau} \frac{d\,\delta x^\nu}{d\tau} \right]$$

$$\xrightarrow[\text{parts } \int]{} \int d\tau \, 2\eta_{\mu\nu} \left(\frac{d^2 x^\mu}{d\tau^2} \right) \delta x^\nu = 0$$

or $\quad \dfrac{d^2 x^\mu}{d\tau^2} = 0$

or $\quad \dfrac{dx^\mu}{d\tau} = $ constant – this is u^μ the 4-velocity

and it says $p^\mu =$ constant for free particle

Now you know $\frac{1}{2}$ of general relativity!

Is I a maximum, or a minimum???

turn I into action S - ~~[illegible]~~

$$S = \int L\,d\tau = \int d\tau \left[\tfrac{1}{2} m \underbrace{u^\mu(\tau) u_\mu(\tau)}_{c^2\,!} + \frac{e}{c} u^\mu A_\mu \right]$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial u^\nu} - \frac{\partial L}{\partial x^\nu} = 0$$

$$(*) \qquad \frac{d}{d\tau}\left[m u_\nu + e A_\nu \right] - e u^\mu \frac{\partial A_\mu}{\partial x^\nu} = 0$$

$$\frac{d}{d\tau} e A_\nu = \frac{\partial A_\nu}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tau} = \partial_\mu A_\nu u^\mu$$

so $(*)$ is $\displaystyle \frac{d}{d\tau}[m u_\nu] - e u^\mu [\partial_\nu A_\mu - \partial_\mu A_\nu] = 0$

which is $\displaystyle \frac{dp_\nu}{d\tau} = \frac{e}{c} F_{\nu\mu} u^\mu$

or shuffle indices

$$\frac{dp^\mu}{d\tau} = \frac{e}{c} F^{\mu\nu} u_\nu$$

$$\frac{d\vec{p}}{dt} = e\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right] \qquad m\frac{du^{\mu}}{d\tau} = \frac{e}{c}F^{\mu\nu}u_{\nu}$$

Motion of charged particle in EM fields

A) Static uniform B-field

energy $\quad \frac{dE}{dt} = 0 \;,\; \frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v}\times\vec{B}$

energy is constant so $|v|$ is constant so γ is constant

$$\omega_B \equiv \frac{eB}{\gamma mc} = \frac{ecB}{\text{Energy}} \equiv \text{cyclotron frequency}$$

$$\frac{d\vec{v}}{dt} = \vec{v}\times\vec{\omega}_B$$

Pick $\vec{B}$ field in z-direction, $\vec{\omega}_B \propto \hat{z}$

We know it's circular motion so let's write

$$\vec{v}(t) = \hat{z}v_z + \omega_B r_0(\hat{x} - i\hat{y})e^{-i\omega_B t}$$

and check:

integrate $\quad \vec{r}(t) = \vec{R}_0 + \hat{z}v_z t + i r_0(\hat{x} - i\hat{y})e^{-i\omega_B t}$

differentiate

$$\frac{d\vec{v}}{dt} = -i\omega_B^2 r_0(\hat{x} - i\hat{y})e^{-i\omega_B t}$$

$$= \omega_B^2 r_0(-i\hat{x} + \hat{y})e^{-i\omega_B t}$$

and $\quad \vec{v}\times\vec{\omega}_B = r_0\omega_B^2 \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{vmatrix} e^{-i\omega_B t}$

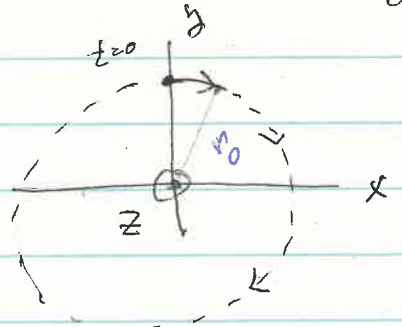
$$= r_0\omega_B^2 e^{-i\omega_B t}\left[-i\hat{x} + \hat{y}\right] \quad \text{— it checks}$$

