Special Relativity done Simply

"Examples of this sort --- suggest that

"The same laws of electrodynamics and optics are valid for all frames of reference for which the laws of mechanics hold good. We will raise this conjecture (~~the purp~~ which will hereafter be called the "Principle of Relativity") to the status of a postulate and also introduce another postulate which is only apparently irreconcilable with the former, namely that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body"

Albert Einstein 1905

The pre-history of special relativity is peculiar.

- People noticed the invariance of the wave eqn under a Lorentz transformation
- People noticed transformation of E + B under change of frame,
- This was all entangled with dynamics.
- Peculiar expts (Michaelson Morley), paradoxes
- The idea that its not something special about electricity and magnetism, but something about space-time is Einstein's.

Nowadays ~~Now~~ we run the story backwards

space-time symmetries plus "internal" symmetries, like gauge invariance ~~symmetries~~ imposed on dynamics give electrodynamics (+ all dynamics)

Long discussion of tests of special rel in Panofsky + Phillips

Discussion of history in Pais "Subtle is the Lord"

Being a little more formal

- "Laws of nature the same in all inertial frames"

  2 inertial frames - means two coordinate frames or 2 observers, origin of #2 is seen by #1 to be at $\vec{r} + \vec{v}(t - t_0)$ with respect to observer #1 - and in terms of observer #1's coordinates & time

  i.e. straight line motion at constant velocity wrt each other

- speed of light independent of source: $c = c$ in all inertial frames, at rest or moving at constant velocity wrt observer

And being still more general: imagine 2 events seen by an observer in an inertial frame - events separated by Δx in space, Δt in time

Observer in another inertial frame sees $\Delta x' \neq \Delta x$, $\Delta t' \neq \Delta t$

Can begin relativity with statement

$$(\Delta s)^2 \equiv c^2(\Delta t)^2 - (\Delta x)^2 = c^2(\Delta t')^2 - (\Delta x')^2$$

is the same in all frames - it is an invariant

Contrast with Galilean invariance

$$x' = x + vt \qquad R^2 = (x_1 - x_2)^2 = (\Delta x)^2 = (\Delta x')^2$$

$$t' = t \qquad \Delta t' = \Delta t$$

Note $F = m\dfrac{d^2x}{dt^2} = m\dfrac{d^2x'}{dt'^2}$ is unchanged under Galilean transf.

(but not under Lorentz transf!)

Special case of light ray,

$$c^2(\Delta t)^2 - (\Delta x)^2 = 0$$

can be used to derive Lorentz transformation - the usual story of lights and mirrors.

Einstein did things in a more complicated way: c = constant implies $c^2(\Delta t)^2 - (\Delta x)^2 = 0$ is invariant, but in principle it could

be that $(\Delta s_1)^2$ could equal $\lambda(v^2)(\Delta s_2)^2$ where λ = some scale factor. Depends on v^2, not v, if space is isotropic, have to consider 3 frames to show $\lambda(v^2) = 1$.

So - if $(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2$ = invariant, how are (x, t), (x', t') related?

Minkowski's way is easiest - analogy with rotation $R^2 = x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$

~~...~~ Specialize to 1-d L.T.

$$y' = y, \; z' = z$$

$$x^2 - c^2t^2 \longrightarrow t = iw$$

$$x^2 + c^2w^2 = x'^2 + c^2w'^2$$

SR
3.1

$$x^2 + c^2 w^2 = x'^2 + c^2 w'^2$$

$$x' = x \cos\theta + cw \sin\theta$$
$$cw' = -x \sin\theta + cw \cos\theta$$

$x^2 - c^2t^2 = x'^2 - c^2t'^2$

$$\Longrightarrow \quad x' = x \cosh\theta - ct \sinh\theta$$
$$ct' = -x \sinh\theta + ct \cosh\theta$$

obviously $x'^2 - c^2t'^2 = x^2 - c^2t^2$

To find θ: suppose origin of S'' is $x' = 0$ in S'

origin of S'' in S is $x = vt$ (origins coincide at $t=0$)

i.e. ~~and the location of origin of S' in S' is $x'=0$~~

$$x' = 0 = x \cosh\theta - ct \sinh\theta \quad \text{@ } x = vt$$

$$\tanh\theta = \frac{x}{ct} = \frac{vt}{ct} = \frac{v}{c} \equiv \beta$$

$$\cosh\theta = \frac{1}{\sqrt{1-\beta^2}} \quad (\equiv \gamma) \qquad \sinh\theta = \frac{\beta}{\sqrt{1-\beta^2}} = \beta\gamma$$

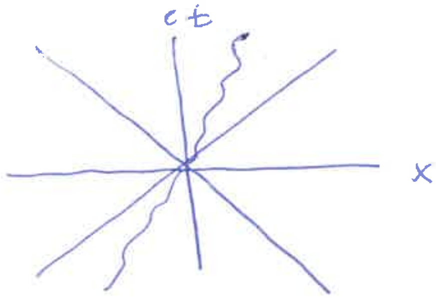
or

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$y' = y$$
$$z' = z$$

for a "boost" along x

Space-time diagrams and a little vocabulary

Plot locations of events ~~on~~ in a "space-time diagram,"
plot location as a function of time (in some frame) - a "world line"



1) world line
2) $x = \pm ct$: "light cone" = world line of light ray emitted at origin.

Note: physical objects travel at $v < c$ ("inside the light cone")

How do space-time locations of events change under a Lorentz transf?
(put one event at origin)

We know $c^2t^2 - x^2 - \cancel{z^2}$ = invariant.

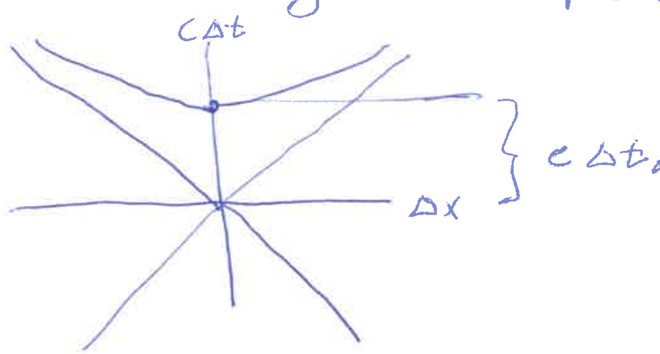
We know $c^2t^2 - x^2 = s^2$ is equation for hyperbola, so under a LT, space-time offset of event is shifted along a hyperbola. For 2 events $(x_1 t_1)$ & $(x_2 t_2)$

$$S_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2$$

$$\equiv c^2(\Delta t_0)^2 \quad \text{where } \Delta t_0 \equiv \text{"proper time"}$$

3 possibilities ~~options S_{12}^2~~

$S_{12}^2 > 0 \equiv$ "lightlike separation" / inside the light cone



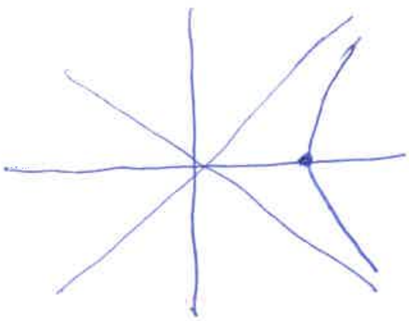
proper time is shortest time you'd measure in any frame - it's the time in the frame where the two events occur at the same location.

LT moves you along hyperbola

Notice in any other frame $\Delta x > 0$, $\Delta t > \Delta t_0$

Notice sense of past & future: ~~Δt always > 0~~
temporal order of events unchanged by L.T.

$s_{12}^2 < 0$ $\equiv$ "spacelike separation" / outside the light cone. There is a frame where $\Delta t = 0$, where $\Delta x = (-s_{12}^2)^{1/2}$.



~~In~~ ~~so~~ overall, time ordering of 2 events is different in different frames: $\Delta t > 0$ in one frame, $\Delta t < 0$ in another frame related by L.T.

("paradoxes" – rocket-in-barn porblem
"loss of simultaneity at a distance"

$s_{12}^2 = 0$: ~~events separated~~ separation is "lightlike"; $\Delta x = c\Delta t$.

Locality: $v \leq c$: no communication possible between spacelike separated events
(Implications for dynamics: QM (QFT...)

To show that Δt_0 is the shortest time interval you can measure, either look at the picture, or compute.

Suppose particle lives time τ, decays.

in frame K', where particle is at rest, τ = proper time

in frame K, where particle has velocity $u(t)$,

i.e. $d\vec{x} = \vec{u}\, dt$, particle lives Δt

$$(c^2 d\tau^2) = (ds)^2 = c^2 (dt)^2 - |d\vec{x}|^2$$

$$= c^2 dt^2 \left[1 - \frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2\right]$$

$$= c^2 (1-\beta^2)\, dt^2$$

$$(c^2 d\tau)^2 = c^2 \frac{dt^2}{\gamma^2}$$

constant

$$d\tau = \frac{dt}{\gamma}$$

γ can be fn of time or of proper time

$$\Delta t = \int_{t_1}^{t_2} dt = \int_{\tau_1}^{\tau_2} \gamma(\tau)\, d\tau$$

$$\Delta t = \int_{\tau_1}^{\tau_2} d\tau\, \gamma(\tau) = \Delta t$$

$\gamma > 1$, thus

$$\Delta t = \int_{\tau_1}^{\tau_2} d\tau\, \gamma(\tau) > \int_{\tau_1}^{\tau_2} d\tau$$ — time dilation (of course)!

