Diffraction

$\equiv$ scattering from a hole in an opaque or conducting screen. A long confusing story ending in simple approximate formulas.

Start with a scalar field first, for simplicity
$\psi(x,t) = \psi(x)\, e^{-i\omega t}$, a solution of Helmholtz eqn.

$$(\nabla^2 + k^2)\psi = 0 \qquad (\omega = ck)$$

Introduce the Green's fn $(\nabla^2 + k^2)\, G(x,x';k) = -\delta^3(x-x')$

recall Green's theorem ($\phi = G$)

$$\int_V \left[G(x,x')\, \nabla'^2 \psi(x') - \nabla'^2 G(x,x')\, \psi(x') \right] d^3x'$$
$$= \int_S \left[G(x,x')\, n' \cdot \nabla' \psi(x') - \psi(x')\, n' \cdot \nabla' G(x,x') \right] dA'$$

$\nabla'^2 \psi = -k^2 \psi$, $\quad \nabla'^2 G = -k^2 G + \delta^3(x-x')$.
k^2 terms cancel. Integrate over δ-fn

$$* \quad \psi(x) = \int_S dA' \left[G(x,x')\, \hat{n}' \cdot \vec{\nabla}' \psi(x') - \psi(x')\, n' \cdot \nabla' G \right]$$

if $x \in V$, $\psi = 0$ otherwise

S' = closed surface, V is inside, ~~$\hat{n}'$~~

$\hat{n}'$ = outward normal to S'

Jackson 10.75 defines $\hat{n}'$ pointing in so sign flip on $*$.

Note this is not a solution – it's an integral eqn. Also, we started with a wave eqn – wave eqn's dont have solutions when we specify both ψ and $\frac{\partial \psi}{\partial n}$ on the boundary. (This is easy to fix; impose Dirichlet or Neumann b.c's – we'll come back to this.)

Now explicitly $G(x, x') = \frac{1}{4\pi} \frac{e^{ikR}}{R}$ & $\vec{R} = \vec{x} - \vec{x}'$

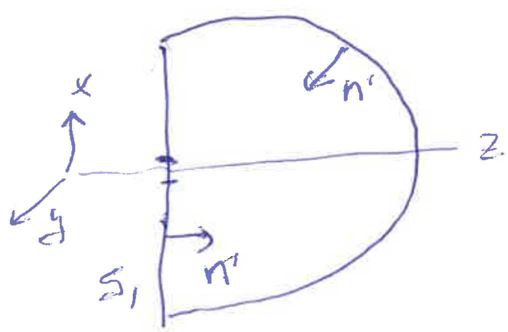
$$\vec{\nabla}' \frac{e^{ikR}}{R} = \left(ik - \cancel{\frac{1}{R}} \right) \frac{e^{ikR}}{R} \vec{\nabla}' R \quad \text{neglect } \frac{1}{R}$$

$$\vec{\nabla}' R = \nabla' \sqrt{(\vec{x} - \vec{x}')^2} = - \frac{\vec{R}}{R}. \quad \text{So}$$

$$\psi(x) = -\frac{1}{4\pi} \int_S dA' \; \hat{n}' \cdot \left[\vec{\nabla}' \psi(x') + \frac{\vec{R}}{R} \left(ik - \cancel{\frac{1}{R}} \right) \psi(x') \right]$$

– sign – inner normal for $\hat{n}'$ & $\frac{\exp(ikR)}{R}$

Now specialize to case where S is an infinite plane screen S_1 (with an aperture) plus an infinite hemisphere S_2. On S_2, $\psi \sim \frac{e^{ikr'}}{r'}$,



$\vec{\nabla}' \psi \sim ik \frac{e^{ikr}}{r}$, $\frac{\vec{R} \cdot \hat{n}'}{R} = -1$

$$\int_{S_2'} = \int \hat{n}' \cdot [\underbrace{ik}_{\nabla'} - \underbrace{ik}_{\frac{R}{R} \text{ term}}] \frac{e^{ikr'}}{r'} dA' = 0$$

