what is this: How does E+M look different/same in different inertial frames, how to translate ~~results~~ variables, results in one ref frame to another, how to see what is frame-independent ...

Now we begin to think about the "covariance of electrodynamics." Begin first with the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

It seems natural to postulate that $\vec{J}$ and $c\rho$ ($x^0 = ct$) form a 4-vector, and that this eqn is

$$\partial_\mu J^\mu = 0$$

Is this sensible? Conservation of charge says

$$Q = \int \rho \, d^3x$$

should be frame-independent. However, d^3x is not invariant - d^4x (4-d volume) is invariant under LT

but $d^4x = dx^0 \, d^3\vec{x}$

so if Q is frame independent, ρ must transform under LT's like dx^0 does - i.e. ρ must be the 4th component of a 4-vector. Yes, this seems sensible.

What about the gauge potentials? In Lorentz gauge

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$$

and the field equations are

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix} = \frac{4\pi}{c}\begin{bmatrix} c\rho \\ \vec{J} \end{bmatrix}$$

and so it is obviously suggestive to assume that Φ and A form a four vector

$$A^\mu = (\Phi, \vec{A})$$

and
$$\partial_\mu A^\mu = 0$$
$$\Box A^\mu = \frac{4\pi}{c} J^\mu$$

However, a moments thought might worry you. The choice of ~~gauge~~ Lorentz gauge was arbitrary. Suppose we had decided to use Coulomb gauge ($\nabla \cdot A = 0$) or any other non covariant gauge? Would the predictions of electrodynamics still be consistent with special relativity?

The answer is ultimately Yes. However, I would like to approach the covariance of electrodynamics in a more modern way than Jackson does - and in a more top-down way. Rather than ask, "Is electrodynamics consistent with special relativity" I would rather ~~ask~~ say. "Nature suggests the presence of certain underlying symmetries - in this case, the two symmetries are ~~the~~ Lorentz ~~invariance~~ covariance and invariance under local gauge transformations. What theoretical description of Nature is consistent with those symmetries?"
Another way of contrasting ~~the~~ my approach is that, rather than taking the equations of motion (Maxwell's equations) as the starting point for electrodynamics, I want to begin with a classical Lagrangian and ~~ask what~~ build it using the constraints the impositions of symmetries have on it - the field equations follow from the Lagrangian as the equation of motion.

Most theorists would agree that this is a superior approach:

1) It puts the symmetries first
2) It sharpens the extent to which a particular set of equations is demanded by symmetries (is electrodynamics a la Maxwell unique?)
3) Relativistic invariance is automatic from the start
4) To construct a quantum theory, you need a Lagrangian (or a Hamiltonian, which can be constructed from the Lagrangian) - not just eqns of motion

Along the way we'll have to consider matter (charges & currents) in addition to the E&M fields. I will not follow Jackson, who works with classical point particles. Instead, I'll consider the matter itself to be described by classical fields. ~~Pe~~ This might appear at first to be too abstract - and often, the fields I'll use will often be ad hoc but

- it's easier in the end
- if you ~~need~~ want to extend our story to the quantum case, point particles are an abstraction, too - particles are excitations of quantum fields, and the first step to doing quantum field theory is classical field theory.

We'll come back later to study point particles in E&M ~~fields~~ fields.

Lagrangian for particles + fields

— particles

$$S = \int dt\, L\left(q_i, \frac{dq_i}{dt}\right)$$

$$\delta S = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad \text{at all } t$$

Fields $\quad S = \int dt \int d^3x\, \mathcal{L}\left(V_i(x,t), \frac{\partial V_i(x,t)}{\partial t}, \vec{\nabla} V(x,t) \right)$

for field variable $V_{i}^{[\,]}(x,t)$ — tensor indices
defined at every space time point — kind of field

$\mathcal{L} \equiv$ "Lagrange density"

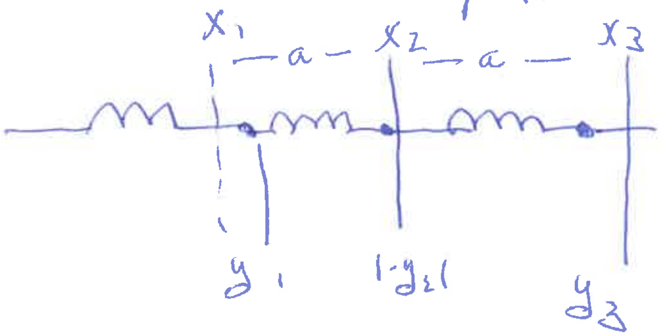
For relativistic invariance, derivatives present as $\partial_\mu V$, $\mathcal{L}$ an invariant under LT.

$$\delta S = 0 \rightarrow \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu V_i)} \right] - \frac{\partial \mathcal{L}}{\partial V_i} = 0$$

at <u>every</u> space time point

Lets see the connection in an iconic example—

1-d set of ~~points~~ mass points and springs



$y_j = y(x_j, t)$ = displacement of mass point near x_j from equilibrium

spacing of points $= a$

$$L = \sum_{j=1}^{N} \frac{1}{2} m \dot{y}_j^2 - \frac{1}{2} k (y_j - y_{j+1})^2 - \tilde{V}(y_j)$$

Equation of motion of jth mass using $\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_j} - \frac{\partial L}{\partial y_j}$

$$m \ddot{y}_j - k\left[y_{j-1} - 2 y_j + y_{j+1}\right] + \frac{\partial \tilde{V}}{d y_j} = 0$$

Now suppose y varies smoothly with x:

$$y_{j+1} = y(x_j + a) = y(x_j) + a \frac{dy}{dx}\Big)_{x = x_j} + \frac{1}{2} a^2 \frac{d^2 y}{dx^2}\Big)_{x = x_j} + \dots$$

EoM: $$m \ddot{y}(x) - k a^2 \frac{d^2 y}{dx^2} - \frac{\partial V}{\partial y(x)} = 0$$

$$\sum_j = \frac{1}{a}\int dx$$

$$L = \frac{1}{a}\int dx \left[\frac{1}{2} m \left(\frac{dy(x)}{dt}\right)^2 - \frac{1}{2} k a^2 \left(\frac{dy}{dx}\right)^2 - \tilde{V}(y(x))\right]$$

$$L = \int dx \left[\mathcal{L}\left(y(x,t), \frac{dy}{dt}(x,t), \frac{dy}{dx}(x,t)\right) \right]$$

EoM from $0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial y}{\partial x}\right)}\right) - \frac{\partial \mathcal{L}}{\partial y}$

Redefinition – replace $y(x,t)$ by $\varphi(x,t)$

rescale m, k, a so

$$L = \int dx \left[\frac{1}{c^2}\left(\frac{\partial \varphi}{\partial t}\right)^2 - \left(\frac{\partial \varphi}{\partial x}\right)^2 - V(\varphi) \right]$$

c = sound velocity, really

$$= \int dx\, \mathcal{L}\left(\varphi, \partial_\mu \varphi\right), \quad \mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - V(\varphi)$$

$$\equiv \frac{1}{2}(\partial_\mu \varphi)^2 - V(\varphi)$$

EoM is $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$

or $\partial^\mu \partial_\mu \varphi - \frac{\partial V}{\partial \varphi} = 0$

refs: Goldstein Ch 11

Peskin & Schroeder Ch 2

Ryder Ch 3

Noether's theorem

Now back to $\mathcal{L}$. Suppose we imagine only a change in the field variables

$$\varphi(x,t) \to \varphi(x,t) + \delta\varphi(x,t) \qquad (1)$$

~~...~~ At the same time the change in the derivative is

$$\partial_\mu \varphi(x,t) \to \partial_\mu \varphi(x,t) + \partial_\mu \delta\varphi(x,t)$$
$$= \partial_\mu \varphi(x,t) + \delta\left[\partial_\mu \varphi(x,t)\right]$$

~~These are called "global" symmetry transformations when $\delta\varphi$ is independent of x,t~~

The change in $\mathcal{L}$ is

$$\delta\mathcal{L} = \sum_j \frac{\partial\mathcal{L}}{\partial\varphi_j}\delta\varphi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_j)}\delta(\partial_\mu\varphi_j)$$

Recall the equation of motion $\frac{\partial\mathcal{L}}{\partial\varphi_j} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_j)}\right]$

so

$$\delta\mathcal{L} = \sum_j \left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_j)}\right)\right]\delta\varphi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_j)}\partial_\mu(\delta\varphi_j)$$

$$= \partial_\mu\left[\sum_j \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_j)}\delta\varphi_j\right]$$

$$\equiv \partial_\mu J^\mu \quad \text{defining } [\;] \text{ as a current.}$$

Now If it happens that the Lagrangian is invariant under the change of φ_j's, then $\delta\mathcal{L}=0$ and

<u>then</u> $\quad \partial_\mu J^\mu = 0$

Recap: $\mathcal{L}(\varphi, \partial_\mu \varphi)$

$$\mathcal{L}(\varphi_i \to \delta\varphi_i) = \mathcal{L}(\varphi)$$

$$\Rightarrow \partial_\mu J^\mu = 0 \ , \quad J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \delta\varphi_i$$

That is, J^μ is a conserved current.