Finally, define (informally),
4-vector ≡ <u>any</u> quantity which transforms under LT like $\vec{x}$, ct

$$A \equiv (\underbrace{A_0}_{\sim ct}, \underbrace{A_1, A_2, A_3})$$

form a 3 vector $\vec{A} \sim \vec{x}$ (mix under rotations as usual)

$$A_0' = \gamma(A_0 - \vec{\beta}\cdot\vec{A})$$

(no meaning to upper/lower indices yet)

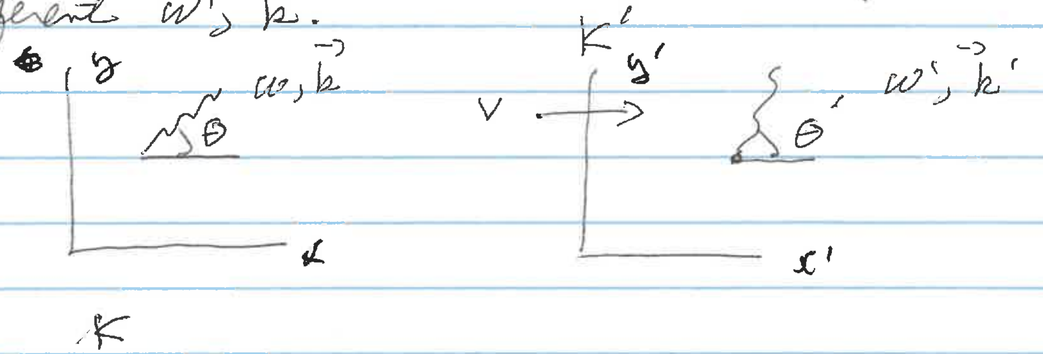
$$\left.\begin{aligned} A_\parallel' &= \gamma(A_\parallel - \beta A_0) \\ A_\perp' &= A_\perp \end{aligned}\right\} \parallel, \perp \text{ wrt boost dir.}$$

scalar product of 2 4 vectors defined as
$A\cdot B \equiv A_0 B_0 - \vec{A}\cdot\vec{B}$ is Lorentz invariant

we'll make this more precise later.
Can "prove" by direct substitution

Relativistic Doppler shift

Begin with plane wave of frequency ω, wave # k, in inertial frame K. Observer in another frame K′ sees different ω', $\vec{k}'$.



To find the relation - imagine it's a pulse - count the wave crests - this is an integer so ϕ (total phase) is an invariant

$$\exp i\phi = \exp i\left[\omega t - \vec{k}\cdot\vec{x}\right] = \exp i\left[\omega' t' - \vec{k}'\cdot\vec{x}'\right]$$

$(ct, \vec{x})$ is a 4-vector, ϕ is invariant, so $\left(\frac{\omega}{c}, \vec{k}\right)$ is also a 4-vector $k \equiv (\omega/c, \vec{k})$ so

$$k_0' = \gamma\left(k_0 - \vec{\beta}\cdot\vec{k}\right)$$
$$k_{\parallel}' = \gamma\left(k_{\parallel} - \beta k_0\right)$$
$$k_{\perp}' = k_{\perp}$$

For light, $\omega = c|\vec{k}| \Rightarrow \omega' = \gamma\omega\left[1 - \beta\cos\theta\right]$

Signs: $\theta = 0°$ $\quad \omega' = \dfrac{\omega(1-\beta)}{\sqrt{1-\beta^2}} = \omega\sqrt{\dfrac{1-\beta}{1+\beta}} < \omega$ if $\beta > 0$

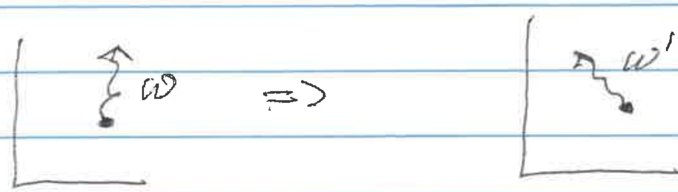
Red shift as source & detector move apart

Direction of light changes (of course)

$$\frac{k'_\perp}{k'_\parallel} = \tan\theta' = \frac{1}{\gamma}\,\frac{k\sin\theta}{\left[k\cos\theta - \beta k_0\right]} = \frac{\sin\theta}{\gamma\left[\cos\theta - \beta\right]}$$

typically $\theta' < \theta$ because of γ

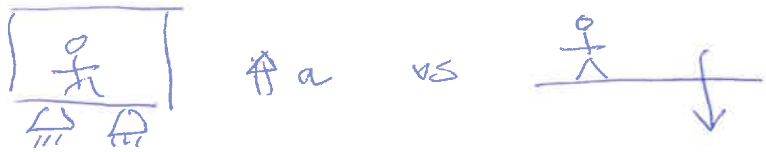
Note there's a transverse doppler shift, too — set $\theta = \pi/2$



$$\omega' = \gamma\omega$$

Aside: special relativity is special!

Equivalence principle – in a small enough region of space, no experiment can distinguish a gravitational field from uniform acceleration.



Case 1: 2 rockets w/ acceleration a, distance d apart

$x_2 = \frac{1}{2}at^2 + d$

$x_1 = \frac{1}{2}at^2$

photon emitted at 0

$\Delta t = \frac{d}{c}$ = time to arrive

$\Delta V = a\Delta t = a\frac{d}{c}$ – change in rockets' velocities at arrival time

Rocket detecting γ moves ΔV faster than frame of emission of γ. Doppler shift is

$$\nu' = \gamma\nu(1-\beta) \simeq \nu(1-\beta) \quad \text{if } \beta \ll 1,\ \gamma \approx 1$$

or $$\frac{\Delta\nu}{\nu} = -\frac{\Delta V}{c} = -\frac{ad}{c^2}$$

Case 2: Tower in gravitational field, height h, photon emitted from ground w/ freq ν

Equivalence principle says $a \to g$, $d \to h$

$$\frac{\Delta\nu}{\nu} = -g\frac{h}{c^2}$$

photon loses energy climbing out of gravitational field

For $h = 10$ cm $\frac{\Delta\nu}{\nu} = \frac{-1\ m^2/s^2}{9\cdot 10^{16}} = 10^{-17}$

(Pound + Rebka 1960, $h = 22.5$ m $\to 2.5\times10^{-15}$)

Case 3 – gravitational red shift in a free fall frame.

$\downarrow a = g$

start photon ν
End in stationary frame
$$\frac{\nu'}{\nu} = \left(1 - \frac{gh}{c^2}\right)$$

Free fall frame $\Delta v = g\Delta t = g\frac{h}{c}$

There's an extra doppler shift $\frac{\Delta\nu'}{\nu'} = \frac{\Delta v}{c} = \frac{gh}{c^2}$
from the extra velocity

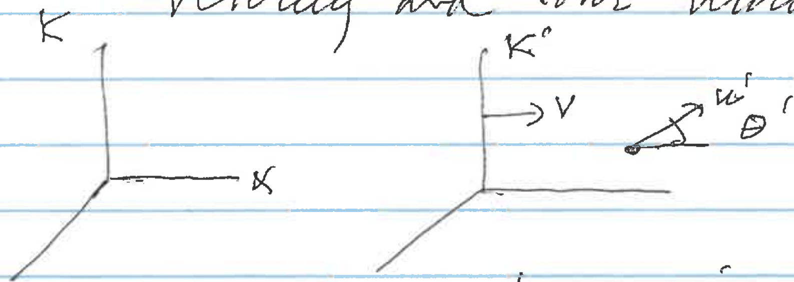
$$\nu'_{FF} = \nu'\left[1 + \frac{gh}{c^2}\right] = \nu\left[1 - \frac{gh}{c^2}\right]\left[1 + \frac{gh}{c^2}\right] = \nu$$

In the free fall frame ν = the same, bottom and top.

$\Rightarrow$ Free fall motion in a local (constant) gravitational field is inertial.

This doesn't make sense when gravity isn't uniform – but local inertial frames defined in small regions where gravity is approximately uniform do make sense. This is the "Special" in special relativity – it is only about physics in inertial frames, but at small enough distances everything is inertial.

Velocity and four-velocity



K' moves with v wrt K
particle has $\vec{u}'$ in K'
$\vec{u}$ seen by observer in K = ?