Kirchhoff integral formula

$$\psi(x) = -\frac{1}{4\pi}\int_{S_1'} dA' \frac{e^{ikR}}{R}\,\hat{n}'\cdot\left[\nabla'\psi(x') + \frac{\vec{R}}{R}\psi(x')\right]$$

$$= -\frac{1}{4\pi}\int_{S_1'} dA'\Big[\hat{n}'\cdot\vec{\nabla}'\psi\, \mathcal{G}(x,x')$$
$$+ \psi(x')\,\hat{n}'\cdot\vec{\nabla}'\mathcal{G}(x,x')\Big]$$

(putting the Greens fn back on -)

Desire - specify information on S_1', compute radiation far away (i.e. specify ψ on aperture)

Issues: 1) math consistency
2) practical calculations
3) scalar → vector

1) Consistency - need Dirichlet or Neumann b.c.

either $\mathcal{G} = \mathcal{G}_D$ with $\mathcal{G}_D = 0$ on S_1

or

$\mathcal{G} = \mathcal{G}_N$ with $\frac{\partial \mathcal{G}_N}{\partial n'} = 0$ on S_1'

trick - S_1 is a flat plane at $z=0$ - use images

$$\mathcal{G}_D(x,x') = \frac{1}{4\pi}\left[\frac{e^{ikR}}{R} - \frac{e^{ikR''}}{R''}\right]$$

$R = |x - x'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$

$R'' = |x - x''| = \sqrt{(x-x'')^2 + (y-y'')^2 + (z-z'')^2}$

$\vec{x}'' = \hat{i}x' + \hat{j}y' - \hat{k}z'$ - mirror image of x'

$$\mathcal{G}_N = \frac{1}{4\pi}\left[\quad '' \quad + \quad '' \quad\right]$$

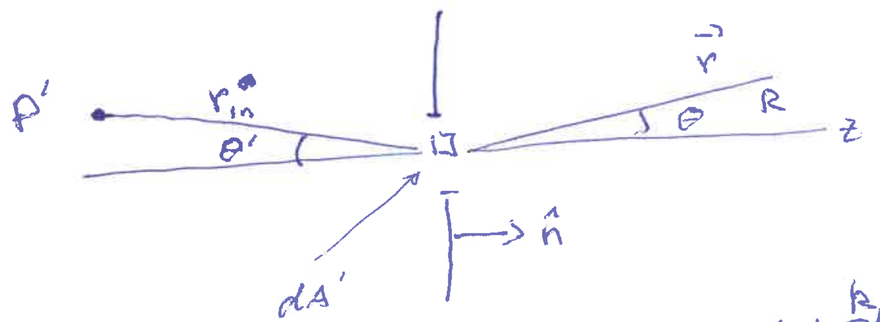
and evaluate on S' - at $z'=0$ - so $R = R''$

so $$\psi_D(x) = \frac{1}{4\pi}\int_{S_1'} dA'\, \psi(x')\; 2\,\frac{\hat{z}\cdot\vec{R}}{R}\,\frac{e^{ikR}}{R}$$

$$\psi_N(x) = \frac{1}{4\pi}\int_{S_1'} dA'\, \frac{e^{ikR}}{R}\; 2\,n'\cdot\nabla'\psi(x')$$

Kirchhoff - average ~~of~~ the two (inconsistent)

Not much difference, practically - consider a source at ~~p~~ P' at $z=0$, observe at large R



on the aperture $\psi_0(r') = \dfrac{e^{ik r_{in}}}{r_{in}}$ - spherical wave;

$r'(r_{in}) \simeq$ ~~distance~~ $r_{in} - \hat{n}'\cdot r'$ (r' on the aperture)

$$n'\cdot\nabla'\psi_0(r') = \hat{z}\cdot\nabla'\frac{e^{ikr_{in}}}{r_{in}} \simeq ik\,\hat{z}\cdot\hat{r}_{in}\,\frac{e^{ikr_{in}}}{r_{in}}$$

$$= ik\,\frac{e^{ikr_{in}}}{r_{in}}\cos\theta'$$

$$\hat{z}\cdot\vec{R}/R = \cos\theta$$

D: $$\psi(r) = \frac{k}{2\pi i}\int_{S_1'} \frac{e^{ikr}}{r}\cos\theta\,\frac{e^{ikr_{in}}}{r_{in}}\,dA'$$

N $$\frac{k}{2\pi i}\int \frac{e^{ikr}}{r}\cos\theta'\,\frac{e^{ikr_{in}}}{r_{in}}\,dA'$$

K $$\frac{k}{2\pi i}\int \frac{e^{ikr}}{r}\,\frac{\cos\theta+\cos\theta'}{2}\,\frac{e^{ikr_{in}}}{r_{in}}\,dA'$$

Huygens (even more inconsistent) 1

The physics point is that if the aperture is reasonably large, $\frac{\Delta x}{\lambda} \gg 1$, and the incident beam is reasonably normally incident, the diffraction pattern will peak forward: $\cos\theta \sim 1$, $\cos\theta' \sim 1$. ~~Also the~~ The integral is dominated by the $e^{ikr}\, e^{ikr_{in}}$ term.