Symmetry implies conservation law
The "charge" associated with the current is also conserved

$$Q \equiv \int d^3x\, J_0$$

Changes in φ could be "internal" - redefine φ at every space-time point.

ex: $\varphi(x,t) \to e^{i\theta(x,t)}\varphi(x,t)$

2 subdivisions: $\theta(x,t)$ = constant: ~~[illegible]~~ in x,t.
example of a global symmetry transf.
ex: $\varphi(x,t) \to \varphi(x,t)+c$ or $\varphi(x,t) \to e^{i\theta}\varphi(x,t)$
or $\theta(x,t)$ varies w/ x,t: "local symmetry transformation"

or $\delta\varphi$ could involve changes in coordinates.
Latter ones lead to "~~[illegible]~~" conserved currents" associated with energy & momentum.

$$\mathcal{L}(\varphi(x+\delta x), \partial_\mu\varphi(x+\delta x)) = \mathcal{L}(\varphi(x), \partial_\mu\varphi(x)) + \delta\mathcal{L}$$

Internal ones are more interesting for the moment.

Ex. - Schrodinger eqn as a "classical field"

$$S = \int dt \int d^3x \left[i\hbar\, \psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} (\vec\nabla \psi^*) \cdot (\vec\nabla \psi) - V(r,t)\, \psi^* \psi \right]$$

general form:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi_j}{\partial t}\right)} + \sum_i \frac{d}{dx_i} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi_j}{\partial x_i}\right)} - \frac{\partial \mathcal{L}}{\partial \varphi_j} = 0$$

$\mathcal{L}$ is a bit unsymmetric - can define more symmetric form by $\int$ by parts.

Let $\varphi = \psi^*$: no $\partial\varphi/\partial t$!

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi - i\hbar \frac{\partial \psi}{\partial t} = 0$$

or $i\hbar \dfrac{\partial \psi}{\partial t} = -\dfrac{\hbar^2}{2m} \nabla^2 \psi + V\psi$.

global phase rotation $\left(\psi \to e^{-\frac{iq\theta}{\hbar c}} \psi \right.$

$\left. \psi^* \to e^{+\frac{iq\theta}{\hbar c}} \psi \right)$

is a symmetry. Infinitesimal form is

$\psi \rightsquigarrow \left[1 - \dfrac{iq\theta}{\hbar c} \right] \psi$ or $\delta\psi = -\dfrac{iq\theta}{\hbar c} \psi$

$$\delta\psi^* = \frac{iq\theta}{\hbar c} \psi^*$$

Conserved Noether current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \delta\psi^*$$

In <u>components</u>

$$J^0 = (i\hbar \psi^*)\left(-\frac{i g \theta}{\hbar c}\right)\psi + 0 \quad \left[\text{no } \frac{\partial \psi^*}{\partial t} \text{ in } \mathcal{L}\right]$$

$$= \frac{g\theta}{c}\psi^*\psi - \frac{g\theta}{c} \text{ times usual probability density.}$$

$$\vec{J} = -\frac{i g\theta}{\hbar c}\left[-\frac{\hbar^2}{2m}(\vec{\nabla}\psi^*)\cdot\psi + \frac{\hbar^2}{2m}\psi^*\vec{\nabla}\psi\right]$$

$$= \frac{g\theta}{c}\left[-\frac{i\hbar}{2m}\left(\psi^*\vec{\nabla}\psi - (\nabla\psi^*)\psi\right)\right]$$

$$= \frac{g\theta}{c} \text{ times usual probability current}$$

And recall - conservation of probability

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$ — can be checked explicitly of course

in this example, follows from invariance of $\mathcal{L}$ under a global phase rotation of ψ

Example

A classical ~~system~~ complex scalar field $\varphi(x,t)$

Example: ① wave fn for a BEC: "classical" ≡ schrodinger wave: just solution of PDE
② ~~A~~ Collection of spins which are restricted to 2-d

$$\varphi(x,t) = \varphi_1(x,t) + i\,\varphi_2(x,t)$$

Symmetry transformation

$$\varphi(x,t)' = e^{i\theta}\varphi(x,t)$$

or
$$\begin{bmatrix}\varphi_1' \\ \varphi_2'\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}$$

Write down a simple $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\varphi_1)^2 + (\partial_\mu\varphi_2)^2\right] - V(\varphi_1,\varphi_2)$$

($(\partial_\mu\varphi)^2 = (\partial_\mu\varphi)(\partial^\mu\varphi)$ sloppy

why $\frac{1}{2}$: $\partial_\mu \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \partial_\mu\partial^\mu\varphi = \dfrac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi$

convention

coeff = 1

$$J^\mu = \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_j)} \delta\varphi_j$$

specifically

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2\right] - V(\varphi_1, \varphi_2)$$

$$\begin{bmatrix} \varphi_1' \\ \varphi_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \underset{\text{small } \theta}{\sim} \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

i.e. $\delta\varphi_1 = \theta\varphi_2$; $\delta\partial_\mu\varphi_1 = \theta\partial_\mu\varphi_2$

$\delta\varphi_2 = -\theta\varphi_1$ $\quad \delta\partial_\mu\varphi_2 = -\theta\partial_\mu\varphi_1$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_1)}\delta(\partial_\mu\varphi_1) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_2)}\delta(\partial_\mu\varphi_2) + \underbrace{\frac{\partial\mathcal{L}}{\partial\varphi_1}\delta\varphi_1 + \frac{\partial\mathcal{L}}{\partial\varphi_2}\delta\varphi_2}$$

Suppose $V(\varphi_1, \varphi_2)$ is a function only of $\varphi_1^2 + \varphi_2^2$ – obviously, "potential" part of $\mathcal{L}$ is unchanged under this transf- and last terms give zero. Kinetic term

$$\delta\mathcal{L} = (\partial^\mu\varphi_1)[\theta\partial_\mu\varphi_2] + (\partial^\mu\varphi_2)[-\theta\partial_\mu\varphi_1]$$

$= 0$, too!

Yes, its a symmetry. What is the associated conserved current?

$$J^\mu = \sum_j \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_j)}\delta\varphi_j = (\partial^\mu\varphi_1)\,\theta\varphi_2 + (\partial^\mu\varphi_2)(\theta\varphi_1)$$

$= \left(\varphi_2\partial^\mu\varphi_1 - \varphi_1\partial^\mu\varphi_2\right)$ up to a ~~n~~ overall constant. $= [\varphi\partial^\mu\varphi^* - \varphi^*\partial^\mu\varphi]$

A lot like ~~Sc~~ Schrodinger eqn ... ~~[illegible]~~

~~For next part, write θ~~ $= q\epsilon$

Note also we could have written

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi^* - V(\varphi, \varphi^*)$$

$$\delta\varphi = i\theta\varphi$$

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\,\delta\varphi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\,\delta\varphi$$

~~For next part, write $\theta = q\epsilon$~~

$\cancel{\delta\varphi = i\theta\varphi}$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}(-i\theta\varphi^*) + \frac{\partial}{\partial(\partial_\mu\varphi)}(i\theta\varphi)$$

For next part, call $\theta = q\epsilon$

So far, we considered a variation of φ

$$\delta\varphi = i\theta\varphi, \quad \delta\varphi^* = -i\theta\varphi$$

with θ a constant over space. We saw that if $\mathcal{L}(\varphi) = \mathcal{L}(\varphi)$, system had a conserved current. What if we now let θ vary from point to point in space?

I.e. we postulate a symmetry

$$\varphi'(x,t) = \exp[i g \epsilon(x,t)]\, \varphi(x,t) \text{ or}$$

$$\delta\varphi(x,t) = i g \epsilon(x,t)\, \varphi(x,t)$$

and we still want $\delta\mathcal{L} = 0$. We know that

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) + (\varphi \to \varphi^*) \overset{?}{\underset{\sim}{=}} 0$$

However, now, $\delta(\partial_\mu\varphi) = \partial_\mu\delta\varphi$

$$\delta(\partial_\mu\varphi) = \underbrace{i g \epsilon\, \partial_\mu\varphi}_{\text{old term}} + \underbrace{i g \phi\, \partial_\mu \epsilon}_{\text{new term}}$$

Eqn of motion $\partial_\mu \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \dfrac{\partial\mathcal{L}}{\partial\varphi}$

allows us to write

$$0 = \delta\mathcal{L} = \epsilon(x,t)\, \partial_\mu J^\mu(x,t) +$$

$$+ (\partial_\mu\epsilon)\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} i g\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}(-ig\varphi^*)\right]$$

$$= \epsilon(x)\, \partial_\mu J^\mu(x) + J^\mu\, \partial_\mu\epsilon(x)$$

First term vanishes because current was conserved,
~~Ep...~~ 2nd term is a current