Use L.T., $x_0 = ct$ $\quad \gamma_v = \dfrac{1}{\sqrt{1 - v^2/c^2}} \quad \beta = \dfrac{v}{c}$

$$dx_0 = \gamma_v (dx'_0 + \beta\, dx'_1)$$
$$dx_1 = \gamma_v (dx'_1 + \beta\, dx'_0)$$
$$dx_2 = dx'_2$$
$$dx_3 = dx'_3$$

$$u'_i = c\frac{dx'_i}{dx'_0}, \quad u_i = c\frac{dx_i}{dx_0}$$

$$u_\parallel = c\frac{dx_1}{dx_0} = \frac{dx'_1 + \beta\, dx'_0}{dx'_0 + \beta\, dx'_1} = \frac{u'_\parallel + v}{1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}} = \frac{\frac{dx'_1}{dx'_0} + \beta}{\frac{dx'_0}{dx'_0} + \beta\frac{dx'_1}{dx'_0}}$$

$$\left(= \frac{u' + v}{1 + \frac{vu'}{c^2}} \text{ for collinear} \right)$$

$$\frac{c + c}{\ } = \frac{2c}{2} = c$$

$$\frac{c + v}{1 + \frac{v}{c}} = c$$

$$u_\perp = c\frac{dx_\perp}{dx_0} = \frac{u'_\perp}{\gamma_v \left[1 + \frac{v \cdot u'}{c^2}\right]} \Leftarrow \frac{dx'_2}{\gamma(dx'_0 + \beta\, dx'_1)}$$

∥ and ⊥ are with respect to $\vec{v}$ of course

$$\tan\theta = \frac{u_\perp}{u_\parallel} = \frac{u' \sin\theta'}{\gamma_v \left[u' \cos\theta' + v\right)}$$

Usual velocity

u is not sort of a 4-vector. However, there is a related 4-vector, the four velocity – basically derivative of four-vector coordinate with respect to invariant proper time – this is a 4-vector by construction

$$(\ldots, \vec{u}) \Rightarrow (U_0, \vec{U})$$

$$U_0 = \frac{dx_0}{d\tau} \qquad \tau = \text{proper time}, \; t = \gamma_u \tau$$

$$= \frac{dx_0}{dt}\left(\frac{dt}{d\tau}\right) = c \cdot \gamma_u \qquad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}$$

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \gamma_u \vec{u}$$

$$U_0^2 - U^2 = \gamma_u^2\left[c^2 - u^2\right] = c^2 \quad \text{invariance checks}$$

$$\therefore U = (c\gamma_u, \vec{u}\gamma_u)$$

$$= c(1, 0) \text{ in rest frame of object}$$

$U_\perp' = U_\perp$,

Lots of algebra to show consistency w/ velocity rule.

For u, u', v as on last page, you can show

$$\gamma_u = \gamma_v \gamma_{u'}\left[1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}\right] \text{ so } \frac{\gamma_u}{1 + \frac{v \cdot u'}{c^2}} = \gamma_v \gamma_{u'}$$

$$\gamma_u u_\parallel = \gamma_u\left[\frac{u_\parallel' + v}{1 + \frac{v u'}{c^2}}\right] \text{ from velocity rule}$$

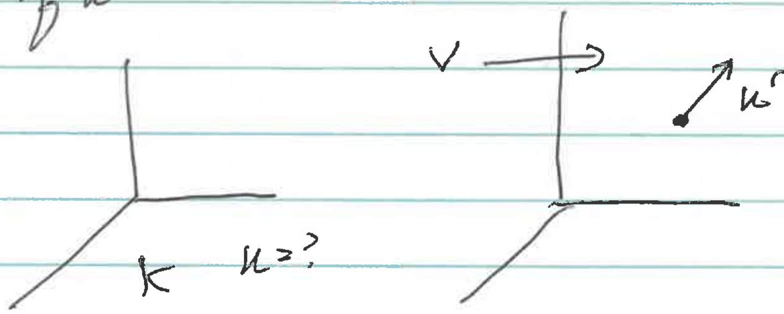
$$= \gamma_v\left[\gamma_{u'} u_\parallel' + \frac{v}{c}\cdot\gamma_{u'} \cdot c\right]$$

$$\gamma_u u_\perp = \gamma_{u'}\left(\gamma_v\left[1 + \frac{v\cdot u'}{c^2}\right]\right)\frac{u_\perp'}{\gamma_v\left(1 + \frac{v\cdot u'}{c^2}\right)}$$

$$= \gamma_{u'} u_\perp' \quad \text{or} \quad U_\parallel = \gamma_v\left[U_\parallel' + \frac{v}{c}U_0'\right]$$

$$U_\perp = U_\perp' \quad \text{as expected}$$

Lots of algebra needed to pass from simple transformation of u to messy transformation of u



There are $\gamma_u = \dfrac{1}{\sqrt{1 - u^2/c^2}}$

$$\gamma_{u'} = \frac{1}{\sqrt{1 - u'^2/c^2}}$$

and $\gamma_v = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ from boost

You need to show $\gamma_u = \gamma_v \gamma_{u'} \left[1 + \dfrac{v \cdot u'}{c^2} \right]$

or $$\frac{\gamma_u}{1 + \frac{v \cdot u'}{c^2}} = \gamma_v \gamma_{u'}$$

then

$$\gamma_u u_{\parallel} = \gamma_u \left[\frac{u'_{\parallel} + v}{1 + \frac{v \cdot u'}{c^2}} \right] = \gamma_v \left[\gamma_{u'} u'_{\parallel} + \frac{v}{c} \gamma_{u'} c \right]$$

$$\gamma_u u_{\perp} = \underbrace{\gamma_{u'} \left[\gamma_v \right] \left[1 + \frac{v \cdot u'}{c^2} \right]}_{\gamma_u} \frac{u'_{\perp}}{\gamma_v \left[1 + \frac{v \cdot u'}{c^2} \right]}$$

$$= \gamma_{u'} u'_{\perp}$$

Check the algebra

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{1}{c^2}\frac{(u'_{\parallel}+v)^2}{\left(1+\frac{v\cdot u'}{c^2}\right)^2} - \frac{u_{\perp}^2(1-v^2/c^2)}{c^2\left(1+\frac{v\cdot u'}{c^2}\right)^2}}}$$

$$= \frac{\left(1-\frac{v\cdot u'}{c^2}\right)}{\left[\left(1+\frac{v\cdot u'}{c^2}\right)^2 - \frac{1}{c^2}(u'_{\parallel}+v)^2 - \frac{1}{c^2}(u'^2-u_{\parallel}^2)\left(1-\frac{v^2}{c^2}\right)\right]^{1/2}}$$

$$v\cdot u' = u'_{\parallel}v$$

Denominator is

$$1 + \cancel{\frac{2u'_{\parallel}v}{c^2}} + \cancel{\frac{u'^2_{\parallel}v^2}{c^4}} - \cancel{\frac{u_{\parallel}^2}{c^2}} - \cancel{\frac{2vu'_{\parallel}}{c^2}} - \frac{v^2}{c^2} - \frac{u'^2}{c^2}\left(1-\frac{v^2}{c^2}\right)$$

$$+\cancel{\frac{u_{\parallel}^2}{c^2}} \Rightarrow \cancel{\frac{v^2u'^2}{c^4}}$$

$$= \left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)$$

$$U = [\gamma c, \gamma \vec{u}] \qquad \frac{d}{d\tau} = \gamma \frac{d}{dt}$$

4-Acceleration a generalized acceleration

$$A = \frac{dU}{d\tau} = \left[\gamma \frac{d(\gamma c)}{dt}, \gamma \frac{d}{dt}(\gamma \vec{u})\right]$$

also a 4 vector

$$\frac{d\gamma}{dt} = \frac{d}{dt} \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{d}{dt} \frac{1}{\sqrt{1-\frac{u \cdot u}{c^2}}} = \frac{\vec{u} \cdot \vec{a}}{c^2} \gamma^3$$

$$\vec{a} = \frac{d\vec{u}}{dt} = \text{"usual" acceleration}$$

$$\frac{d}{dt}(\gamma \vec{u}) = \gamma \vec{a} + \vec{u} \gamma^3 \frac{u \cdot a}{c^2}$$

$$A = \left[\gamma^4 \frac{\vec{u} \cdot \vec{a}}{c}, \gamma^4 \left(\frac{u \cdot a}{c^2}\right) \vec{u} + \gamma^2 \vec{a}\right]$$

Note that in the rest frame with $\vec{u} = 0$, $\gamma = 1$

$$A = [0, \vec{a}]$$

$$U = [c, 0]$$

$A \cdot U = 0$ (4 vector dot product)

This is true in any other inertial frame (of course)

Use: potentially applicable in relativistic generalization of Newton's laws.

Note weasel word potentially

Use (sometimes)

observer at rest has 4 velocity
which is purely timelike

$$V = c[1, 0, 0, 0]$$

Need the Timelike component of 4 vector

$$A = [A_0, A_1, A_2, A_3]$$

gotten from $A_0 = \frac{1}{c}(V \cdot A)$ (obviously) in
rest frame

And - it's an invariant

$V \cdot A$ same in all frames.

seems like overkill ... until you do GR!

Relativistic generalization of momentum

All that was just kinematics. Now we want to discuss dynamics - this is much harder - easy to define relativistic extensions of NR energy and momentum. However it is a separate question (which must be decided by experiment) whether or not our definitions are useful / sensible / correct.

Relativity introduces constraints. - think QM for a second.
$$\psi(q+\delta q) = \psi(q) + \delta q \frac{\partial \psi}{\partial q} \equiv \hat{S}\psi(q)$$

$$\hat{S} = 1 + \delta q \frac{\partial}{\partial q} . \quad [p, q] = -i\hbar \Rightarrow$$

$$\hat{S} = 1 - \frac{i}{\hbar} \delta \vec{q} \cdot \hat{\vec{p}}$$ Momentum is generator of spatial translation

$$\psi(t+\delta t) = \psi(t) + \delta t \frac{\partial \psi}{\partial t} = \hat{U}\psi(t)$$

$$\hat{U} = 1 - \frac{i}{\hbar} \delta t H$$ - H is generator of time translation.