It checks! Note

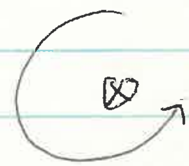
$$\mathrm{Re}\left\{ i r_0 (\hat{x} - i\hat{y})(\cos\omega_B t - i \sin\omega_B t) \right\}$$

$$= \quad \hat{x}\, r_0 \sin\omega t + \hat{y}\, r_0 \cos\omega t$$

From ahead

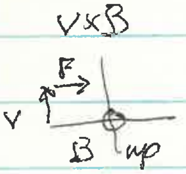


or

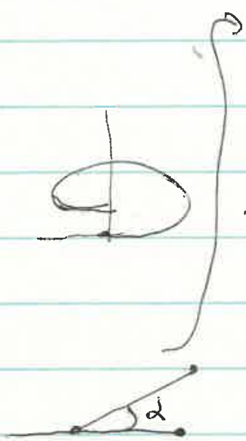


from behind along direction of B

so a + charge rotates counter clockwise about field lines viewed in direction of B (rh rule for electrons)



Motion is a helix – radius r_0



pitch angle $\tan\alpha = \dfrac{v_z}{\omega_B r_0} = \dfrac{v_z}{v_\perp}$

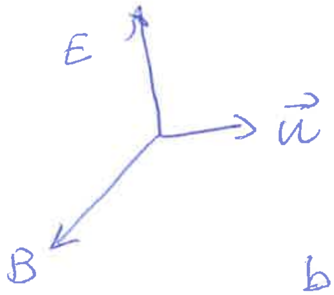
And the bend radius gives the $\perp$ momentum (and vice versa)

$$P_\perp = \gamma m |v_\perp| = \gamma m \omega_B r_0 = \frac{eBr_0}{c}$$

$$\boxed{cP_\perp = eBr_0}$$

Obvious practical use!

B) Motion in crossed fields; $\vec{E}\cdot\vec{B}=0$



With $\vec{E}\neq 0$, energy is not constant in time

But for this special case we can boost to a frame where either $\vec{E}'=0$ or $\vec{B}'=0$.

Then solve $\frac{dp'}{dt} = e\left[E' + \frac{v'\times B'}{c}\right]$. (boost back if necessary)

Recall
$$\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})$$
$$\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

boost along direction orthogonal to $\vec{E}$ & $\vec{B}$ – no last term. Then try to eliminate E' or B'.

2 cases:

a) $|E|<|B|$. $\quad \frac{\vec{u}}{c} = \frac{\vec{E}\times\vec{B}}{B^2}$ kills E'

check: $\vec{\beta}\times\vec{B} = \frac{(\vec{E}\times\vec{B})\times\vec{B}}{B^2}$

numerator is $\epsilon_{ijk}[\epsilon_{jlm}E_l B_m]B_k$

$$= -\epsilon_{jik}\epsilon_{jlm}E_l B_k B_m$$
$$= -[\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}]E_l B_k B_m$$
$$= -\vec{E}B^2 + \vec{B}(\cancel{E\cdot B})^{0}$$

$$E' = \gamma\left(E - \frac{EB^2}{B^2}\right) = 0$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E})$$

$$(E\times B)\times E = -[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}]E_l E_k B_m$$

$$= -E(\cancel{E\cdot B}^{\,0}) + \vec{B}E^2$$

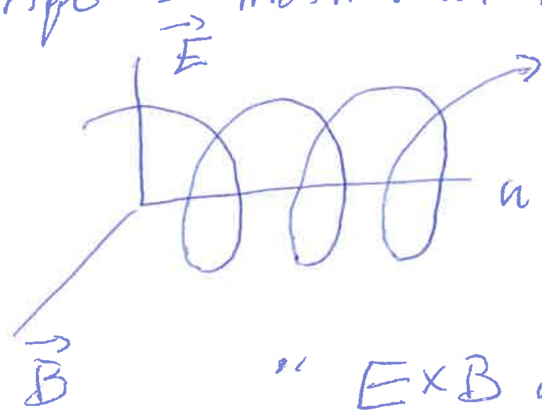
$$B' = \gamma\vec{B}\left[1 - \frac{E^2}{B^2}\right]$$

and $\frac{u}{c} = \frac{E\times B}{B^2} \Rightarrow \frac{u^2}{c^2} = \frac{E^2}{B^2}$

$$\Rightarrow 1 - u^2 = 1 - \frac{E^2}{B^2} = \frac{1}{\gamma^2}$$

$$B' = \frac{1}{\gamma}\vec{B}$$

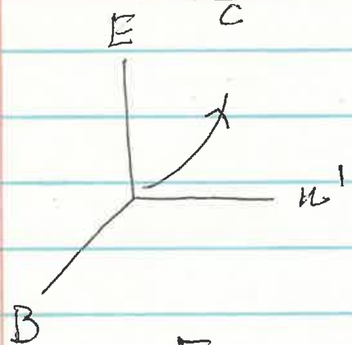
In this frame, pure $B' \rightarrow$ circular motion about field line, in original frame, "drift" = motion w/ velocity $\vec{u}$