The only way to make the 2nd term zero is to add new fields to $\mathcal{L}$, whose variation cancels the $\partial_\mu \epsilon$ term. The current is a 4-vector so let's add in a vector field A_μ ~~[illegible]~~ and ask that the simultaneous variation leave $\mathcal{L}$ unchanged:

$$\delta\varphi(x) = ig\,\epsilon(x)\,\varphi(x) \qquad \delta\varphi^* = -ig\epsilon\varphi^*$$

AND $$\delta A_\mu(x) = \partial_\mu \epsilon(x)$$

i.e. $$\vec{A}'(x) = \vec{A}(x) + \vec{\nabla}\epsilon$$
$$A_0'(x,t) = A_0(x,t) + \frac{\partial \epsilon}{\partial t}$$

Now $$\delta\mathcal{L}' = \delta\mathcal{L}_{matter} + \frac{\partial \mathcal{L}}{\partial A_\lambda}\delta A_\lambda + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)}\partial_\nu \delta A_\lambda$$

$$= (\partial_\lambda \epsilon) J^\lambda + \frac{\partial \mathcal{L}}{\partial A_\lambda}\partial_\lambda \epsilon + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)}\partial_\nu \partial_\lambda \epsilon$$

This is zero as long as

$$\frac{\partial \mathcal{L}}{\partial A_\lambda} = -J^\lambda \, ;$$ unique specification of coupling of new field to the conserved current

and $$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)} = -\frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\nu)}$$ so that the 2 $\partial_\nu\partial_\lambda\epsilon + \partial_\lambda\partial_\nu\epsilon$ bits cancel

No new conservation laws, but conditions on couplings of fields

Summary

1) In our example, we had a complex "matter" field ~~and~~ ~~bosons~~ $\varphi(x,t)$ and $\underline{\text{if}}$ its $\mathcal{L}$ is invariant under the global transformation

$$\delta\varphi(x,t) = i\epsilon q\varphi(x,t)$$

there is a conserved current J_μ.

This is a global symmetry transformation.

2) Replace global symmetry by local symmetry transformation

$$\delta\varphi(x,t) = iq\epsilon(x,t)\,\varphi(x,t)$$

To make this a symmetry, $\delta\mathcal{L} = 0$

a) you need to add a gauge field

$$A_\mu(x) \; ; \; \delta A_\mu(x,t) = \partial_\mu\epsilon(x,t)$$

b) gauge field couples to conserved current J^μ

$$\mathcal{L}_I = -J_\mu A^\mu$$

c) $\mathcal{L}$ involves $\partial_\mu A_\nu - \partial_\nu A_\mu \equiv F_{\mu\nu}$

$$\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu\varphi) - J_\mu A^\mu + \mathcal{L}(F_{\mu\nu})$$

The transformation $A'_\mu = A_\mu + \partial_\mu \epsilon(x)$
is called a <u>local gauge transformation</u>. Theories which are invariant under local gauge transformations are called <u>gauge theories</u> and include

Electrodynamics (what we just did)

$\cancel{\phi(x) \to e^{iq\epsilon(x)}\phi(x)}$

as well as matrix generalizations (ϕ a column vector or set of fields

$$\phi(x) \to R(x)\,\phi(x)$$

$$\cancel{A'_\mu(x) = R(x) A_\mu(x) R^{-1}}$$

A_μ becomes a <u>matrix</u>

including QCD (strong interactions)

Weinberg - Salam model (unification of weak + electromagnetic int.)

i.e. all of Nature (even gravity, though long story)

Can work with * on GR but there is something better —

To satisfy local gauge invariance

$$\mathcal{L} = \mathcal{L}(\varphi, D_\mu\varphi, F_{\mu\nu})$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D_\mu\varphi = \partial_\mu\varphi - iqA_\mu\phi$$ ≡ "covariant derivative"

Covariant derivative.

Recall $\varphi' = \exp i q \epsilon(x) \varphi$, $A'_\mu = A_\mu + \partial_\mu \epsilon$

$\mathcal{L}_{matter} \propto (\partial_\mu \varphi)^* (\partial^\mu \varphi) - V(\varphi^* \varphi)$.

Replace $\partial_\mu \varphi$ by $D_\mu \varphi = \partial_\mu \varphi - i q A_\mu \varphi$

then $\mathcal{L}(\varphi, D_\mu \varphi, A) = \mathcal{L}(\varphi', D'_\mu \varphi', A')$

is gauge invariant. The idea is, φ & $D_\mu \varphi$ transform identically under a gauge transformation.

Check: $D'_\mu \varphi' = \partial_\mu \varphi' - i q A'_\mu \varphi'$

$= \partial_\mu [e^{iq\epsilon} \varphi] - iq [A_\mu + \partial_\mu \epsilon] e^{iq\epsilon} \varphi$

$= e^{iq\epsilon} \{ \partial_\mu \varphi + iq \varphi \partial_\mu \epsilon - iq A_\mu \varphi - iq \varphi \partial_\mu \epsilon \}$

$= e^{iq\epsilon} [\partial_\mu \varphi - iq A_\mu \varphi] =$

$= e^{iq\epsilon} D_\mu \varphi$

so $\mathcal{L}_{matter} = \frac{1}{2} (D'_\mu \varphi')^* (D_\mu \varphi') - V(|\varphi'|^2)$

$= e^{-iq\epsilon} e^{iq\epsilon} \frac{1}{2} (D_\mu \varphi)^* (D_\mu \varphi) - V(|\varphi|^2)$

is gauge invariant.

$\mathcal{L} = \mathcal{L}_{matter} + \mathcal{L}(F)$

And finally, what is a good candidate $\mathcal{L}$ for the new electrodynamic gauge degrees of freedom?

$\mathcal{L}$ is a scalar – contract indices

$$\mathcal{L} = c_1 F_{\mu\nu} F^{\mu\nu} + c_2 (F_{\mu\nu} F^{\mu\nu})^2 + \cdots$$

The Lagrange density which gives Maxwell's equations is "the simplest one you can write"

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$$

+ A-independent terms.

Obviously rel. invariant from index contraction
Arbitrary coefficients ~~gives~~ chosen to give CGS conventions for field equations.

Why isn't there more? Hold that thought!

Let's just check that this $\mathcal{L}$ gives Maxwell's equations.

Equation of motion

$$\partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} - \frac{\partial \mathcal{L}}{\partial A^\alpha} = 0$$

Write $\mathcal{L} = -\frac{1}{16\pi}\, g_{\lambda\mu}\, g_{\nu\sigma} \left[\partial^\mu A^\sigma - \partial^\sigma A^\mu\right]\left[\partial^\lambda A^\nu - \partial^\nu A^\lambda\right]$
$- \frac{1}{c} J_\alpha A^\alpha$

$$\frac{\partial \mathcal{L}}{\partial[\partial^\beta A^\alpha]} = -\frac{1}{16\pi}\, g_{\lambda\mu}\, g_{\nu\sigma} \Big[\delta_\beta^{\ \mu} \delta_\alpha^{\ \sigma} F^{\lambda\nu} - \delta_\beta^{\ \sigma} \delta_\alpha^{\ \mu} F^{\lambda\nu} + \delta_\beta^{\ \lambda} \delta_\alpha^{\ \nu} F^{\mu\sigma} - \delta_\beta^{\ \nu} \delta_\alpha^{\ \lambda} F^{\mu\sigma} \Big]$$

= First term: $\lambda = \mu = \beta$, ~~ν~~ $\nu = \sigma = \alpha$, $F_{\beta\alpha}$

~~$\frac{24}{16\pi}$~~ 2nd $\lambda = \mu = \alpha$, $\nu = \sigma = \beta$, $-F_{\alpha\beta} = F_{\beta\alpha}$

2 more - exactly similarly

$$= -\frac{4}{16\pi} F_{\beta\alpha} \qquad \frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c} J_\alpha$$

$$-\partial^\beta \frac{1}{4\pi} F_{\beta\alpha} + \frac{1}{c} J_\alpha = 0$$

$$\boxed{\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c} J_\alpha}$$

And $\frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c} J_\alpha$

or $\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c} J_\alpha$

Now the field strength tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ is nothing more than the E and B fields:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \epsilon_{ijk} \partial_j A_k$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad E_i = -\partial_i A_0 - \partial_0 A_i$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & +B_z & 0 & -B_x \\ -E_z & -B_y & +B_x & 0 \end{bmatrix}$$

so the inhomogeneous field equation is just $(J^\mu = (c\rho, \vec{J})$)

$\alpha = 0 \quad \nabla \cdot E = 4\pi\rho$

$\alpha = i \quad \nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$

2 of 4 maxwell eqns!