The discussion in the optics literature is a bit more general / complicated than in Jackson. We need $\psi(x')$ in the aperture. It comes from a source on the far side of the opening.

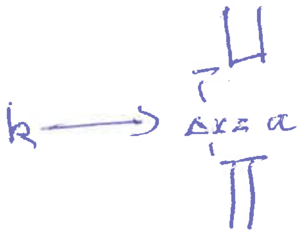
$$*. \quad \psi(x') = t(x,x_0)\,\psi_{source}(x_0)$$

$t(x,x_0)$ propagates source to aperture. This is not discussed in Jackson - nor does he discuss possibility that $\psi(x')$ is not uniform across the aperture. To use $*$, need $\Delta x \gg \lambda$ so aperture does not affect most of the incoming wave.

You can see, things are starting to get complicated ...

There are two important special cases which we can motivate by uncertainty principle arguments:

At aperture, beam has width $\Delta x = a$. This introduces a $\Delta k_x \sim \frac{1}{\Delta x}$ ($\Delta k_x \Delta x \sim 1$) — beam spreads.



Look out a distance r:

~~Compare with intrinsic width of beam~~

$$\frac{\Delta x_r}{r} = \frac{\Delta k_x}{k} \; ; \quad \Delta x_r = \frac{\Delta k_x}{k} r = \frac{1}{ak} r.$$

Beam spreads — combine spread with intrinsic width, $\Delta x_0 \sim a$

$$\Delta x_{TOT} = \left[a^2 + \left(\frac{r}{ak} \right)^2 \right]^{1/2}$$

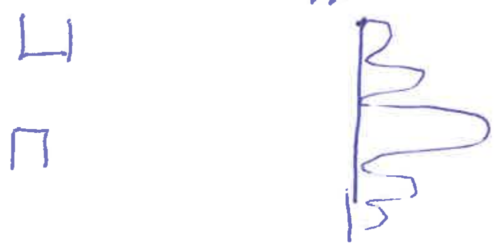
Two extreme limits!

1) $\Delta x_{TOT} \approx a$ if $\frac{r}{ak} \ll a$ or $\frac{r}{k} \ll a^2$

or ($k = \frac{2\pi}{\lambda}$) $\underline{\lambda r \ll a^2}$

This is called the Fresnel limit — basically, image given by geometric optics w/ ripples on the edges

2) If $\frac{r}{ak} \gg a$ or $\lambda r \gg a^2$, beam spread is all $\Delta k \, \Delta x = 1$. This is called Fraunhofer diffraction



Mathematical difference between the 2 is how e^{ikR} is expanded

$R = |x - x'|$; $|x| = r$, $|x'| = r' \sim a$

$$kR = kr - k\hat{n}\cdot\vec{x}' + \frac{k}{2r}\left[r'^2 - (\hat{n}\cdot\vec{x}')^2\right] + O\left(\frac{ka^3}{r^2}\right)$$

Size $\quad kR + \left[ka \sim \frac{a}{\lambda}\right] + \left[\frac{ka^2}{r} \sim \frac{a^2}{\lambda r}\right] + ka\left(\frac{a}{r}\right)^2$

If $r \gg a$, last term is always small

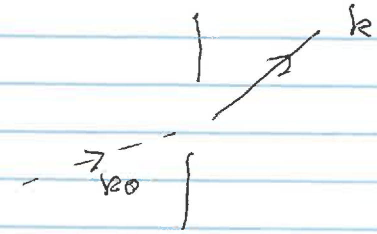
Fraunhofer is $\frac{a^2}{\lambda r} \ll 1$ so only keep first 2

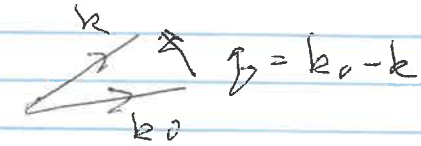
$$\frac{e^{ikR}}{R} = \frac{e^{ikr}}{r} e^{-ik\hat{n}\cdot\vec{x}'}$$

Just like the far field limit for antenna or scattering.