"E×B drift"

b) if $|E| > |B|$, go to a frame with no $\vec{B}$

$$\frac{\vec{u}'}{c} = \frac{\vec{E} \times \vec{B}}{E^2} \qquad \gamma = \frac{E^2}{E^2 - B^2}$$



$$\vec{B}' = 0, \quad E_\parallel' = 0, \quad E_\perp' = \frac{1}{\gamma} E$$

$$= \vec{E} \sqrt{\frac{E^2 - B^2}{E^2}}$$

For general solution, see Jackson problem 12-3 (which you'll do)

But - if $p_y(0) = 0$, $p_z(0) = 0$, $p_x(0) = 0$, $E = \hat{x} E_0$

we can integrate $\frac{dp_x}{dt} = e E_0$ (drop primes)

$$p_x(t) = e E_0 t$$

$$E(t) = \gamma m c^2 = \sqrt{p^2 c^2 + (mc^2)^2}$$

$$mc^2 \sqrt{\left(\frac{e E_0 t}{mc}\right)^2 + 1}$$

$$\text{so} \quad \gamma = \sqrt{\left(\frac{e E_0 t}{mc}\right)^2 + 1}$$

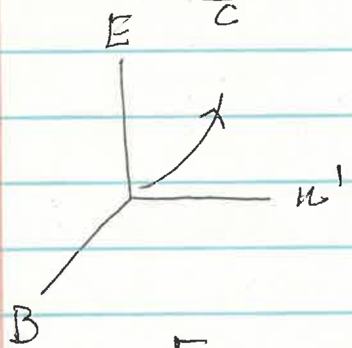
$$\frac{v_x(t)}{c} = \frac{c\, p_x(t)}{E(t)} = \frac{e E_0 t}{mc \sqrt{\left(\frac{e E_0 t}{mc}\right)^2 + 1}}$$

$$x(t) = \int v_x(t')\, dt' \qquad \text{etc}$$

note $\frac{v}{c} \to 1$ as $t \to \infty$

b) if $|E| > |B|$, go to a frame with no $\vec{B}$

$$\frac{\vec{u}'}{c} = \frac{\vec{E} \times \vec{B}}{E^2} \qquad \gamma = \frac{E^2}{E^2 - B^2}$$



$$\vec{B}' = 0, \quad E_\parallel' = 0, \quad E_\perp' = \frac{1}{\gamma} E$$

$$= \vec{E}\sqrt{\frac{E^2 - B^2}{E^2}}$$

For general solution, see Jackson problem 12.3 (which you'll do)

But - if $p_y(0) = 0$, $p_z(0) = 0$, $p_x(0) = 0$ $\quad E = \hat{x} E_0$

we can integrate $\dfrac{dp_x}{dt} = eE_0$ (drop primes)

$$p_x(t) = eE_0 t$$

$$E(t) = \gamma mc^2 = \sqrt{p^2c^2 + (mc^2)^2}$$

$$mc^2 \sqrt{\left(\frac{eE_0 t}{mc}\right)^2 + 1}$$

so $\gamma = \sqrt{\left(\dfrac{eE_0 t}{mc}\right)^2 + 1}$

$$\frac{v_x(t)}{c} = \frac{c\,p_x(t)}{E(t)} = \frac{eE_0 t}{mc\sqrt{\left(\frac{eE_0 t}{mc}\right)^2 + 1}}$$

$$x(t) = \int v_x(t')\,dt' \qquad \text{etc}$$

note $\frac{v}{c} \to 1$ as $t \to \infty$

Approximate Solutions

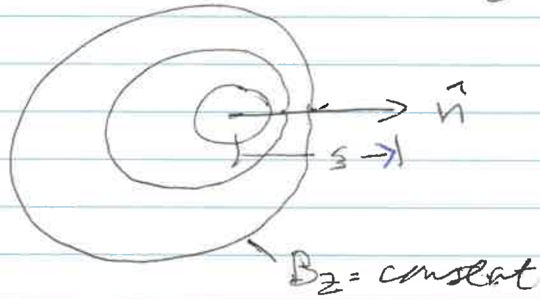
Drifting in nonuniform but static fields

$$\vec{B}(x) = \vec{B}(x_0) + (\delta\vec{x}\cdot\vec{\nabla})\vec{B}(x_0)$$

Assume variation in $\vec{B}$ is all perpendicular to its direction (we have $B_z(x,y)$, for example)

Motion will be (roughly) circular motion around field lines with a local cyclotron frequency.