The homogeneous equations vanish by construction

($\nabla \cdot B = 0 \Leftrightarrow B = \nabla \times A$)

~~Presentation~~ In covariant ~~constant~~ language they are

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0 \qquad \alpha \neq \beta \neq \gamma$$

or $\partial_\alpha \mathcal{F}^{\alpha\beta} = 0$ where $\mathcal{F}^{\alpha\beta} \equiv$ "dual field strength tensor"

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ & 0 & E_z & -E_y \\ & & 0 & E_x \\ & & & 0 \end{bmatrix}$$

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} \partial_\alpha F_{\mu\nu}$$

$$= \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} \partial_\alpha \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right)$$

$= 0$ because ϵ is antisymmetric under exchange of α, μ or $\alpha\nu$

$\epsilon^{\alpha\beta\gamma\delta} = 1$ for one order of $\alpha\beta\gamma\delta$

-1 for opposite

0 if any 2 equal

Lagrangian and Hamiltonian

$$L = \int d^3x \, \mathcal{L}$$

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = \frac{1}{8\pi}\left(E^2 - B^2\right)$$

Hamiltonian? $\quad H = \int d^3x \, \mathcal{H}$

Write $\quad \mathcal{L} = -\frac{1}{8\pi}\left[\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right]\frac{\partial A^\mu}{\partial x^\nu}$

Need canonical coordinates – an obvious choice is the 4 A_μ's (generally true in a field theory $x \to \varphi$!)

Canonical momenta

$$k = 1, 2, 3 \qquad \pi^k = \frac{\partial \mathcal{L}}{\partial \dot{A}_k} = -\frac{1}{8\pi} \cdot 2 \cdot \left[\frac{\partial A_k}{\partial x^0} - \frac{\partial A_0}{\partial x^k}\right]$$

$$= \frac{E_k}{4\pi}$$

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \qquad \left(E_k = -\nabla_k \Phi - \frac{1}{c}\frac{\partial A}{\partial t}\right.$$

$$\mathcal{H} = \sum_{k=1}^{3} \pi^k \dot{A}_k - \mathcal{L} = \frac{1}{4\pi}\left[\frac{1}{2}\left(E^2 + B^2\right) + \vec{E} \cdot \vec{\nabla}\Phi\right]$$

At this point things get a bit tricky.
Note (first of all) that there is no momentum conjugate to A_0: A_0 has no independent dynamics

Second, in the absence of sources, Maxwell's equations (which we could as well derive as a set of Hamilton equations) require

$$\vec{\nabla} \cdot \vec{E} = 0$$

so the 3 E's are not independent

The system has constraints. They are related to gauge symmetry; not all the A's are independent variables
To the best of my knowledge, this is not a problem for the classical theory, but it is a problem in constructing the quantum theory using "canonical quantization":

1) write down classical $\mathcal{L}$
2) construct classical H; identify p, q
3) Impose canonical quantization conditions
$$[q_i, p_j] = i\hbar\delta_{ij}$$
4) Construct ~~classical~~ quantum H by replacing coordinates & momenta by coord & mom. operators.

To proceed beyond point 3) it is necessary to fix a gauge ~~to have eigenfunctions~~ even

to have energy eigenfunctions → and in order to make H time independent the gauge must be noncovariant, like Coulomb gauge. But then the formulas all look noncovariant (there is an instantaneous Coulomb interaction, for example) and then it is a lot of work to show that the quantum theory is covariant - all the noncovariant parts cancel.

Transformation properties:

The A^μ's form a 4-vector, under a Lorentz transformation they mix $A'^\mu = \Lambda^\mu{}_\nu A^\nu$

or $\begin{pmatrix} A'_0 \\ \vec{A}' \end{pmatrix} = \Lambda \begin{pmatrix} A_0 \\ \vec{A} \end{pmatrix}$ (matrix)

E & B are elements of the second rank tensor $F^{\mu\nu}$

It transforms as

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} F^{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu}$$

or (matrix notation)

$$F' = \Lambda F \Lambda^T$$

Multiply out a boost along the x-axis

$$\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$xyz \to 1\,2\,3$

$$E'_1 = E_1 \,, \quad B'_1 = B_1$$

$$E'_2 = \gamma(E_2 - \beta B_3) \qquad B'_2 = \gamma(B_2 + \beta E_3)$$

$$E'_3 = \gamma(E_3 + \beta B_1) \qquad B'_3 = \gamma(B_3 - \beta E_2)$$

or $$\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{E})$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{B})$$

$$\gamma - \frac{\gamma^2\beta^2}{\gamma+1} = \frac{\gamma^2+\gamma-\gamma^2\beta^2}{1+\gamma} = \frac{\gamma^2(1+\beta^2)+\gamma}{1+\gamma}$$

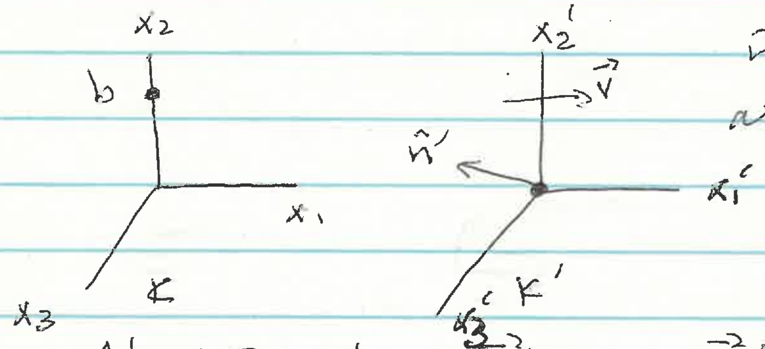
$$\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & +B_x & 0 \end{bmatrix}\right] \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -\beta\gamma E_x & \gamma E_x & E_y & E_z \\ -\gamma E_x & +\beta\gamma E_x & -B_z & B_y \\ -\gamma E_y - \beta\gamma B_z & \beta\gamma E_y + \gamma B_z & 0 & -B_x \\ -\gamma E_z + \beta\gamma B_y & \beta\gamma E_z - \gamma B_y & +B_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \gamma^2(1-\beta^2)E_x = E_x & \gamma(E_y + \beta B_z) & \gamma(E_z - \beta B_y) \\ & 0 & -\gamma(B_z + \beta E_y) & \gamma(B_y - \beta E_z) \\ & & 0 & -B_x \\ & & & 0 \end{bmatrix}$$

Example: point charge at origin in frame K'; point charge moving with $\vec{v} = +\hat{x}v$, in frame K what?

In K, observer is at $(x,y,z) = (0,b,0)$ but



in K', observer is at $(x',y',z') = (-vt', b, 0)$

Also in K', $\vec{B}' = 0$, $\vec{E}' = q\dfrac{\hat{n}'}{(r')^2}$

or $E_1' = -\dfrac{qvt'}{(r')^3}$ $\quad E_2' = -\dfrac{qb}{(r')^3}$ $\quad E_3' = 0$

and $(r')^2 = b^2 + v^2 t'^2$.

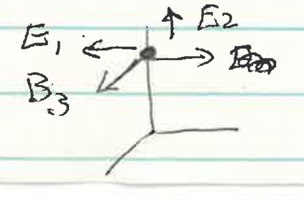
→ We have to convert both the coordinates and the fields. To convert the coords, $t' = \gamma t$ so

$$E_1' = \frac{q(-\gamma v t)}{\left[b^2 + (\gamma v t)^2\right]^{3/2}} \qquad E_2' = \frac{qb}{\left[b^2 + (\gamma v t)^2\right]^{3/2}}$$

Now we transform the fields - refer to table,

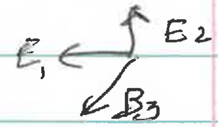
$$E_1 = E_1' = -\frac{q\gamma v t}{\left[b^2 + (\gamma v t)^2\right]^{3/2}}, \quad E_2 = \frac{qb\gamma}{\left[b^2 + (\gamma v t)^2\right]^{3/2}} \quad (= \gamma E_2')$$

$$B_3 = \beta\gamma E_2' = \beta E_2$$



Note transverse B - B_3

Note NR limit



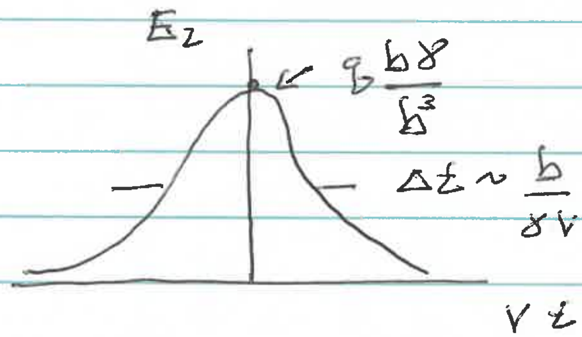
$$\vec{B} = \frac{q}{c}\,\frac{\vec{v}\times\vec{r}}{r^3} \qquad \frac{v}{c} \text{ in CGS}$$

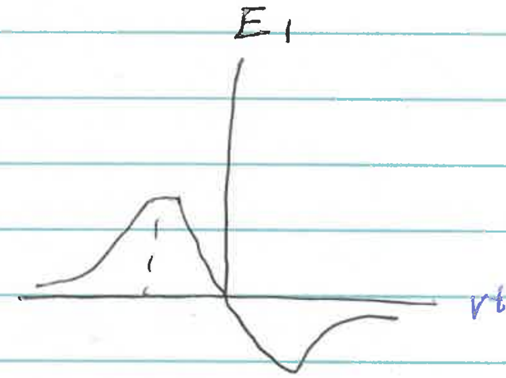
(Biot-Savart: $q\vec{v} \to I\,d\vec{l}$)