In NR systems, no direct connection between space & time - so no connection between H & p - but relativity says $\Delta x, \Delta t \Leftrightarrow \Delta x', \Delta t'$ are related by L.T. - there must be a connection between momentum and energy.

Conventional to begin w/ noninteracting systems.

We know, for free particle (NR), $\vec{p} = m\vec{u}$

$$E = E(0) + \frac{p^2}{2m}$$

Relativistic energy & momentum should reduce to these at low velocity

Kinematics, minimizing algebra

Einstein, "Meaning of relativity": (1922)

NR $\vec{p} = m\dfrac{d\vec{x}}{dt} = m\vec{v}$ "must generalize to" $p = m\dfrac{dx}{d\tau} = mV$

(4 vector) since mV is the unique 4-vector whose 3 vector piece reduces to NR $\vec{p}$ in the NR limit.

Immediate consequence: energy & momentum conservation in one Lorentz frame automatically implies conservation in all inertial frames

$$\underline{P} = mV = (\gamma mc, \gamma m\vec{v}) = \left(\frac{E(p)}{c}, \gamma m\vec{v}\right)$$

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} = mc^2 + \frac{1}{2}mv^2 + \cdots \quad *$$

$$p = \gamma m\vec{v} = m\vec{v} + \cdots$$

Recall billiard ball scattering in NR limit: elastic

~~KE conserved~~ masses don't change, KE conserved

$\rightarrow$ E conserved in *

Now think of a reaction $a + b \to c + d + e + \cdots$

and suppose there is one frame where

$$\sum_{\text{initial}} \vec{p}_j - \sum_{\text{final}} \vec{p}_k = 0 \equiv \sum \vec{p}_i = 0$$

and $\sum E_j - \sum E_k = 0 \equiv \sum E_i = 0 \quad (E_i = \gamma m_i c^2)$

Under a LT
$$E' = \gamma(E - \beta c p_\parallel)$$
$$p'_\parallel = \gamma(p_\parallel - \beta E/c)$$
$$p'_\perp = p_\perp$$

Then $\sum E_i' = \gamma(\sum E_i - \beta c \sum p_{\parallel i})$

$\sum p_{i\parallel}' = \gamma(\sum p_{\parallel} - \frac{\beta}{c} \sum E_i)$ *

$\sum \vec{p}_{\perp}' = \sum \vec{p}_{\perp}$

and obviously conservation of E, p in unprimed frame gives conservation in primed frame.
Generalization for Einstein was that energy involved mass.

Jackson ~~based the derivation I got hold of~~ is basically (*) but without associative lino of algebra!
Is this defn. unique? As long as you assume no dependence of conserved quantities ~~on~~ x, or accelerations, yes — V is only 4 vector which reduces to velocity, ~~yes~~ (so unique for free particles)

Note $\underline{P} = \left(\frac{E}{c}, p\right) = (\gamma mc, \gamma m\vec{v})$ if $m \neq 0$

implies $\boxed{\frac{E}{mc^2} = \gamma}$ $\frac{cp}{E} = \frac{\gamma m v}{\gamma m c} = \beta$ $\boxed{\frac{cp}{E} = \beta}$

useful fact

invariant: $\frac{E^2}{c^2} - p^2 = \gamma^2 m^2 (c^2 - v^2) = m^2 c^4$

$E^2 - p^2c^2 = (mc^2)^2$

$$\left\{ \begin{array}{l} \text{at } p = 0,\ E = mc^2 \\ \text{at } m = 0 \quad E = pc \end{array} \right\}$$

$x = (ct, \vec{r})$ $\quad k = (\frac{\omega}{c}, \vec{k})$ $\quad V = (\gamma c, \gamma \vec{v})$

$p = (\frac{E}{c}, \vec{p}) \underset{m \neq 0}{=} (\gamma m c, \gamma m \vec{v})$

Joke (!) version of all this:

1) From discussion of rel. doppler effect

$$\left(\frac{\omega}{c}, \vec{k}\right) = \text{4-vector for description of}$$

wave motion consistent w/ relativity

2) Nature described by quantum fields, modes in free field theory are plane waves (generalization of free particle) or soln of wave eqn)

deBroglie relation $\vec{p} = \hbar \vec{k}$

$$\Rightarrow E = \hbar \omega$$

$$\left(\frac{\hbar\omega}{c}, \hbar\vec{k}\right) = \text{4 vector} = \left(\frac{E}{c}, \vec{p}\right)$$

$$\frac{E^2}{c^2} - p^2 = \text{invariant}$$

call this invariant $(mc)^2$

and $\quad E^2 - p^2c^2 = (mc^2)^2$

Note E is not E_{kinetic} – this is an artificial distinction in relativistic context

Practical facts

$$\frac{E}{mc^2} = \gamma \quad \text{gives } \gamma$$

$$\frac{pc}{E} = \frac{v}{c} = \beta \quad \text{gives } \beta$$

$$E^2 - p^2c^2 = (mc^2)^2 \quad \text{relates } E \text{ \& } p$$

Practical problems typically about relating physics in different frames

2 approaches: 1) Lorentz transform 4-vectors $(E, c\vec{p})$

2) use invariants

Background for relativistic kinetics problems:

- some reaction involves incoming particles, outgoing particles (which might be different)
- Interaction dynamics presumed to be consistent w/ special rel. (we have not talked about that yet).
- Might be interested in moving from frame to frame (physical intuition simple in one frame, but experimental setup simpler in another frame
- Might be interested in frame-~~depend~~ independent statements extracted from measurements in one frame

Techniques involve either / both of

1) Lorentz transformation
2) use of invariants

L.T. needs γ, β. If particle of ~~m~~ mass M has 4-vector momentum $p = \left(\frac{E}{c}, \vec{p}\right)$, you know

1) $E^2 - (pc)^2 = (Mc^2)^2$
2) L.T. for going to rest frame where $p_r = (mc, 0)$ involves $\gamma = \frac{E}{mc^2}$, $\beta = \frac{cp}{E}$.

Invariants: product of 2 4-vectors the same in all frames

$$p_1 = (E_1, \vec{p}_1) \qquad p_2 = (E_2, \vec{p}_2) \text{ in some frame}$$

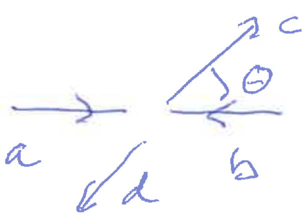
$$p_1 \cdot p_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 \text{ in that frame}$$

$$(p_1 + p_2)^2 = p_1^2 + 2 p_1 \cdot p_2 + p_2^2$$

$$= m_1^2 + 2 p_1 \cdot p_2 + m_2^2$$

and in all frames, even though E's, $\vec{p}$'s differ.

Example: CM to Fixed target $(c=1)$



equal mass ~~(equal mass means all E's the same~~

$$p_a' = (E', p', 0, 0)$$

$$p_b' = (E', -p', 0, 0)$$

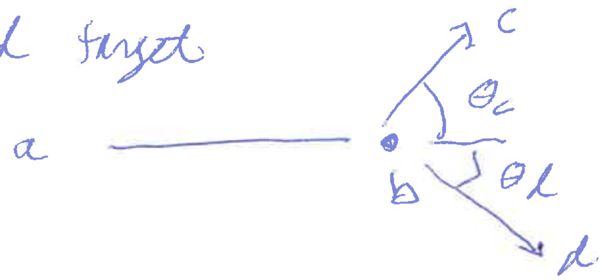
$$p_c' = (E', p' \cos\theta, p' \sin\theta, 0)$$

$$p_d' = (E', -p' \cos\theta, -p' \sin\theta, 0)$$

$$\sum \vec{p}_{in} = 0 \Rightarrow \sum \vec{p}_{out} = 0, \quad E'^2 - p'^2 = m^2$$

m the same $\Rightarrow$ all E' the same

Fixed target



$$p_a = (E, p, 0, 0)$$

$$p_b = (m, 0, 0, 0)$$

$$p_c = (E_c, p_c \cos\theta_c, p_c \sin\theta_c, 0)$$

$$p_d = (E_d, p_d \cos\theta_d, p_d \sin\theta_d, 0)$$

We know $p_i^2 = m_i^2$, $E_i^2 = p_i^2 + m_i^2$, $p_c \sin\theta_c = p_d \cos\theta_d$

etc.

Q1: how are lab E, p, E_c, p_c, θ_c related to CM E', p', θ ?

answer 1: boost left by $\gamma = \frac{E'}{m}$, $\beta = \frac{p'}{E'} = \sqrt{1 - \frac{m^2}{E'^2}}$

answer 2: use invariants!

$$(p_a \pm p_b)^2 = (p_a' \pm p_b')^2$$

ex. $(p_a + p_b)^2 = (p_a' + p_b')^2$

$$= p_a^2 + 2p_a \cdot p_b + p_b^2 = p_a'^2 + 2p_a' \cdot p_b' + p_b'^2$$

$$= m^2 + 2mE + m^2 = 2m^2 + 2[E'^2 + p'^2]$$

$$2m^2 + 2mE = 2[E'^2 + (E'^2 - m^2)] + 2m^2$$

$$2m^2 + 2mE = 4E'^2 \equiv (E_{cm})^2$$

or easier: $p_a' + p_b' = (2E', 0)$!