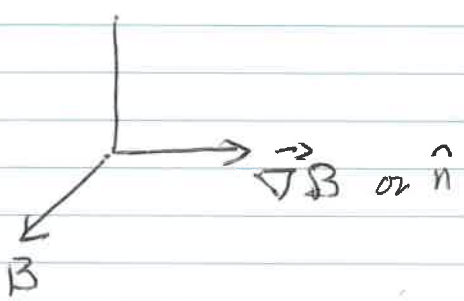
Suppose $\vec{\nabla}_\perp(B_z)$ points in direction $\hat{n}$, coordinate labelled by ξ



$$\vec{\nabla}_\perp B_z = \frac{\partial B}{\partial \xi}\hat{n}$$

$$\vec{\omega}_B(\vec{x}) = \frac{e}{\gamma mc}\vec{B}(x) = \vec{\omega}_0\left[1 + \frac{1}{B_0}\frac{\partial B}{\partial \xi}\Big)_0 \hat{n}\cdot\vec{x}\right]$$

$$\equiv \vec{\omega}_0[1+\delta]$$



Dominant motion is still in-plane. Solve by successive approximation. Begin with

$$\frac{d\vec{V}_\perp}{dt} = \vec{V}_x \times \vec{\omega}_B(x) \quad ; \quad \left\{ \begin{array}{l} \vec{V}_\perp = \vec{V}_0 + \vec{V}_1 + \cdots \\ \vec{\omega}_B = \vec{\omega}_0 + \vec{\omega}_1 + \cdots \\ \omega_1 = \delta \cdot \omega_0 \\ \delta = \dfrac{1}{B_0}\dfrac{dB}{ds} \end{array} \right)$$

$$\frac{d}{dt}\left[\vec{V}_0 + \vec{V}_1\right] = (\vec{V}_0 + \vec{V}_1) \times (1+\delta)\,\vec{\omega}_0$$

$$= \vec{V}_0 \times \vec{\omega}_0 + (\vec{V}_0\delta + \vec{V}_1) \times \vec{\omega}_0$$

Zeroth order: $\dfrac{d\vec{V}_0}{dt} = \vec{V}_0 \times \vec{\omega}_0$

Circular rotation about a center - call it $\vec{X}$

$$\left\{ \begin{array}{l} \vec{V}_0(t) = -\vec{\omega}_0 \times \left[\vec{x}_0(t) - \vec{X}\right] \\ \vec{x}_0(t) - \vec{X} = \dfrac{1}{\omega_0}\left\{\vec{\omega}_0 \times \vec{V}_0(t)\right\} \end{array} \right.$$

First order $\dfrac{d\vec{V}_1}{dt} = \left[\vec{V}_1 + \vec{V}_0\delta\right] \times \vec{\omega}_0$ $\quad *$

physics: first term is periodic motion, already included in $\vec{V}_0$. Solution will be

$$\vec{V}_1 = \text{periodic} + \text{constant}.$$

To expose the constant, do a time average on $*$, periodic motion averages to zero. "constant" is $\dfrac{d\vec{V}_1}{dt} = 0$

Then $\langle \vec{v}_1 + \vec{v}_0 \delta \rangle_{\text{time ave}} \times \vec{\omega}_0 = 0$

and

$\vec{v}_1 = \cancel{\ldots} - \langle \vec{v}_0 \delta \rangle \qquad v_0 = -\cancel{\omega_0 x_0} \; -\vec{\omega}_0 \times \vec{x}_0$

$= \dfrac{-1}{B_0} \dfrac{dB}{d\xi} \langle \hat{n} \cdot \hat{x}_0 \left[-\vec{\omega}_0 \times \vec{x}_0 \right] \rangle$ ($\omega \times \bar{x}_0$ averages to zero) time ave.