Thorn + Blandford have pictures of patterns for $\lambda r \ll a^2$ to $\lambda r \gg a^2$

With $\psi_0(x') = A e^{i\vec{k}_0\cdot\vec{x}'}$





$$\psi_D = \frac{k}{2\pi i}\frac{e^{ikr}}{r}\cos\theta \cdot A \int_{\text{aperture}} e^{i\vec{q}\cdot\vec{x}'}\, dA'$$

Easy example - square hole of size L, normal incidence - $k_0 = (0, 0, k)$

$$x' = (x', y', 0)$$

$$k = (k_x, k_y, k_z)$$

$$\hat{x}'\cdot\vec{k} = k_x x' + k_y y'$$

$$\int_{\text{aperture}} dA' e^{ik\cdot x'} = \int_{-L/2}^{L/2} dy' e^{ik_y y'} \int_{-\frac{L}{2}}^{L/2} dx' e^{ik_x x'}$$

$$= L^2 \left[\frac{\sin k_y L/2}{k_y L/2}\right]\left[\frac{\sin k_x L/2}{k_x \frac{L}{2}}\right]$$

$|\psi|^2$ = intensity - look along one axis -

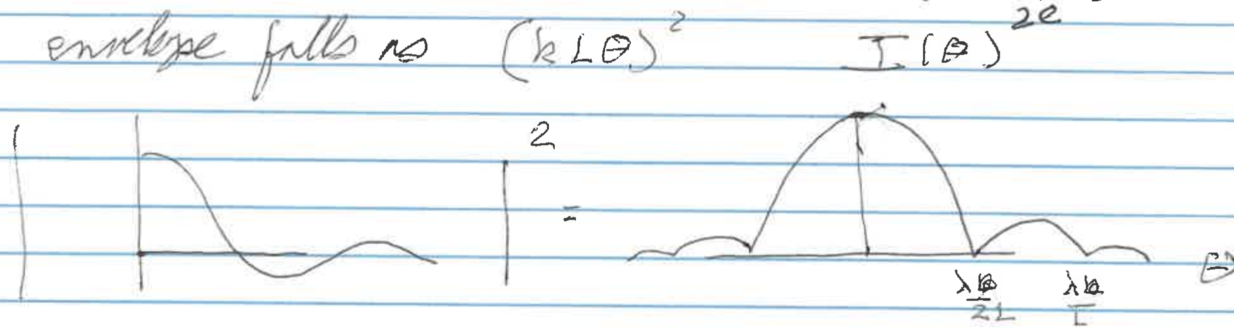
$$k_y = 0,\ k_x = k\sin\theta \sim k\theta$$

$$I \sim \frac{\sin^2 \frac{kL\theta}{2}}{\left(\frac{kL\theta}{2}\right)^2}$$

nodes at $\frac{kL\theta}{2} = \pi, 2\pi \ldots = \frac{2\pi}{\lambda} L\theta = \pi, 2\pi \ldots$

$\theta = \frac{\lambda}{2L} \cdot 1, 2, \ldots$

envelope falls as $(kL\theta)^2$



everything happens at small $\theta \sim \frac{\lambda}{L}$

Note also - a k_x is induced by the aperture

$k_x = k \sin\theta = \Delta k_x = k\theta$

$\Delta x \sim L$

$\Delta k_x \Delta x \sim 1 \quad - (kL\theta \sim 1)$.

~~application~~ restricting size of wave induces nonzero wave number -

classical version of uncertainty principle.

$\Rightarrow$ resolving power of optical devices ...