Angles not so easy, still true that

$$(p_a - p_c)^2 = (p_a' - p_c')^2$$

$$2m^2 - 2(EE_c - pp_c \cos\theta_c) = 2m^2 - 2(E'^2 - p'^2 \cos\theta)$$

$$(p_b - p_c)^2 = 2m^2 - 2mE_c = 2m^2 - 2(E'^2 + p'^2 \cos\theta)$$

$+ \; E'^2 - p'^2 = m^2$ --- messy

$-1 < \cos\theta < 1 \Rightarrow$ range of θ_c in lab?

RK4

Threshold for a reaction?

$$\underset{E,\ m}{\overset{a}{\longrightarrow}} \quad \underset{m \text{ at rest}}{\bullet} \quad \Longrightarrow \quad \begin{array}{l} M \\ M \end{array}$$

§, $M > m$, E_{min}?

In CM, before

$$\underset{p_a = (E, p)}{\longrightarrow} \quad \underset{p_b = (E, -p)}{\longleftarrow} \quad \Longrightarrow \quad \begin{array}{l} p_c = (M, \vec{0}) \\ p_d = (M, 0) \end{array}$$

$$(p_a' + p_b')^2 = (p_c' + p_d')^2 = 4M^2 = (p_a + p_b)^2$$

$$= m^2 + 2mE + m^2$$

Higgs discovery via decay mode $h \to \gamma\gamma$

At LHC, h <u>not</u> produced at rest

γ, $p_1 = (E_1, \vec{p}_1)$

$E_1 = |p_1|$

$h \Rightarrow$

p_h

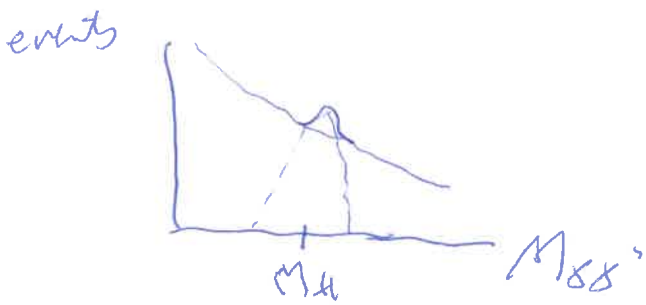
γ p_2 $\qquad p_2 = (E_2, p_2)$

but $\quad p_h = p_1 + p_2$

$$p_h^2 = (p_1 + p_2)^2$$

$$M_H^2 = 2(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2) \equiv \text{"invariant mass of } \gamma\gamma\text{"}$$

$M_{\gamma\gamma}^2$



Now need to develop technology to automatically deal with relativistic transformations -

Differential geometry for Special Relativity

Describe space + time in terms of 4-d space with coordinates

$$x^\mu = (x^0, x^1, x^2, x^3)$$

Imagine existance of transformation rule

$$x^\mu \to x^{\mu'}(x)$$

under which observables (typically functions of coordinates, momenta, fields) transform simply: $f(x) \to f'(x')$

Observables will be defined by Tensors - objects of rank k, typically depending on space-time point x

Rank 0 ≡ Scalars are unchanged $s' = s$

Rank 1: $x^{\mu'}$ = linear function of x^μ

$$x' = \gamma x + \beta\gamma t \quad ; \quad x = \gamma x' - \beta\gamma t'$$
$$t' = \gamma t + \beta\gamma x \quad ; \quad t' = \gamma t - \beta\gamma x'$$

$$x' = \frac{\partial x'}{\partial x} x + \frac{\partial x'}{\partial t} t = \gamma x + \beta\gamma t$$

or $$x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} x^\nu$$

Notation - sum repeated indices (more rules follow)
but note ~~indices~~ index placement

$$x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} x^{\nu} \equiv \underbrace{\Lambda^{\mu'}{}_{\nu}}_{\text{the Lorentz transf.}} x^{\nu}$$

upper in denom
lower in numerator

Rule: "contract" ≡ sum indices ~~Rule~~ ~~stuff~~ upper against lower.
grammar: only two ν's! (ν is a dummy index)

Vector defined to transform in same way

components $A^{\mu} = (A^0, A^1, A^2, A^3)$

$$A^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} A^{\nu} = \frac{\partial x^{\mu'}}{\partial x^{0}} A^{0} + \frac{\partial x^{\mu'}}{\partial x^{1}} A^{1} + \cdots = \Lambda^{\mu'}{}_{\nu} A^{\nu}$$

[Pause - index free notation ~~adds~~ adds ~~components~~ basis vector (like $\vec{V}$)

$$V = V^{\mu} \hat{e}_{(\mu)}$$

$$V = V^{\mu} \hat{e}_{(\mu)} = V^{\nu'} \hat{e}_{(\nu')} = \Lambda^{\nu'}{}_{\mu} V^{\mu} \hat{e}_{(\nu')}$$

can think of V as not changing (like passive rotation ... but eq's changes)

$$\hat{e}_{(\mu)} = \Lambda^{\nu'}{}_{\mu} \hat{e}_{(\nu')}$$
]

Also $\Lambda^{\mu'}{}_{\nu}$ = Lorentz trans (unprimed → primed)

$$V^{\mu} = \Lambda^{\mu}{}_{\nu'} V^{\nu'} = \Lambda^{\mu}{}_{\nu'} \left(\Lambda^{\nu'}{}_{\rho} V^{\rho} \right)$$

$$\Lambda^{\mu}{}_{\nu'} \Lambda^{\nu'}{}_{\rho} = \delta^{\mu}{}_{\rho}$$ {- Kroniker δ
- $\Lambda^{\nu'}{}_{\mu}$ is inverse LT}

Another one-index object: "covariant 4-vector"
"dual vector"
"one-form"
like $|\psi\rangle$ & $\langle\psi|$.

components B_μ defined so

$$B_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}} B_\nu = \Lambda^\nu{}_{\mu'} B_\nu$$

[point at $\Lambda^{\mu'}{}_\nu A^\nu$]

~~Think like a. Purpose~~

Think of it as a function of a vector, designed to build scalars.

Has a basis $B = B_\mu \hat{\theta}^{(\mu)}$

Can (and do) require $\hat{\theta}^{(\nu)} \hat{e}_{(\mu)} = \delta^\nu_\mu$

to relate V (vector) to V (dual vector)

Components of A^μ & A_μ will be related.

[like in QM - expansion coeffs]

Example of B_μ: Gradient

Chain rule $\frac{\partial \phi}{\partial x^{\alpha'}} = \frac{\partial x^\beta}{\partial x^{\alpha'}} \frac{\partial \phi}{\partial x^\beta}$ or $\frac{\partial}{\partial x^{\alpha'}} = \frac{\partial x^\beta}{\partial x^{\alpha'}} \frac{\partial}{\partial x^\beta}$

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$$

$$\partial_\alpha \phi = \frac{\partial \phi}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)\phi$$

Cartesian tensors : "rank" = $k+\ell$ or (k,ℓ)

$$T \equiv T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

defined by ~~transformation~~ independent transf. on each index

$$T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_\ell'} = \Lambda^{\mu_1'}{}_{\mu_1} \Lambda^{\mu_2'}{}_{\mu_2} \cdots \Lambda^{\mu_k'}{}_{\mu_k}$$

$$\times \Lambda^{\nu_1}{}_{\nu_1'} \Lambda^{\nu_2}{}_{\nu_2'} \Lambda^{\nu_\ell}{}_{\nu_\ell'} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

ex: $F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\alpha} \Lambda^{\nu'}{}_{\beta} F^{\alpha\beta}$ $(2,0)$

$G_{\mu'\nu'} = \Lambda^{\alpha}{}_{\mu'} \Lambda^{\beta}{}_{\nu'} G_{\alpha\beta}$ $(0,2)$

Contractions

$V \cdot A \equiv V^{\mu} A_{\mu}$ is a <u>scalar</u> $(= V_\mu A^\mu)$

$$\cancel{A^{\mu}{}_{\alpha'} \ldots}$$

$$V' \cdot A' = \left(\frac{\partial x^\nu}{\partial x^{\mu'}} V_\nu \right)\left(\frac{\partial x^{\mu'}}{\partial x^\alpha} A^\alpha \right)$$

restore sum $\quad \sum_{\mu'} \frac{\partial x^\nu}{\partial x^{\mu'}} \frac{\partial x^{\mu'}}{\partial x^\alpha} = \frac{dx^\nu}{dx^\alpha} = \delta^\nu{}_\alpha$ $\quad = V_\alpha A^\alpha$

$$V' \cdot A' = \left(\frac{dx^\nu}{dx^\alpha}\right) V_\nu A^\alpha = \delta^\nu{}_\alpha V_\nu A^\alpha = \underline{V \cdot A}$$

scalar

also $V^\mu = T^{\mu\nu} A_\nu$ is a vector or $(1,0)$ tensor

etc

These are general results for differential geometry. Now specialize to special relativity and rectangular coordinates

Define invariant interval

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$
$$\equiv g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$g_{\alpha\beta} \equiv$ metric tensor - note symmetric $g_{\alpha\beta} = g_{\beta\alpha}$

here $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$

Often called $\eta_{\alpha\beta}$ to distinguish from arbitrary metric like 3-d spherical coords $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$