Note as $v \to c$ $|B_3| = |E_2|$

Note extreme relativistic limit

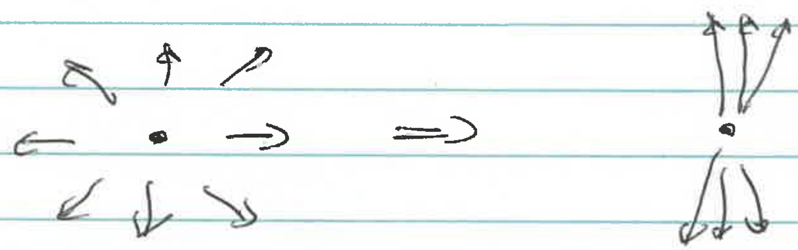
$E_2 \propto \gamma$, $E_1 \sim 1$





at $\gamma v t = b$, $E_1 = \frac{q b}{b^3\, 2^{3/2}} \sim \frac{q}{b^2}$

Fields become a pulse of transverse wave



q at rest

q at $v \sim c$

- If detector averages over times $T > \frac{b}{\gamma v}$, $\langle E_1 \rangle = 0$
- Can exploit analogy between field of real particle and plane wave - "Weizsäcker-Williams approx - see Sec 15.4

Why $F_{\mu\nu}F^{\mu\nu}$?

Why is $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$ and nothing else?

Purely classical answer - its the only thing which predicts superposition: $\mathcal{L} = E^2 - B^2$, linear E & M

Real answer - there is ~~something~~ more but corrections are small at low energy.

Quantum

Dimensional analysis. Set $\hbar = c = 1$ $\left\{ \frac{\text{mass}}{} \right.$

$\hbar c =$ energy $\times$ length $\rightarrow$ $[\text{energy}] = \frac{1}{[\text{length}]}$.

Lagrange density is $\mathcal{L} = \frac{[\text{energy}]}{[\text{length}^3]} = \frac{1}{[\text{length}]^4}$

$F_{\mu\nu} \sim \vec{E}$. $E \sim e/r^2$ $\qquad \frac{e^2}{\hbar c} = \frac{1}{137}$ is dim-less

$[E] = \frac{1}{[\text{length}]^2}$ so

$$\underbrace{\mathcal{L}}_{\frac{1}{[\text{length}]^4}} = (\text{dimensionless } \#) \underbrace{F_{\mu\nu}F^{\mu\nu}}_{\frac{1}{[\text{length}]^4}} + \cdots$$

What about other terms? Gauge invariance plus Lorentz invariance require

$\mathcal{L} = \mathcal{L}(F_{\mu\nu}F^{\mu\nu} \text{ or } F\tilde{F})$.

Imagine $\mathcal{L}$ as a polynomial in $F_{\mu\nu}F^{\mu\nu}$

Each new term needs a dimensionful coefficient

$$\mathcal{L} = c_1 F_{\mu\nu} F^{\mu\nu} + c_2 [\text{length}]^4 (F_{\mu\nu} F^{\mu\nu})^2$$
$$+ c_3 [\text{length}]^8 (F^2)^3 + \cdots$$

$$[\text{length}]^4 = \left(\frac{1}{\text{energy scale}}\right)^4$$

$$\mathcal{L} = c_1 F_{\mu\nu} F^{\mu\nu} + \frac{c_2}{\Lambda^4} (F_{\mu\nu} F^{\mu\nu})^2 + \cdots$$

What is Λ? This is a scale where some new physics, not in Maxwell's eqns, appears. The simplest new physics is quantum: exchange of virtual particles as intermediate states -

This suggests $\Lambda \sim m_e = \frac{1}{2}$ MeV.

~~Next, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$~~

~~$\sim k_\mu A_\nu - k_\nu A_\mu$ in momentum space~~

$\Rightarrow$ $\cancel{F_{\mu\nu} F^{\mu\nu} \sim k^2}$

$$\frac{1}{\Lambda^4} \cancel{(F_{\mu\nu} F^{\mu\nu})^2 \sim \frac{k^4}{\Lambda^4}} = \cancel{\frac{k^4}{\Lambda^4}}$$

For $k \ll \Lambda$, corrections from the 2nd term are completely negligible.
Of course, at $k \sim m_e$ these processes are important – a classical description fails completely.

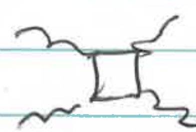
There was a calculation supporting this statement, by 2 of Heisenberg's students, Euler and Kockel, in 1935.

They started with QED (photons and electrons) and "integrated out" the electrons to derive an effective theory of photons, valid for $E_\gamma \ll m_e$,

$$\mathcal{L} = \frac{1}{2}(E^2-B^2) + \frac{e^4}{360\pi^2 m_e^4}\left\{(E^2-B^2)^2 + 7(\vec{E}\cdot\vec{B})^2\right\} + \cdots$$

Correction isn't $\frac{1}{m_e^4}$, but $\frac{\alpha^2}{m_e^4}$, 10^{-4} smaller!

The new term includes a 4-γ term – scattering of light by light



QED is itself an incomplete description of Nature at very high energy, too – it is only part of the "electroweak" interacti

"Mass of the photon"

Of course, in classical E+M there are no ~~ph~~ photons, they only appear after the theory has been quantized.

"Mass of a photon" is a poetic shorthand for how $w(k)$ depends on k. This comes from wave equation - in E+M and in free space

(1) $$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} = 0 \quad + \quad \vec{A} \sim e^{i(\vec{k}\cdot x - wt)}$$

$$\left(-\frac{w^2}{c^2} + k^2\right) A = 0 \Rightarrow w = ck$$

$$\hbar w = \hbar c k$$

as opposed to ($c=1$)

(2) $$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\Phi = 0$$

$$-w^2 + k^2 + m^2 = 0, \quad w = \pm\sqrt{k^2+m^2}$$

$$(\hbar w)^2 = (\hbar c k)^2 + (mc^2)^2$$

$$E^2 = p^2 + m^2$$

(2) comes from a Lagrange density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2 \quad (-J\Phi \text{ for source})$$

$$\partial^\mu\frac{\partial\mathcal{L}}{\partial\partial^\mu\Phi} - \frac{\partial\mathcal{L}}{\partial\Phi} = 0 : \quad \partial^\mu\partial_\mu\Phi + m^2\Phi - J = 0$$

Also - static solution $(-\nabla^2 + m^2)\Phi = J$, $\Phi \sim \frac{e^{-mr}}{r}$

~~m^2 term~~ Φ falls off exponentially w/ distance, NOT pure power law

Maxwell action
(~~?~~ is a complicated combination of $\partial_\mu A^\nu \partial_\lambda A^\sigma$ terms) with no A_μ^2 term. Thus we say, $\mu^2=0$, photon mass $=0$.

Indeed, requiring that the theory be invariant under local gauge transformations would seem to preclude a photon mass, since gauge invariance requires that the action be built of $F_{\mu\nu}$ and a term

$$A_\mu A^\mu$$

is not GI - thus excluded from the start.

Nevertheless we ~~?~~ might want to imagine models with "massive "photons" - for several reasons

1) As a straw man for doing precise experiments
2) Because they really exist in Nature.
 - superconductivity - Meissner effect 1933
 - W, Z particles

At the same time, we do not want to sacrifice ~~?~~ gauge invariance as a symmetry. We already saw that GI implied current conservation, and a naive ~~?~~ breaking of GI ~~might~~ would cost us charge conservation. This might be an expensive price to pay.

(Jackson's discussion is very naive ---)

~~In fact masslessness is a very special~~

"Spontaneous Breaking of Gauge Symmetry"

We can get a taste for the modern discussion of the photon mass (and related issues) if we diverge from pure electrodynamics to consider two related topics (which we will do by example):

Goldstone's theorem

the Higgs effect – gives a massive gauge boson

Let's consider once again a classical field with n components

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{n} (\partial_\mu \varphi_j)(\partial^\mu \varphi_j) - V(\varphi)$$

where the "potential" is taken for simplicity to be

$$V(\varphi) = \frac{\mu^2}{2} \varphi^2 + \frac{\lambda}{4!} (\varphi^2)^2$$

and $\varphi^2 = \sum_{j=1}^{n} \varphi_j^2$

These models arise in a variety of contexts

– In condensed matter physics φ might represent the average value of the spin of a patch of a system of atoms (in a magnet). The term $V(\varphi)$ measures the energy of the patch due to self-interactions (does φ want to be large or small?) The $(\partial_\mu \varphi)(\partial^\mu \varphi)$ term measures the interaction between patches of spins, separated in space. ["Ginzberg-Landau model"]

– In microscopic systems showing quantum behavior: as a description of the condensate in a Bose Einstein gas or in liquid helium: here Ψ $(\equiv \varphi)$ is complex, the wave/fn.