Note also - if $\frac{\lambda}{L} \to$ large $\left(\frac{L}{\lambda} \to \text{small}\right)$

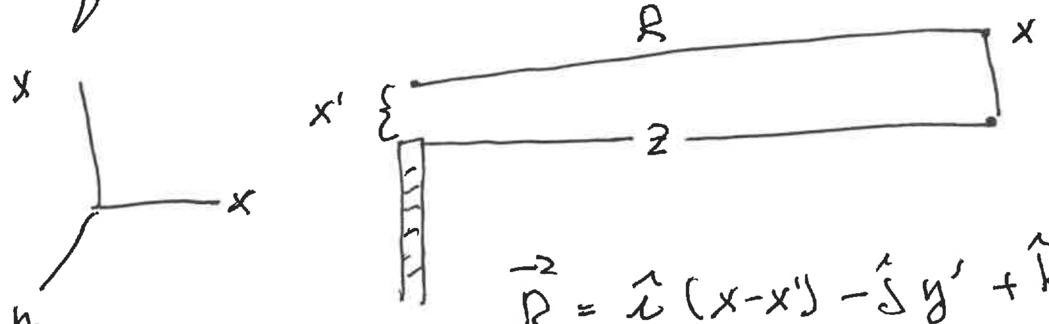
the diffraction minimum moves out to $\theta = \frac{\pi}{2}$

JDJ: "The assumption of unperturbed fields in the aperture breaks down."

2) Fresnel diffraction keeps the $\frac{ka^2}{r} \gg 1$ term.

Best described by example - prob. 2 of homework

Useful to eliminate the linear term with a change of variables



$$\vec{R} = \hat{i}(x-x') - \hat{j}y' + \hat{k}z$$

$$R = \left[(x-x')^2 + y'^2 + z^2\right]^{1/2}$$

$$\simeq z + \frac{1}{2z}\left[(x-x')^2 + y'^2\right]$$

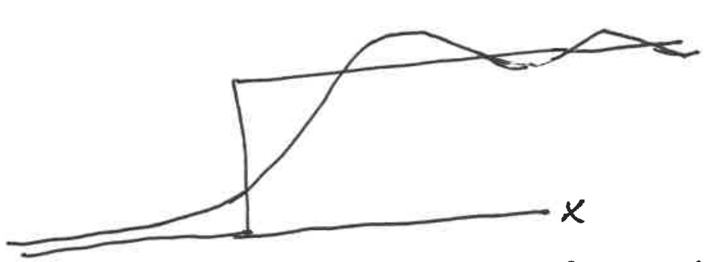
and $\hat{n}' \cdot \hat{R} = \hat{n}' \cdot \hat{z} = 1$

$$\psi_0(x') = I_0^{1/2} e^{i k_x x'} \rightarrow I_0^{1/2} \text{ for normal incidence}$$

$\vec{k_0}$

$$\psi(x) = \frac{k}{2\pi i} \frac{e^{ikz}}{z} I_0^{1/2} \int_0^\infty dx' \int_{-\infty}^{\infty} dy'$$

$$\times \exp \frac{ik}{2}\left[\frac{(x-x')^2}{2z} + \frac{y'^2}{2z}\right]$$



Intensity rises as the observer moves above the knife-edge

Vector formula is called the "vector Smythe-Kirchhoff formula"

$$\vec{E}(x) = \frac{1}{2\pi}\vec{\nabla}\times\int_{\text{Aperture}} \frac{e^{ikR}}{R}\,\hat{n}\times\vec{E}(x')\,dA'$$

and in Fraunhofer limit

$$\vec{E}(x) = \frac{ik}{2\pi}E_0\frac{e^{ikr}}{r}\,\hat{n}\times(\hat{z}\times\hat{\epsilon}_0)\int_{\text{Aperture}} dA'\,e^{i(\vec{k}_0-\vec{k})\cdot x'}$$

what's new: polarization info

long, difficult derivation: physics issues are basically

(1) due to fact that, really, radiation does not just come from the hole - it comes from the walls. Think about it:

$\vec{E}_i\,\vec{B}_i$

$\vec{E}=0 = \vec{E}_i + \vec{E}_{sc}$
$\vec{B}=0 = \vec{B}_i + \vec{B}_{sc}$

screen w/ no aperture

field from screen caused by charges + currents in screen

"Official" derivation only for conducting screen

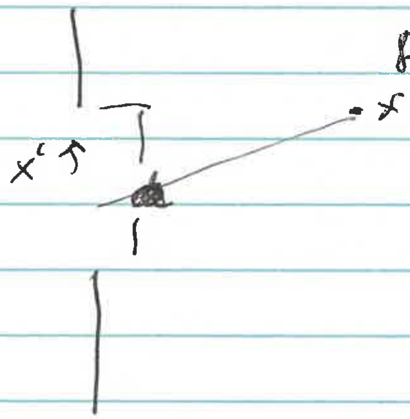
(2) We don't want to deal with the screen, we only want to deal with the hole, so lots of rewriting - Turns out, can only do this for $\vec{E}$: $\vec{B}$ involves integral over complementary region (the screen)