$g_{rr} = 1$ $g_{\theta\theta} = r^2$ $g_{\varphi\varphi} = r^2 \sin^2\theta$

$g_{\mu\nu}$ is (0,2) tensor (special)

$$g_{\mu\nu} V^\mu W^\nu = \text{scalar}$$

Raising & lowering: Define $V_\mu = g_{\mu\nu} V^\nu$ to make 1-form from vector

Inverse metric $g^{\nu\lambda}$ defined so $g_{\mu\nu} g^{\nu\alpha} = \delta_\mu{}^\alpha$

~~$g^{\mu\nu} \equiv$~~ For SR, in rectangular coords, $g^{\alpha\beta} = g_{\alpha\beta}$ (not so in any other case!)

repeat

$g^{\alpha\beta}$ defined by $g_{\mu\nu} g^{\nu\alpha} = \delta_\mu{}^\alpha$ = Kronecker δ
(= 1 if $\mu = \alpha$, 0 if $\mu \neq \alpha$)

For us, $g^{\alpha\beta} = g_{\alpha\beta}$

Metric tensor converts upper to lower indices

$$x_\mu = g_{\mu\nu} x^\nu$$

$$A^\mu = g^{\mu\nu} A_\nu$$

$$F_{---}^{---\alpha} = g^{\alpha\beta} F_{---\beta}^{---}$$

Thus - components: if a vector has components

$$A^\mu = (A^0, A^1, A^2, A^3)$$

its covariant partner is $A_\mu = (A^0, -A^1, -A^2, -A^3) = g_{\mu\nu} A^\nu$

$$A \cdot B = A_\mu B^\mu = A^0 B^0 - \vec{A} \cdot \vec{B}$$

Derivatives: $\dfrac{\partial}{\partial x'^\alpha} = \dfrac{\partial x^\beta}{\partial x'^\alpha} \dfrac{\partial}{\partial x^\beta}$ is covariant
wrt x^α

$$\equiv \partial_\alpha \qquad = \Lambda_\alpha{}^\beta \partial_\beta$$

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, + \vec{\nabla} \right)$$

$$\partial^\alpha = \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, - \vec{\nabla} \right)$$

$$\partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A}$$ is invariant

Laplacian $\partial_\alpha \partial^\alpha = \dfrac{\partial^2}{\partial x^{0\,2}} - \vec{\nabla}^2$ also invariant
$\equiv \Box$, sometimes

Matrix representation of Lorentz group

If all you have are vectors and rank-2 tensors, you can replace Greek letter geometry w/ matrices.

[Careful: $x^{\nu} = F^{\mu\nu} b_{\mu}$ - order doesn't matter
$x_i = F_{ij} c_j$ order does!

$x^{\mu} \rightarrow$ column vector $\underline{x} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$

Metric tensor $g_{\alpha\beta} \rightarrow$ matrix $\underline{\underline{g}}$ $\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$

$y_{\mu} = g_{\mu\nu} x^{\nu}$ or $y = g \cdot x$

$A_{\mu} B^{\nu} = A^{\mu} B^{\nu} g_{\mu\nu} \rightarrow \underline{A}^T \cdot \underline{\underline{g}} \cdot B$

or $(a \cdot b) = (a, gb) = (ga, b) = a^T g b$

useful, but be careful w/ indices!

Physics question: what's the most general set of transformations preserving the norm?

$x' = Ax$ or $x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}$

$(x', gx') = (x, gx)$

$(x^T \Lambda^T) g (\Lambda x) = x^T g x$

or $\Lambda^T g \Lambda = 1$

$g_{\mu\nu} x^{\mu'} x^{\nu'} = g_{\rho\sigma} x^{\rho} x^{\sigma}$

$g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} x^{\rho} x^{\sigma} = x^{\rho} x^{\sigma}$

$g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$

Classify as "proper": Λ has infinitesimal limit, improper - no

$\det \Lambda^T g \Lambda = \det g (\det \Lambda)^2 = \det g$ or

$\det \Lambda = \pm 1$. Proper L.T's are $\Lambda = 1 + \epsilon + \cdots$ or $\det \Lambda = 1$.

example: pure boost $\Lambda = \begin{bmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{bmatrix}$

(as $v \to 0$, $\gamma = 1 +$ tiny
$\beta\gamma =$ tiny)

Improper ones include spatial inversions $\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$

but this is not a unique description: space inversion AND time reversal has $\Lambda = -\mathbb{1}$, $\det \Lambda = 1$ - only an annoyance

Let's focus on the proper ~~ones~~ ones

$$\Lambda = 1 + \epsilon \quad \text{or} \quad \Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu$$

and plug in to $O(\epsilon)$

$$g_{\mu\nu}\left[\delta^\mu{}_\rho + \epsilon^\mu{}_\rho\right]\left[\delta^\nu{}_\sigma + \epsilon^\nu{}_\sigma\right] \stackrel{?}{=} g_{\rho\sigma}$$

$$= g_{\rho\sigma} + \underbrace{g_{\mu\nu}\epsilon^\mu{}_\rho \delta^\nu{}_\sigma}_{g_{\mu\sigma}\epsilon^\mu{}_\rho} + \underbrace{g_{\mu\nu}\delta^\mu{}_\rho \epsilon^\nu{}_\sigma}_{g_{\rho\nu}\epsilon^\nu{}_\sigma}$$

$$= g_{\rho\sigma} + \epsilon_{\sigma\rho} + \epsilon_{\rho\sigma} = g_{\rho\sigma}$$

$\therefore$ Need $\epsilon_{\sigma\rho} = -\epsilon_{\rho\sigma}$: ϵ is antisymmetric

$\Longrightarrow$ ϵ has 6 components (3 for boosts, 3 for rotations!)

What is the most general $\Lambda^{\mu}{}_{\nu}$ ~~[illegible]~~ for which

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

and

$$x'_{\mu} x'^{\mu} = x_{\mu} x^{\mu} ?$$

Consider infinitesimal $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon^{\mu}{}_{\nu}$, $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}$$

$$\delta x^{\mu} = \epsilon^{\mu}{}_{\nu} x^{\nu} \equiv \frac{i}{2} \epsilon^{\rho\sigma} L_{\rho\sigma} x^{\mu}$$

$$L_{\mu\nu} = i \left[x_{\mu} \partial_{\nu} - x_{\nu} \partial_{\mu} \right]$$

$$\partial_{\mu} = \left[\frac{\partial}{\partial t}, \vec{\nabla} \right] , \quad L_{\mu\nu} = -L_{\nu\mu} \Rightarrow 6\ L\text{'s}$$

Jackson's discussion of transformations is really specific to vector quantities. It's possible to do this in greater generality. To be fair, all we need ~~is the~~ to ~~stay~~ do classical E&M is the story for vectors, and we basically have that with the Lorentz transformations, but lets be more general. We will learn some interesting things about relativistic field theory!

Begin with our transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = [\delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu}] x^{\nu}$$

$$= x^{\mu} + \delta x^{\mu} \qquad \delta x^{\mu} = \epsilon^{\mu}{}_{\nu} x^{\nu} = \underline{\epsilon^{\mu\lambda} x_{\lambda}}$$

keeps

(Recall, $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ to preserve $x^{\mu} x_{\mu}$.)

Write $\qquad \delta x^{\mu} = \epsilon^{\mu}{}_{\nu} x^{\nu}$

$$= \frac{1}{2} \epsilon^{\rho\sigma} L_{\rho\sigma} x^{\mu}$$

where $L_{\mu\nu} = i[x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}]$ is a sort of generalized angular momentum. $\partial_{\mu} = \left[\frac{\partial}{\partial t}, \vec{\nabla}\right]$

(note $L_{\nu\mu} = -L_{\mu\nu}$ - there are six L's)

Recall $\epsilon^{\rho\sigma}$ is a parameter. This is like with,

$$\delta \vec{x} = i(\vec{\theta} \cdot \vec{L}) \cdot \vec{x}$$

for a rotation.

Check this improbable result

$$\delta x^{\mu} = \frac{1}{2} \epsilon^{\rho\sigma} L_{\rho\sigma} x^{\mu} = \frac{1}{2}(i)\epsilon^{\rho\sigma}[x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}] x^{\mu}$$

$$\partial_{\sigma} x^{\mu} = \delta^{\mu}_{\sigma}, \quad \partial_{\rho} x^{\mu} = \delta^{\mu}_{\rho}$$

$$\delta x^{\mu} = -\frac{1}{2} \epsilon^{\rho\sigma}[x_{\rho}\delta^{\mu}_{\sigma} - x_{\sigma}\delta^{\mu}_{\rho}]$$

$$= -\frac{1}{2}[\epsilon^{\rho\mu} x_{\rho} - \epsilon^{\mu\sigma} x_{\sigma}] = \epsilon^{\mu\nu} x_{\nu}$$ — flip indices on first term.

Now the point is, ϵ is a number, $L_{\mu\nu}$ is an operator. What is its algebra? It is easy though tedious to grind out the commutator - it is

$$[L_{\mu\nu}, L_{\rho\sigma}] = i g_{\nu\rho} L_{\mu\sigma} - i g_{\mu\rho} L_{\nu\sigma}$$

$$- i g_{\nu\sigma} L_{\mu\rho} + i g_{\mu\sigma} L_{\nu\rho} \qquad (*)$$

The L's are the generators of a Lie algebra. Actually, some of the entries are familiar. Look at μ, ν spacelike, define $L_i = \frac{1}{2} \epsilon_{ijk} L_{jk}$

$$[\text{ex } L_1 = \tfrac{1}{2}(L_{23} - L_{32}) = L_{23} - L \text{ is antisymmetric}$$

$$\cancel{[L_1, L_2]} = [L_{23}, L_{31}] \underset{= [L_1, L_2]}{=} i g_{33} L_{21} = -i L_{21} = i L_{12} = i L_3$$

$$(\mu, \nu, \rho, \sigma = 2331)$$

which is the usual angular momentum commutator.