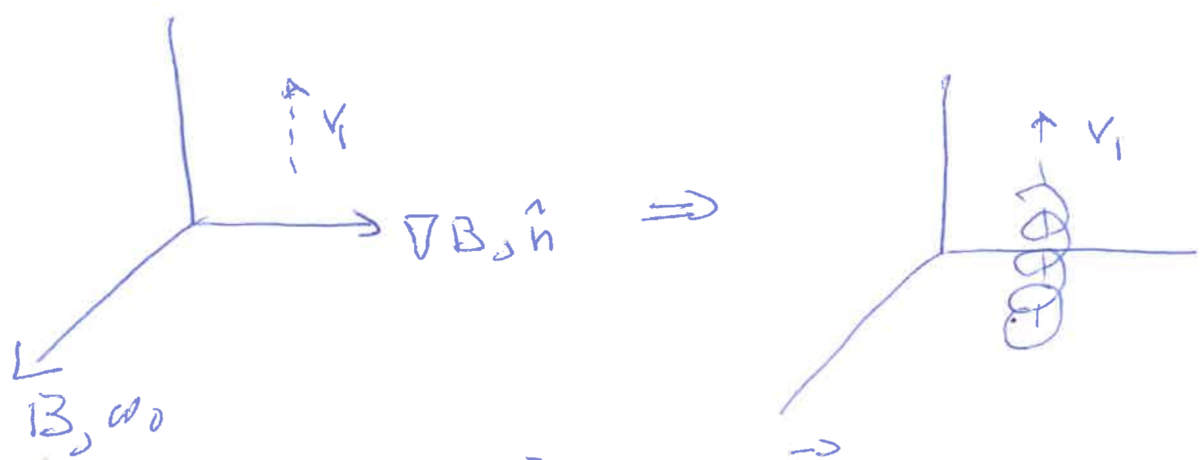
$= \dfrac{1}{B_0} \dfrac{dB}{d\omega} \vec{\omega}_0 \times \langle \vec{x}_0 (\hat{n} \cdot \vec{x}_0) \rangle_{\text{time ave}}$

let $\hat{x}_0 = a [\hat{\imath} \cos \omega t + \hat{\jmath} \sin \omega t]$

$\hat{n} = x$

$\vec{x}_0 (n \cdot x_0) = \hat{n} \, a^2 \cos^2 \omega t \rightarrow \frac{1}{2} \hat{n} a^2$

$v_1 = \dfrac{1}{B_0} \dfrac{dB}{d\xi} \dfrac{a^2}{2} \left[\vec{\omega}_0 \times \hat{n} \right] = \dfrac{\omega_B a^2}{2B^2} \left[\vec{B} \times \vec{\nabla}_\perp B \right]$

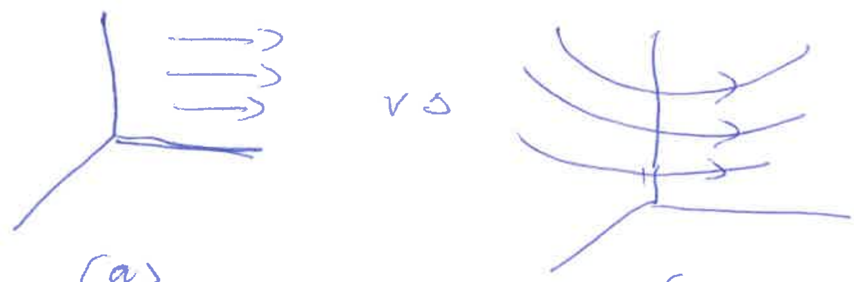


$\vec{v}_1 \perp \vec{B}$ and $\perp \vec{\nabla}_\perp B$

"gradient drift" — particle orbits field lines and move up at constant velocity

$\perp$ to B and ∇B

Next consider curvature of field lines

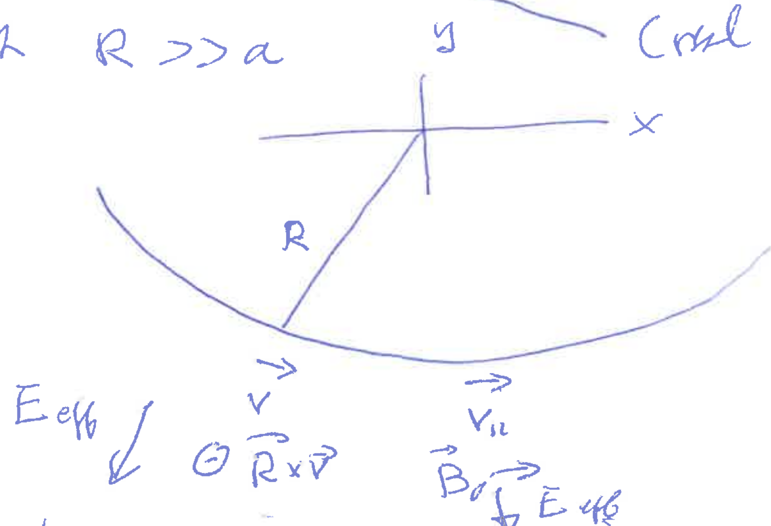


(a) (b)