– As models for real fundamental particles – the Higgs

What does the potential term do to the field equations?

$$-\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial V}{\partial \varphi}$$ provides a "force" which

attempts to drive φ towards minima of V. A particularly interesting and simple case to study is the case when V has a minimum in φ, and we simply linearize the equations of motion (or expand $\mathcal{L}$ quadratically) about that minimum. Then

$$V(\varphi) = V(\varphi_0) + \frac{1}{2} V''(\varphi_0)(\varphi-\varphi_0)^2 + \underbrace{\cdots}_{\text{neglect}}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}V''(\varphi_0)(\varphi-\varphi_0)^2 + \text{constant } V(\varphi_0)$$

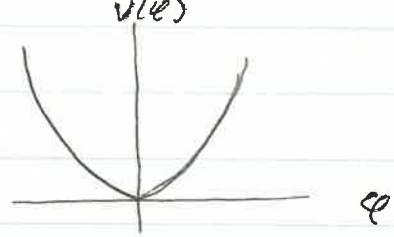
$$= \frac{1}{2}\partial_\mu(\varphi-\varphi_0)\,\partial^\mu(\varphi-\varphi_0) - \frac{1}{2}\underbrace{V''(\varphi_0)}_{-\mu^2}(\varphi-\varphi_0)^2$$

$\boxed{\therefore \text{mass}^2 = \text{2nd derivative of } V \text{ at minimum.}}$

— Hierarchy of examples —

<u>example</u> $D=1$ $V = \frac{1}{2}\mu_0^2\varphi^2 + \frac{\lambda}{4!}\varphi^4$

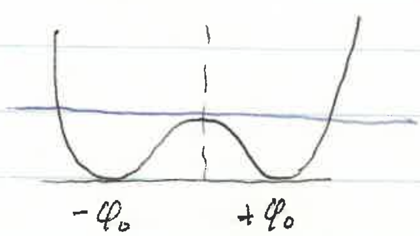
a) $\mu_0^2 > 0$



minimum is at $\varphi_0 = 0$

$$\mu^2 = \mu_0^2$$

b) $\mu_0^2 < 0$ ← it's just a parameter $\left\{ \begin{array}{l} \text{in a magnet we might} \\ \text{parametrize it as } \mu_0^2 \propto T-T_c) \\ \text{~~in a magnet } \mu_0^2 \propto T-T_c~~ \end{array}\right.$



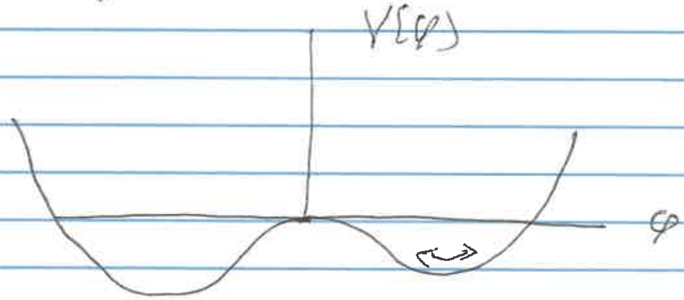
$$\left.\frac{\partial V}{\partial \varphi}\right|_{\varphi_0} = 0 = \mu_0^2\varphi_0 + \frac{\lambda}{6}\varphi_0^3$$

$$\varphi_0^2 = \frac{-6\mu_0^2}{\lambda}$$

$$\left.\frac{\partial^2 V}{\partial \varphi^2}\right)_{\varphi=\varphi_0} = \left[\mu_0^2 + \frac{\lambda}{2}\varphi_0^2\right] = \mu_0^2 - 3\mu_0^2$$

$$= -2\mu_0^2$$

(Recall μ_0^2 was set < 0) $\Rightarrow$ $\text{mass}^2 = \mu^2 = -2\mu_0^2$



μ^2 measures "rocking"

Notice that the original model had a discrete symmetry $\varphi(x,t) \to -\varphi(x,t)$

If the system minimizes its potential – we say that the system chooses one vacuum φ_0 – we say that the symmetry is broken

Indeed, write $\varphi(x,t) = \varphi_0 + \chi(x,t)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - V(\chi + \varphi_0)$$

$$V(\chi+\varphi_0) = \frac{1}{2}\mu_0^2[\chi+\varphi_0]^2 + \frac{\lambda}{24}[\chi+\varphi_0]^4$$

$$= \frac{1}{2}\mu_0^2\left[\chi^2 + 2\chi\varphi_0 + \dots\right]$$

$$+\frac{\lambda}{24}\left[\chi^4 + 4\chi\varphi_0^3 + 6\chi^2\varphi_0^2 + \dots\right]$$

$$= \chi^2\left[\frac{1}{2}\mu_0^2 + \frac{\lambda}{4}\varphi_0^2\right] + \chi \cdot 0 + \dots \chi^3 + \frac{\lambda}{4!}\chi^4$$

It's not obvious that $\chi \to -\varphi_0 - [\chi+\varphi_0]$ is a symmetry! "An ant living in a magnetized ferromagnet has a hard time realizing that the underlying system is rotationally invariant"

Phenomenon called "spontaneous symmetry breaking"

Recap

$$\mathcal{L} = \sum_i \frac{1}{2}\partial_\mu \varphi_i \partial^\mu \varphi_i - V(\phi)$$

If $V(\phi)$ has a minimum

$$\varphi = \varphi_0 + \chi \quad , \quad \frac{1}{2} m_\chi^2 = \frac{1}{2}\frac{\partial^2 V}{\partial \phi^2}\Big)_{\phi=\phi_0}$$

If $V(\phi)$ has multiple degenerate minima φ_i

- Minimum energy field configuration is

  $\varphi = \varphi_i + \chi(x,t)$

- Hard to recognize presence of symmetry

$-\varphi_0 \quad \varphi_0$

$$\varphi \to -\varphi$$

$$\varphi_0 + \chi \to -(\varphi_0 + \chi)$$

"spontaneous symmetry breaking"

"Symmetry is broken" by choice of φ_0

2nd example: $j=2$, or φ is 2-dimensional.

$$V(\varphi) = \frac{1}{2}\mu_0^2\left[\varphi_1^2+\varphi_2^2\right] + \frac{\lambda}{4!}\left[\varphi_1^2+\varphi_2^2\right]^2$$

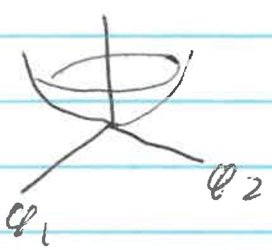
Note the continuous symmetry

$$\begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix} = \begin{bmatrix}\cos\Omega & \sin\Omega\\ -\sin\Omega & \cos\Omega\end{bmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

or $\varphi' = R\varphi$. A global rotation of (φ_1, φ_2) leaves $\mathcal{L}$ invariant. There is an associated conserved Noether current, as we found earlier.

~~Now if $\mu_0^2<0$ the potential surface looks like a sombrero~~

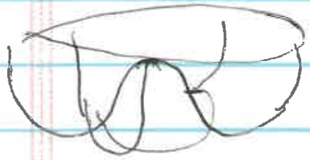
If $\mu^2>0$ the potential is concave up with a minimum at $\varphi_1=\varphi_2=0$ -



$$\frac{1}{2}\frac{\partial^2 V}{\partial\varphi_1^2}\bigg|_{\varphi_1=\varphi_2=0} = \frac{1}{2}\frac{\partial^2 V}{\partial\varphi_2^2}\bigg|_{\varphi_1=\varphi_2=0} = \frac{1}{2}\mu_0^2$$

and $\displaystyle \frac{1}{2}\frac{\partial^2 V}{\partial\varphi_1 d\varphi_2}\bigg|_{\varphi_1=\varphi_2=0} = 0$

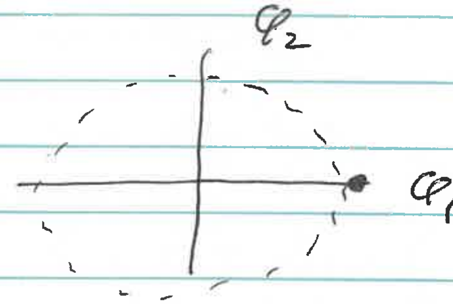
But if $\mu_0^2<0$, the potential surface looks like a sombrero or the bottom of a wine bottle -



Defining $\varphi_1^2+\varphi_2^2 = e^2$, the potential has a minimum at any point on the circle $e^2 = -\dfrac{6\mu_0^2}{\lambda}$

Let's arbitrarily suppose that the vacuum chooses to break the symmetry by setting

$$\varphi_1 \equiv \varphi_0 = \sqrt{\frac{-6\mu_0^2}{\lambda}}$$



$$\varphi_2 = 0$$

What is the spectrum?