L doesn't know about spin, but (in complete analogy to orbital vs spin angular momentum) we can imagine that we have states which are characterized by a set of internal labels, ~~spin~~ affected by operators with the same ~~~~ commutation relations as $*$, call them $S_{\mu\nu}$, and

$$[L_{\mu\nu}, S_{\mu\nu}] = 0$$

Then the most general representation of the generator is

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$$

and we have a generalized infinitesimal rotation matrix $D(\epsilon) = 1 + \frac{i}{2} \epsilon^{\mu\nu} M_{\mu\nu}$

$L_{\mu\nu} = i x_\mu \partial_\nu - x_\nu \partial_\mu$; $M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$ GG L

$[M_{\mu\nu}, M_{\rho\sigma}] = \dots$

Now for the miracle. Define

$J_i \equiv \frac{1}{2}\epsilon_{ijk} M_{jk}$ 3 J's

$K_i = M_{0i}$ 3 K's

and two linear combinations

$A_i = \frac{1}{2}[J_i + iK_i]$

$B_i = \frac{1}{2}[J_i - iK_i]$

discover

$[A_i, B_j] = 0$

$[A_i, A_j] = i\epsilon_{ijk} A_k$

$[B_i, B_j] = i\epsilon_{ijk} B_k$

$[J_i, J_j] = i\epsilon_{ijk} J_k$

$[J_i, K_j] = i\epsilon_{ijk} K_k$

$[K_i, K_j] = -i\epsilon_{ijk} J_k$

J's generate rotations
K's generate boosts

A & B obey the algebra of SU(2).

Recall how eigenstates of angular momentum behave - states are labeled by a J which tell us its transformation properties under rotations - in fact, there are 2J+1 states, labelled by m, $-J \le m \le J$, which mix under rotations.

For the Lorentz group there are 2 such indices. States are labelled by pairs of integers a & b

collections of dynamical variables, cmbo, ... fields

$$A^2 |\psi_{AB}\rangle = a(a+1)|\psi_{AB}\rangle$$

$$B^2 |\psi_{AB}\rangle = b(b+1)|\psi_{AB}\rangle$$

$a, b = 0, \frac{1}{2}, 1 \dots$

Interesting - and useful - this is where "intrinsic spin of particle" comes from.

since $\vec{J} = \vec{A} + \vec{B}$, usual spin of rep is $a+b$

$L_{\mu\nu} = i[x_\mu \partial_\nu - x_\nu \partial_\mu]$, $L_{0i} \to K$ GϕD
$L_{ij} \to J$

Note under parity - J is an axial vector, K is a vector

$J_i \to J_i$

$K_i \to -K_i$

this means - reflection is equivalent to exchange A & B

∘ Irreducible representations of Lorentz groups are <u>not</u> necessarily parity eigenstates (unless A = B)

Example - left handed neutrino - A = 1/2 B = 0
(σ·p = -s)

~~Since 2J = 2A & B & usual spin of~~ ~~J~~ = a + b

* States conveniently labelled

$\begin{pmatrix} - \\ - \\ - \\ \hline - \\ - \\ - \end{pmatrix}$ ← 2a+1 entries for A quantum #'s
← 2b+1 entries for B

Spin-$\frac{1}{2}$ $a = 1/2$ $b = 0$ or $a = 0$ $b = \frac{1}{2}$ $\left(\begin{matrix}2\\ \text{comps}\end{matrix}\right)$

<u>not</u> parity eigenstate. Dirac particle <u>is</u> parity eigenstate $(a,b) = (\frac{1}{2}, 0) + (0, \frac{1}{2})$ direct product

$\begin{pmatrix} 1 \\ 2 \\ \hline 3 \\ 4 \end{pmatrix}$ 4-component spinor

Spin-0   (0,0) is overkill - just a one-component state, $\Phi(\Lambda x) = \Phi(x)$

How do states transform?

$$D(\epsilon) = 1 + \frac{i}{2} \epsilon^{\mu\nu} M_{\mu\nu} = 1 + i(\vec{\theta}_A \cdot \vec{A}) + i(\vec{\theta}_B \cdot \vec{B})$$

$$\longrightarrow \left[\begin{array}{c|c} e^{iA\cdot\theta_A} & 0 \\ \hline 0 & e^{iB\cdot\theta_B} \end{array}\right]$$

easy to transform in (A,B) basis.

Also $\vec{A} = \frac{1}{2}(\vec{J} + i\vec{K})$ $\vec{B} = \frac{1}{2}(\vec{J} - i\vec{K})$

$$D = 1 + \frac{i}{2}\vec{J}\cdot(\vec{\theta}_A + \vec{\theta}_B) - \frac{\vec{K}}{2}\cdot(\vec{\theta}_A - \vec{\theta}_B)$$

$$= 1 + i J\cdot\vec{\omega} - \vec{K}\cdot\vec{\zeta}$$

3 ω's, 3 ζ's

and we are back to Jackson's notation

<u>Vector fields</u> are usually treated more simply.
The analogy is like cartesian vectors $\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$

vs spherical vectors $\begin{pmatrix} A_+ = \frac{A_x + iA_y}{\sqrt{2}} \\ A_0 = A_z \\ A_- = \frac{A_x - iA_y}{\sqrt{2}} \end{pmatrix}$

Case of pure boost, $\vec{\omega} = 0$, $\vec{\zeta} \neq 0$ is interesting.

$\delta x^\mu = \epsilon^{\mu\nu} x_\nu$ is easiest starting point - let only $0,1$ dir be non zero

$\delta x^0 = \epsilon^{01} x_1 = -\epsilon^{10} x_1 = +\epsilon^{10} x^1$

$\delta x^1 = \epsilon^{10} x_0 = \epsilon^{10} x^0$

i.e. $\delta\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \epsilon^{10}\begin{pmatrix} x^1 \\ x^0 \end{pmatrix} = \epsilon^{10}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$

call $\epsilon^{10} \equiv \delta\zeta)$

infinitesimal $D(\delta\zeta) = 1 + \zeta\sigma_x \xrightarrow{\text{finite}} \exp(\zeta\sigma_x)$

$$= \begin{bmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{bmatrix}$$

or $V' = \begin{bmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} V$

for boost along x — as we found at the start.

Finally: $[J_i, J_j] = i\epsilon_{ijk} J_k$ —

rotations don't commute
difference = rotation

$[J_i, K_j] = i\epsilon_{ijk} K_k$

rotation + boost don't commute
difference is a boost

$[K_i, K_j] = -i\epsilon_{ijk} J_k$

non collinear boosts don't commute —

there is a rotation.

Refs: Ryder p. 38-43
Peskin - Schroeder Sec 3.1
Srednicki p. 15-21
Zee p. 114-117

Thomas Precession

Recall electron magnetic moment – a la Goudsmit & Uhlenbeck 1925

$$\vec{\mu} = \frac{ge}{2mc}\vec{S}$$

$$g = 2$$
$$S_z = \pm \hbar/2$$

$g=2$ explains anomalous Zeeman effect

$$H = \frac{e}{2mc}(\vec{L} + \underbrace{2}_{g=2}\vec{S}) \cdot B$$

but Fine structure too big by $\times 2$

Twenty years later, Einstein heard something about the Lorentz group that greatly surprised him. It happened while he was in Leiden. In October 1925 George Eugene Uhlenbeck and Samuel Goudsmit had discovered the spin of the electron [U1] and thereby explained the occurrence of the alkali doublets, but for a brief period it appeared that the magnitude of the doublet splitting did not come out correctly. Then Llewellyn Thomas supplied the missing factor, 2, now known as the Thomas factor [T1]. Uhlenbeck told me that he did not understand a word of Thomas's work when it first came out. 'I remember that, when I first heard about it, it seemed unbelievable that a relativistic effect could give a factor of 2 instead of something of order v/c. . . . Even the cognoscenti of the relativity theory (Einstein included!) were quite surprised' [U2]. At the heart of the Thomas precession lies the fact that a Lorentz transformation with velocity $\vec{v}_1$ followed by a second one with a velocity $\vec{v}_2$ in a different direction does not lead to the same inertial frame as one single Lorentz transformation with the velocity $\vec{v}_1 + \vec{v}_2$ [K1]. (It took Pauli a few weeks before he grasped Thomas's point.*)

Pais, "Subtle is the Lord"

Recall derivation of fine structure (1): electron moves with velocity $\vec{v}$ in external fields $\vec{E}$ & $\vec{B}$

In NR limit moving e^- sees $B' \sim \left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}\right)$ – we'll derive this later

Cast of characters (somewhat redundant)