For uniform B (case (a)) unperturbed motion is

- drift along field lines w/ $v_\parallel$
- orbit w/ cyclotron freq. ω_B, radius a

Hand waving calculation – assume field line is curved with $R \gg a$ (real calculation in Jackson)



There's a centrifugal acceleration along $\vec R$, $A = \frac{v_\parallel^2}{R}$.

So there's a force $F = \gamma m A$. Think of it as arising from an E-field. $E_{eff} = \frac{F}{q} = \frac{\gamma m A}{q} = \frac{\gamma m v_\parallel^2}{q}\frac{\vec R}{R^2}$

This is along $\vec R$ which is $\perp$ to $\vec B$ – think E×B!

There is an effective E×B drift up or $\perp$ curvature of field lines

$$\frac{\vec v_c}{c} = \frac{\gamma m}{q} v_\parallel^2 \frac{\vec R \times \vec B_0}{R^2 B_0^2} \Rightarrow v_c = \frac{v_\parallel^2}{\omega_B R}\left[\frac{\vec R \times \vec B_0}{R B_0}\right]; \quad \omega_B = \frac{q B_0}{\gamma m c}$$

These are motions $\perp B$. For motion along B

use "adiabatic invariants"

If $q + p$ are a conjugate pair and if they are periodic, there is a quantity

$$J = \oint_{\text{one period}} p\, dq = \text{"the action"}$$

Action integrals are a constant for mechanical systems with specified initial conditions. If parameters of system change slowly ($\equiv$ adiabatic) integrals are unchanged (= invariant)

$$J = \oint p \cdot dl$$

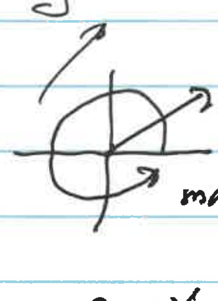
$$p = \gamma m \vec{v} + \frac{e\vec{A}}{c}$$

$$J = \oint \gamma m \underbrace{\vec{v}}_{\omega_B a} \cdot \underbrace{d\vec{l}}_{a\, d\theta} + \frac{e}{c}\oint A \cdot dl$$

$$\int \gamma m\, \omega_B a\, \hat{e} \cdot a\, \hat{e}\, d\theta + \frac{e}{c}\oint A \cdot dl$$

$$+ \frac{e}{c}\int \vec{B} \cdot \hat{n}\, dA$$

$\uparrow$ RH rule



is c (+ charge - LH rule)

$$2\pi \gamma m \omega_B a^2 - \frac{e}{c}\int B \cdot n\, dA$$

$$J = \gamma m \omega_B a^2 \cdot 2\pi - \frac{e}{c} B \cdot \pi a^2$$

$$\omega_B = \frac{eB}{\gamma mc} \quad \text{so} \quad J = 2\pi \frac{eBa^2}{c} - \pi \frac{eBa^2}{c}$$

$$J = \pi a^2 B \cdot \frac{e}{c} = \frac{e}{c} \times \text{flux through orbit}$$

If J is invariant the following are invariant

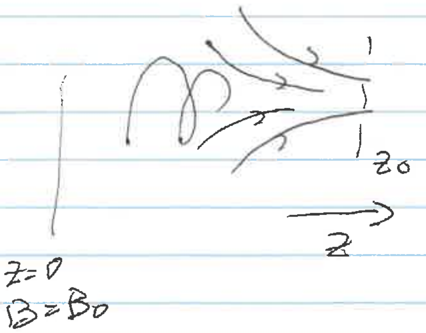
Ba^2 flux thru orbit $= \Phi_B$

$$\frac{\gamma e \omega_B a^2}{2c} = I \times \frac{\text{area}}{c} = \gamma \cdot \text{magnetic moment}$$

$$\frac{(\text{First term})^2}{\text{2nd term}} = \frac{(\gamma m \omega_B a^2)^2}{Ba^2} = \frac{(\gamma m \omega_B a)^2}{B} = \frac{p_\perp^2}{B}$$

If B changes slowly, or if $v_\parallel$ is small and the particle moves into a region of different B, radius of orbit changes so Φ_B enclosed by the orbit remains unchanged.