Write $\varphi_1 = \varphi_0 + \chi_1(x,t)$

$\varphi_2 = \chi_2(x,t)$

$$V(\varphi_1, \varphi_2) = \frac{1}{2}\mu_0^2\left(\varphi_1^2 + \varphi_2^2\right) + \frac{\lambda}{24}\left[\varphi_1^2 + \varphi_2^2\right]^2$$

"When a global continuous symmetry is spontaneously broken, there is an accompanying massless mode"

≡ Goldstone's theorem

Massless ≡ $E(k) \propto k$ or arbitrary long λ costs zero energy

← rocking mode has restoring force

rolling mode has no restoring force

easy to see from picture ...

Massless mode is called a "Goldstone Boson"

example: spin wave in a magnetic system can support arbitrarily long-wavelength, low energy excitations, with spins precessing around the ordered direction

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This is $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with $\langle \varphi_3 \rangle = v$

...

More complicated symmetries - same general story, more complicated math

- $V = -\frac{\mu_0^2}{2} \Phi^2 + \lambda (\Phi^2)^2$; $\Phi^2 = \sum_{i=1}^{N} \varphi_i^2$

  O(N) symmetry breaks to O(N-1) .. (N=3, 2 GB's)

- QCD
- Electroweak

$\frac{3 \cdot 2}{2} = 3 \quad - \quad \frac{2 \cdot 1}{2} = 2$

$$= \frac{1}{2}\mu_0^2\left[(\varphi_0+\chi_1)^2+\chi_2^2\right]+\frac{\lambda}{24}\left[(\varphi_0+\chi_1)^2+\chi_2^2)\right]^2$$

$$= \frac{1}{2}\mu_0^2\left[\ \varphi_0^2+2\varphi_0\chi_1+\chi_1^2+\chi_3^2\right]$$

$$+\ \frac{\lambda}{24}\left[\varphi_0^2+2\varphi_0\chi_1+\chi_1^2+\chi_2^2\ \right]^2$$

$$= \chi_1\ \left[\varphi_0\mu_0^2+\frac{\lambda}{24}\cdot 2\cdot 2\varphi_0\cdot\varphi_0^2\right]$$

$$+\ \chi_1^2\left[\frac{1}{2}\mu_0^2+\frac{\lambda}{24}(2\varphi_0^2+4\varphi_0^2)\right]$$

$$+\chi_2^2\ \left[\frac{1}{2}\mu_0^2+\frac{2\lambda}{24}(\varphi_0^2)\right]$$

$$+\ \chi_1\chi_2\cdot 0$$

+ higher orders

Now $\varphi_0^2 = -\frac{6\mu_0^2}{\lambda}$ >0

$$\chi_1\ \varphi_0\left(\mu_0^2+\frac{\lambda}{6}\varphi_0^2\right) \qquad = \chi_1\cdot 0$$

$$+\chi_1^2\left(\frac{1}{2}\mu_0^2+\frac{\lambda}{4}\left(\frac{-6\mu_0^2}{\lambda}\right)\right) \qquad = -\frac{1}{2}\mu_0^2\chi_1^2$$ (no surprise)

$$+\chi_2^2\ \left[\frac{1}{2}\mu_0^2+\frac{\lambda}{12}\varphi_0^2\right] \qquad =\chi_2^2\cdot 0$$

$\mu_0^2<0$

quadratic part is $-\frac{1}{2}\mu_0^2\chi_1^2+0\cdot\chi_2^2$

The χ_2 mode is massless. This is
- magical, special
- general

Higgs Effect.

This is still not a massive photon. Recall that electrodynamics appeared when we tried to enforce Local phase rotations as a symmetry.

$$\phi(x) \to e^{i\theta(x)}\varphi(x) \qquad \varphi \text{ complex}$$

$$\text{or} \quad \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} \to \begin{pmatrix}\cos\theta(x) & \sin\theta(x)\\ -\sin\theta(x) & \cos\theta(x)\end{pmatrix}\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix}$$

Let's treat ϕ as complex, write

$$\mathcal{L} = \left[(\partial_\mu + ieA_\mu)\varphi\right]\left[(\partial^\mu - ieA^\mu)\phi^*\right] + \mu_0^2\phi^*\phi - \lambda(\varphi^*\varphi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

and ~~try to~~ look at the broken symmetry case.

Note: $+\mu_0^2$ corresponds to sombrero (to avoid annoying signs)

(Factor and μ, λ, $\frac{1}{4}$ convention of Ryder, Quantum Field Theory p. 301)

This is called the "Abelian Higgs model"

The minimum of V is at $|\phi| = \sqrt{\dfrac{\mu_0^2}{2\lambda}} \equiv \dfrac{a}{\sqrt{2}}$

Now write $\phi(x) = \dfrac{a + \chi_1(x) + i\chi_2(x)}{\sqrt{2}}$

i.e. move slightly off the minimizing circle and re-express in terms of the physical (fluctuating) fields χ_1 & χ_2.

Re-insert: (aside) and look only at the quadratic terms

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2a^2A_\mu A^\mu + \frac{1}{2}(\partial_\mu\varphi_1)^2 + \frac{1}{2}(\partial_\mu\varphi_2)^2 - 2\lambda a^2\phi_1^2 + \sqrt{2}\,e a A^\mu\partial_\mu\varphi_2 + \text{cubic} + \text{quartic}$$

$[\varphi^* A_\mu \varphi]$

Lets check what we have

a) potential term

$$V(\phi) = -\mu_0^2 \left| \frac{a + \chi_1 + i\chi_2}{\sqrt{2}} \right|^2 + \frac{\lambda}{4}\left(\left| a + \chi_1 + i\chi_2 \right|^2 \right)^2$$

$$= -\frac{\mu_0^2}{2}\left[a^2 + 2a\chi_1 + \chi_1^2 + \chi_2^2 \right]$$

$$+ \frac{\lambda}{4}\left(a^2 + 2a\chi_1 + \chi_1^2 + \chi_2^2 \right)^2$$

expand, keep linear + quadratic, discard the rest

$$V(\varphi) = \chi_1\left[-a\mu_0^2 + 4a^3\frac{\lambda}{4} \right] \longrightarrow \underset{0}{a^2 = \mu_0/\lambda}$$

$$+ \chi_1^2\left[-\frac{\mu_0^2}{2} + (4a^2 + 2a^2)\frac{\lambda}{4} \right] \longrightarrow -\frac{\mu_0^2}{2} + \frac{3}{2}\lambda a^2 = \frac{\lambda a^2}{2}$$

$$+ \chi_1\chi_2 \cdot 0$$

$$+ \chi_2^2 \cdot \left[-\frac{\mu_0^2}{2} + 2a^2\frac{\lambda}{4} \right] \longrightarrow 0$$

$$V(\varphi) = \frac{\lambda a^2}{2}\chi_1^2 + 0 \cdot \chi_2^2$$ as expected –

χ_2 is massless Goldstone

χ_1 is massive

Now the kinetic term

$$D_\mu = \partial_\mu - ieA_\mu \quad , \quad \varphi = \frac{\chi_1 + a + i\chi_2}{\sqrt{2}}$$

$$D_\mu \varphi = \frac{1}{\sqrt{2}}\Big[\partial_\mu \chi_1 - ieA_\mu \chi_1 - ieA_\mu a$$
$$+ i\partial_\mu \chi_2 - eA_\mu \chi_2\Big]$$
$$= \frac{1}{\sqrt{2}}\Big[\underbrace{\partial_\mu \chi_1 - eA_\mu \chi_2}_{(2)} + i(\partial_\mu \chi_2 - eA_\mu \chi_1 \underbrace{- eA_\mu a}_{(1)})\Big]$$

$$(D_\mu \varphi)^*(D^\mu \varphi) \doteq \frac{1}{2}\left[\partial_\mu \chi_1 \partial^\mu \chi_1 + \partial_\mu \chi_2 \partial^\mu \chi_2\right]$$

$$+ \frac{1}{2} e^2 a^2 A_\mu A^\mu \qquad \text{square of (1)}$$

$$+ \frac{2}{\sqrt{2}} e a A_\mu \partial^\mu \chi_2 \qquad \text{1-2 cross term}$$

$$+ \text{non quadratic}$$

Hmm - $A_\mu A^\mu$ - photon seems to have acquired a mass! Also χ_1 is massive, χ_2 is massless from before

But - what is the funny $A_\mu \partial^\mu \chi_2$ term?