Anomalous Zeeman effect: in external magnetic field

$$H = -\vec{\mu}\cdot\vec{B} \quad , \quad \vec{\mu} = \vec{\mu}_L + \vec{\mu}_S$$

$$\mu_L = \frac{-e}{2mc}\vec{L} \qquad \mu_S = \frac{-ge}{2mc}\vec{S} \quad ; \; g = 2$$

$$H = \frac{e}{2mc}\left[\vec{L} + 2\vec{S}\right]\cdot\vec{B}$$

Fine structure: electron moves w/ velocity v in external E-field. In NR limit (we'll derive this later)

$$\vec{B}' \sim \left(\vec{B} - \frac{\vec{v}}{c}\times\vec{E}\right)$$

$$\left.\frac{d\vec{S}}{dt}\right)_{\text{rest frame of } e^-} = \vec{\mu}_S \times \vec{B}'$$

or $$U' = -\vec{\mu}_S\cdot\vec{B}' \quad \cancel{\text{for}}$$

$$V = -\frac{Ze^2}{r} \; , \quad e\vec{E} = -\frac{\vec{r}}{r}\frac{\partial V}{\partial r}$$

$$B' = B + \frac{\vec{v}\times\vec{r}}{e\,c\,r}\frac{\partial V}{\partial r} \quad ; \quad L = \vec{r}\times m\vec{v}$$

$$B' = \vec{B} - \frac{1}{emc}\frac{1}{r}\frac{\partial V}{\partial r}\vec{L}$$

$$U' = \frac{ge}{2mc}\vec{S}\cdot\vec{B} + \underbrace{\frac{g}{2m^2c^2}\frac{1}{r}\frac{\partial V}{\partial r}}_{\frac{g}{2}\frac{Ze^2}{m^2c^2}\frac{1}{r^3}}\vec{L}\cdot\vec{S}$$

$$\frac{g}{2}\frac{Ze^2}{m^2c^2}\frac{1}{r^3}\vec{L}\cdot\vec{S}$$

Issue: $L\cdot S$ 2x too big.

The source of the discrepancy is that the electron's rest frame rotates with time w/ frequency $\vec{\omega}_T$

Classical $\left.\frac{dG}{dt}\right)_{\text{nonrotating}} = \left.\frac{dG}{dt}\right)_{\text{rot}} + \vec{\omega}_T \times \vec{G}$

$$\left.\frac{d\vec{s}'}{dt}\right)_{nr} = \vec{s}' \times \left[\frac{ge\vec{B}'}{2mc} - \vec{\omega}_T\right] \quad (s \times \omega = -\omega \times \vec{s})$$

$$U = U' + \vec{s} \cdot \vec{\omega}_T$$

Where does this rotation come from?

Suppose at time t, velocity of e^- rest frame wrt lab is

$$\vec{V}(t) = c\vec{\beta} \Rightarrow x' \text{ (} e^- \text{ rest frame)}$$
$$= A_{boost}(\beta)\, x \text{ (lab)}$$

boost carries lab coord to e^- rest frame coord

At a later time

$$\vec{V}(t+\delta t) = c(\vec{\beta}+\delta\vec{\beta}) \Rightarrow \vec{x}'' = A_{boost}(\beta+\delta\beta)\,\vec{x}$$

These A's are pure boosts. But, what is relation of 2 points on electron's orbit?

$$x'' = A_T\, x' \quad ?$$

$$A_T = A_{boost}(\vec{\beta}+\delta\vec{\beta})\, A^{-1}_{boost}(\vec{\beta})$$

$$= A_{boost}(\vec{\beta}+\delta\beta)\, A_{boost}(-\vec{\beta})$$

We'll find that the product is a pure boost and a pure rotation: $A(\vec{\beta}+\delta\vec{\beta})A(-\vec{\beta}) = R(\Delta\Omega)A(\Delta\beta)$

pure boost gives $B' = B - \frac{v}{c}\times E$.

Rotation contaminates the NR equation of motion — what we really want to think about is the set of rest frame coordinates at time $t+\delta t$, found by a boost alone

$$x''' = A_{boost}(\Delta\beta)x' = R(-\Delta\Omega)A_b(\beta+\Delta\beta)\cdot x$$

If we can find $\Delta\Omega$ we can get ω_T:

Proper time precessional rate is $\frac{dG}{d\tau} = \vec{\omega}_T \times G$

$$\omega_T = -\frac{\Delta\Omega}{\delta t}, \quad \frac{dG}{dt} = \frac{1}{\gamma}\frac{dG}{d\tau}$$

and $U' = -\vec{S}' \cdot \left[\frac{geB'}{2mc} - \frac{\vec{\omega}_T}{\gamma}\right]$; $\gamma = 1$ for NR

A tedious way to ~~get~~ proceed is to follow Jackson, multiplying two 4×4 matrices. A quicker way to proceed is to look at the product of two infinitesimal boosts (this only works for NR motion)

$$A\cdot A' = \exp\left[-\vec{\zeta}(\beta+\delta\beta)\cdot\vec{K}\right]\exp\left[\zeta(\beta)\cdot K\right]$$

and compare it to a single boost without rotation

(is A_T) $A'' = \exp\left[\left(-\vec{\zeta}(\beta+\delta\beta)+\zeta(\beta)\right)\cdot\vec{K}\right]$

K = 4x4 matrix generator of L.T.

Call $\vec{\zeta} = \vec{\zeta}(\beta+\delta\beta)$, $\vec{\zeta}\,' = \vec{\zeta}(\beta)$

For small ζ, ζ', expand

$$A\cdot A' = \left[1 - \zeta\cdot K + \frac{1}{2}(\zeta\cdot K)^2 + \cdots\right]\left[1 + \zeta'\cdot K + \frac{1}{2}(\zeta'\cdot K)^2 + \cdots\right]$$

$$= 1 + (\zeta'-\zeta)\cdot K + K_i K_j \left(\frac{1}{2}\zeta_i\zeta_j - \zeta_i\zeta'_j + \frac{1}{2}\zeta'_i\zeta'_j\right)$$

The pure boost is

$$A'' = 1 + (\zeta'-\zeta)\cdot K + \frac{1}{2}K_iK_j(\zeta'-\zeta)_i(\zeta'-\zeta)_j$$

$$A''-AA' = \frac{1}{2}K_iK_j\{\zeta'_i\zeta'_j - \zeta'_i\zeta_j - \zeta_i\zeta'_j + \zeta_i\zeta_j - \zeta'_i\zeta'_j + 2\zeta_i\zeta'_j - \zeta_i\zeta_j\}$$

$$= \frac{1}{2}K_iK_j[\zeta_i\zeta'_j - \zeta'_i\zeta_j]$$

$$= \frac{1}{2}\zeta_i\zeta'_j[K_iK_j - K_jK_i] \quad \text{flipping a dummy index}$$

But $[K_i, K_j] = -i\epsilon_{ijk}S_k$ so $A''-AA' = -\frac{i}{2}S\times\cdots$

$$A\cdot A' \simeq A''\left(1 + \frac{i}{2}\vec{S}\cdot(\vec{\zeta}\times\vec{\zeta}\,')\right)$$

This is a rotation by $\Delta\vec{\Omega} = \frac{1}{2}\vec{\zeta}\times\vec{\zeta}\,'$

Now for small β, $\vec{\zeta}\,' = \hat{\beta}\tanh\beta \simeq \vec{\beta}$

$\vec{\zeta} \simeq \vec{\beta} + \delta\vec{\beta}$

$\zeta\times\zeta' = \delta\beta\times\beta$

$$\Delta\Omega = -\frac{1}{2}\vec{\beta}\times\delta\vec{\beta}$$

The $\frac{1}{2}$ is the NR limit of $\frac{\gamma^2}{\gamma+1}$.

So $$\omega_T = -\lim_{\delta t \to 0} \left(\frac{1}{2} \frac{\vec{v}}{c} \times \frac{\delta \vec{v}}{c\,\delta t} \right) = \frac{1}{2} \frac{\vec{a} \times \vec{v}}{c^2}$$

$$\vec{a} = \text{acceleration} = \frac{\vec{F}}{m} = -\frac{\vec{r}}{mr} \frac{\partial V}{\partial r}$$

$$\omega_T = -\frac{1}{2mc^2} \vec{r} \times \vec{v} \frac{1}{r} \frac{\partial V}{\partial r} = -\frac{1}{2m^2c^2} \vec{L} \frac{1}{r} \frac{\partial V}{\partial r}$$

$$U = \frac{-ge}{2mc} \vec{S} \cdot \vec{B} + \frac{(g-1)}{2mc^2} \vec{L} \cdot \vec{S} \frac{1}{r} \frac{\partial V}{\partial r}$$

and $g = 2$: $2 - 1 = 1$, this is the $\frac{1}{2}$ of Thomas

Comments:

1) For more sophisticated treatment see discussion of "BMT" equation in Jackson

$$\frac{d}{dt} \left[\hat{\beta} \cdot \vec{S} \right] \propto \frac{g-2}{2} \vec{S} \cdot (\vec{\beta} \times \vec{B})$$

precession of longitudinal polarization can be used to measure $g-2$, or calibrate $\vec{B}$ using $g-2$.

2) In electromagnetism, Φ = 4th component of 4 vector. For other Φ's (a scalar) - can only get Thomas term, $U \sim -\frac{1}{2mc^2} L \cdot S \frac{\partial V}{r \partial r}$ - inverted multiplets

3) Very easy to pop Thomas $\frac{1}{2}$ out of NR reduction of Dirac equation