Application – one end of a "magnetic bottle"



Write $\vec{v} = v_\parallel + v_\perp$ $\quad \vec{v}_\perp \cdot \hat{z} = 0$

$v^2 = v_\parallel^2 + v_\perp^2 =$ constant due to energy conservation in B field

At $z=0$ $\quad v_\parallel = v_{\parallel 0}$, $v_\perp = v_{\perp 0}$, $v^2 = v_0^2$

Now $\dfrac{p_\perp^2}{B} = \dfrac{\gamma^2 m^2 v_\perp^2}{B}$ is invariant

$$\frac{v_\perp^2(z)}{B(z)} = \frac{v_{\perp 0}^2}{B_0}$$

$$v_\parallel^2(z) = v_0^2 - v_\perp^2(z) = v_0^2 - \frac{B(z)}{B_0} v_{\perp 0}^2$$

If $B(z)$ becomes sufficiently large, $v_\parallel(z) = 0$ and the particle is reflected.

Check: if $B_z = B(z)$

$$\nabla \cdot B = 0 \qquad \frac{1}{\rho}\frac{\partial \cancel{B}}{\partial \rho}\rho B_\rho = -\frac{\partial B}{\partial z}$$

so $B_\rho \approx -\frac{1}{2}\rho \dfrac{\partial B_z}{\partial z}$ $\quad$ (check: $-\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\dfrac{1}{2}\rho^2 \dfrac{\partial B_z}{\partial z} = -\dfrac{1}{\cancel{2}}\dfrac{\partial B_z}{\partial z}$

$$\gamma m \ddot{z} = q(v \times B)_z = -q[\rho \dot{\phi} B_\rho]$$

$$= \frac{q}{2}\rho^2 \dot{\phi} \frac{\partial B}{\partial z}$$

$$\rho^2 \dot{\phi} = -a^2 \omega_B = -\frac{(a\omega_B)^2}{\omega_B} = -\frac{v_{\perp 0}^2}{\omega_B} = -\frac{v_{\perp 0}^2}{\left(\frac{qB_0}{\gamma m c}\right)}$$

$\ddot{z} = -\dfrac{v_{\perp 0}^2}{2B_0}\dfrac{\partial B_z}{\partial z}$ is force vs z

energy $\quad v_\parallel^2 = -\dfrac{v_{\perp 0}^2}{2}\displaystyle\int \frac{1}{B_0}\frac{dB}{dz}dz = v_0^2 - v_{\perp 0}^2 \frac{B(z)}{B}$

Check suppose $B_z = B(z)$

$$\nabla \cdot B = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho} \rho B_\rho = -\frac{\partial B}{\partial z}$$

so $B_\rho \sim -\frac{1}{2}\rho \frac{\partial B_z}{\partial z}$

Check: $-\frac{1}{\rho}\frac{\partial}{\partial \rho}\frac{1}{2}\rho^2 \frac{\partial B_z}{\partial z} = -\frac{\rho}{\rho}\frac{\partial B_z}{\partial z}$

$$\gamma m \ddot{z} = q(v \times B)_z = -q[\rho \dot{\phi} B_\rho]$$

$$= \frac{q}{2}\rho^2 \dot{\phi} \frac{\partial B}{\partial z}$$

but $\rho^2 \dot{\phi} = -a^2 \omega_B = -\frac{(a\omega_B)^2}{\omega_B} = -\frac{v_{\perp 0}^2}{\omega_B}$

$$= \frac{v_{\perp 0}^2}{\left[\frac{qB_0}{\gamma m c}\right]}$$

$\ddot{z} = -\frac{v_{\perp 0}^2}{2B_0}\frac{\partial B_z}{\partial z}$ is force vs z

energy: $v_\parallel^2 = -\frac{v_{\perp 0}^2}{2}\int \frac{1}{B_0}\frac{dB_z}{dz}dz$

$$= v_0^2 - v_{\perp 0}^2 \frac{B(z)}{B}$$