To make the answer less confusing, make a change of variables - a local gauge transformation

go to expand & collect terms -

Gauge transformation is $A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \Lambda$

$$\phi' = \exp i\Lambda(x)\cdot\phi \sim [1 + i\Lambda]\phi$$

$$= (1+i\Lambda)\left[\frac{a + \chi_1 + i\chi_2}{\sqrt{2}}\right]$$

$$= \frac{a + (\chi_1 - \Lambda\chi_2) + i[\chi_2 + \Lambda\chi_1 + a\Lambda]}{\sqrt{2}}$$

$$= \frac{a + \chi_1' + i\chi_2'}{\sqrt{2}}$$

Chose Λ so that $\chi_2' = 0$: $\chi_2 + \Lambda\chi_1 + a\Lambda = 0$

In that gauge

$$\mathcal{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}e^2a^2A'_\mu A^{\mu\prime}$$
$$+ \frac{1}{2}[D'_\mu\chi_1']^2 - \frac{\lambda a^2}{2}\chi_1^2$$
$$+ \text{ non-quadratic}$$

Relabel A' as A, χ' as χ. This is a Lagrangian

for A_μ = massive gauge boson, $M_A^2 = e^2a^2 = \frac{e^2\mu_0^2}{\lambda}$

χ_1 = massive scalar boson - ~~the Higgs field~~

the Higgs field, $M_H^2 = \lambda a^2$

The physics we have developed is called the Higgs effect

Contrast

Goldstone mode: spontaneous breaking of global $U(1)$

2 massive scalar fields when symmetry unbroken $\Rightarrow$ 1 massive scalar, 1 massless scalar when broken

Higgs mode (SB of <u>local</u> $U(1)$)

2 massive scalars + 1 massless photon $\Rightarrow$ 1 massive scalar, 1 massive photon (3 pol. states)

$2 + 1 \cdot \underset{\text{states of photon}}{2\,\text{helicity}} = 4$ modes $= 1 + 1 \cdot 3\ m_J\text{'s} = 4$ modes

"the photon has eaten the scalar mode and acquired a mass"

N.B. The original gauge invariance is still a symmetry of the Lagrangian, though <u>not</u> of the ~~minimum action~~ vacuum - there is still a conserved current.

Examples: ① Meissner effect in superconductor

see S. Weinberg Prog Theor Phys Suppl <u>86</u> 43 (1986) or Ryder p.305

② W and Z mass ($U(1) \to SU(2) \times U(1)$)

More complicated group theory; 4 ~~higgs~~ scalar components

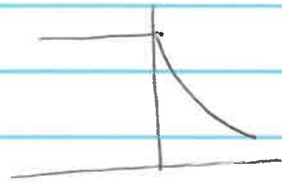
3 GB's eaten $\to m_{W^\pm}, m_Z$

1 left as physical massive higgs - ~~whose [illegible]~~

the particle at 125 GeV

Meissner effect

Superconductivity is always accompanied by Meissner effect: E & M fields vanish exponentially inside sc.



. Exponential suppression is basically mass generation for photon.

Below T_c, electron-phonon interactions (int with lattice vibrations) lead to ~~effective~~ attractive int. of electrons - system can lower its energy from usual free electron gas by forming bound state (Cooper pairs)

$\Psi_s(r)$ = wave fn of SC state

$$|\Psi_s(r)|^2 = n_{pairs}(r) = \frac{n_s(r)}{2}$$ for electrons in SC state

at $T > T_c$ $n_s = 0$. Write a free energy function of SC in terms of ψ, free energy density is function of ψ_s, $\nabla\psi_s$. ψ_s is a charged field (with charge $e^* = 2e$, mass $m^* = 2m$) so F depends on $|\psi|^2$ and canonical momentum

$$\frac{\hbar}{i}\nabla \to \frac{\hbar}{i}\nabla - \frac{e^* A}{c}$$

$$F(\Psi_s) = F_{normal}(T>0) + \int d^3r \, \frac{B(r)^2}{8\pi}$$

$$+ \int d^3r \left[a|\Psi_s|^2 + \frac{b}{2}|\Psi_s|^4 + \cdots \right]$$

$$+ \int d^3r \left[\frac{1}{2m^*} \left| \left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}(r) \right) \Psi_s(r) \right|^2 + \cdots \right.$$

and $B = \nabla \times A$. We want to minimize F by varying Ψ. This is a "typical" variational problem — it should be no surprise that the min of F happens ~~at~~ if eqns of motion are satisfied, namely

$$\vec\nabla \times \vec B = \frac{4\pi}{c}\vec J_s \quad \text{where} \qquad 1)$$

$$\vec J_s = \frac{e^*\hbar}{2m^* i}\left(\psi_s^* \vec\nabla \psi_s - (\vec\nabla \psi_s)\,\psi_s\right) \qquad 2)$$

$$- \frac{e^2}{m^* c}|\psi_s|^2 \vec A(r)$$

diff F wrt A

$$\left\{ \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}A\right)^2 + b|\psi_s|^2 \right\}\psi_s = -a\psi_s \qquad 3)$$

Notice ~~that~~ the 2nd term in the current; It comes from

$$\frac{\partial \mathcal{F}}{\partial A_\mu} = -J_\mu$$

where $\mathcal{F}$ is the free energy density = 1) is

$$0 = \partial_\mu \frac{\partial \mathcal{F}}{\partial^\mu A_\nu} - \frac{\partial \mathcal{F}}{\partial A_\nu} = 0$$

in (3), $-a$ is "like" an energy eigenvalue and the $b|\psi_s|^2$ is "like" a repulsive self interaction which tries to spread Ψ_s over the whole volume.

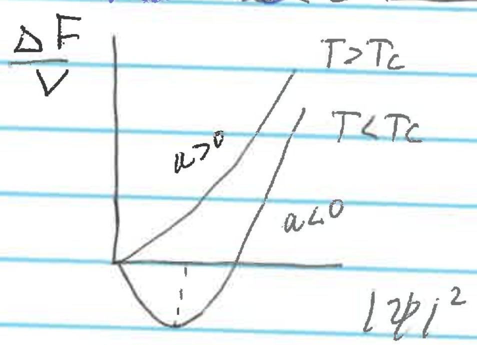
Now for solutions: deep inside the SC, $\vec{B}=0$, n_s is a constant. Then the free energy shift from the normal state is

$$\Delta F = F_S - F_N = V\cdot\left(a|\psi_s|^2 + \frac{b}{2}|\psi_s|^4\right)$$

with $(a + b|\psi_s|^2)\Psi_s = 0$ (3)

$b>0$ so if $a>0$, then $\Psi_s = 0$, the system is in its normal state

but if $T<T_c$, to get a nontrivial ground state, need $a<0$



$$|\psi_s|^2 = \frac{n_s}{2} = -\frac{a}{b} > 0$$

$$\frac{\Delta F}{V} = -\frac{a^2}{2b} \text{ is } < 0$$

$$a\left(-\frac{a}{b}\right) + \frac{b}{2}\left(\frac{a}{b}\right)^2$$

Now imagine adding a small magnetic field B. The SC current is (ψ_s is constant in space so no gradients)

$$\vec{J}_s = -\frac{e^{*2}}{m^*c}|\psi_s|^2\vec{A} = -\frac{e^2 n_s}{m^*c}\vec{A}$$

$$\text{so } \vec{\nabla}\times\vec{J} = -\frac{e^2 n_s}{m^*c}\vec{B}$$

Now $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$, take curl again

$$\nabla\times(\nabla\times B) = \nabla(\cancel{\nabla\cdot B})^0 - \nabla^2 B = \frac{4\pi}{c}\nabla\times J = -\frac{4\pi n_s e^2}{m^* c^2}\vec{B}$$

$$\nabla^2\vec{B} = \frac{1}{\lambda^2}\vec{B}, \quad \lambda = \sqrt{\frac{m^* c^2}{4\pi n_s e^2}}$$

ie - a magnetic field is screened.

λ is called the "London penetration depth"
but you can see it's basically ~~a~~ an inverse photon mass.

$$\nabla \times E = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times A$$

$$= \frac{1}{c}\frac{d}{dt}\frac{m^* c}{e^2 n_s}\vec{\nabla} \times \vec{J}$$

$$E = \frac{m^*}{e^2 n_s}\frac{d\vec{J}}{dt}$$ - London eq.

Superconductivity if $m\frac{d\vec{v}}{dt} = e\vec{E}$

$\vec{J}_s = e n_s \vec{v}_s$ ~~electrons ...~~

so London eqn

$$\frac{d\vec{J}}{dt} = \frac{n_s e^2}{m}\vec{E}$$

replaces Ohm's law $\vec{J} = \sigma\vec{E}$

In a normal conductor, electric field energy goes into dissipation, $\vec{v}_n = \frac{\sigma}{e n_e}\vec{E}$ = constant for constant E

Here

$\Rightarrow$ $B = \nabla \times A$

Pick a gauge $\nabla \cdot A = 0$ to determine $\vec{A}$ uniquely

and recall $\vec{J}_s = -\frac{e^2 n_s}{mc}\vec{A}$

$\Rightarrow$ $\nabla \times E = -\frac{1}{c}\frac{dB}{dt}$,

$\Rightarrow$ $\underline{\nabla \times J} = -\frac{e^2 n_s}{mc}\nabla \times A = \underline{-\frac{e^2 n_s}{mc}B}$ Meissner

$$\nabla \times E = -\frac{1}{c}\frac{d}{dt}\nabla \times A = +\frac{1}{c}\frac{d}{dt}\frac{mc}{e^2 n_s}\nabla \times J$$

so $\vec{E} = \frac{m}{n_s e^2} d\vec{J}/dt$ London eq.