Example: Diffraction by Circular Aperture

Use the Smythe-Kirchhoff formula with

$\vec{E}(x) = \vec{E}_0(x)$ the incident field

$$\vec{E}(x) = \frac{1}{2\pi}\,\vec{\nabla}\times\int dA'\,\frac{e^{ikR}}{R}\,\hat{z}\times\vec{E}_0(x)$$



Fraunhofer:

$$\frac{e^{ikR}}{R} = \frac{e^{ikr}}{r}\,e^{-ikn\cdot x'}$$

$$\vec{\nabla} \to ik\hat{n}$$

$$\vec{E} = \hat{\epsilon}_0 E_0 e^{i\vec{k}_0\cdot x'}$$

$$E(x) = \frac{ik}{2\pi}E_0\frac{e^{ikr}}{r}\,\hat{n}\times(\hat{z}\times\hat{\epsilon}_0)\int_{\text{aperture}} dA'\exp i(\vec{k}_0 - k\vec{n})\cdot x'$$

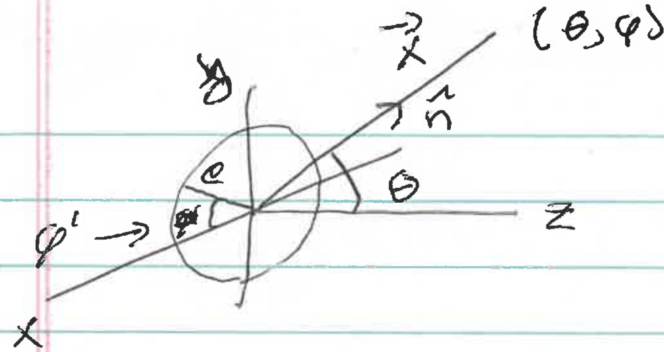
Let's assume $\vec{k}_0 = \hat{z}k_0$ normal incidence for simplicity.

Pick $\hat{\epsilon}_0 = \hat{x}$: $\hat{n}\times(\hat{z}\times\hat{x}) = (\hat{n}\times\hat{y})$

Aperture in x-y plane

Note $\vec{k}_0\cdot\vec{x}' = 0$

Focus on $\underline{I} = \int_{\text{aperture}} e^{-i\vec{k}\cdot\vec{x}'}\,dA'$



$$\hat{n} = [\hat{x}\cos\varphi + \hat{y}\sin\varphi]\sin\theta + \hat{z}\cos\theta$$

$$x' = \rho'[\hat{x}\cos\varphi' + \hat{y}\sin\varphi']$$

$$0 < \rho' < a$$

$$\hat{n}\cdot\vec{x}' = \rho'\sin\theta\cos(\varphi-\varphi')$$

$$I = \int_0^a \rho' d\rho' \int_0^{2\pi} d\varphi' \, e^{-ik\rho'\sin\theta\cos(\varphi-\varphi')}$$

Now it's just special functions

- shift variables to $d(\varphi-\varphi')$
- $J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi' \, e^{\pm ix\cos\varphi}$

$$x = k\rho'\sin\theta \equiv q\rho'$$

$$\vec{E}(x) = ik\, E_0 [\hat{n}\times\hat{y}]\frac{e^{ikr}}{r}\int_0^a \rho' d\rho' J_0(k\rho'\sin\theta)$$

$$= ik\, E_0 [\hat{n}\times\hat{y}]\frac{e^{ikr}}{r}\frac{1}{q^2}\int_0^{qa} x J_0(x)\, dx$$

More identities

$$x J_0(x) = x J_1'(x) + J_1(x)$$

$$\int_0^{x_0} dx\, [\underbrace{x J_1'}_{\text{parts}} + J_1] = \int_0^{x_0} dx\, [-J_1 + J_1] + x_0 J_0(x)$$

$$\vec{E}(\vec{x}) = i k E_0 (\hat{n}\times\hat{y}) \frac{e^{ikr}}{r} a^2 \frac{J_1(\xi a)}{\xi a}$$

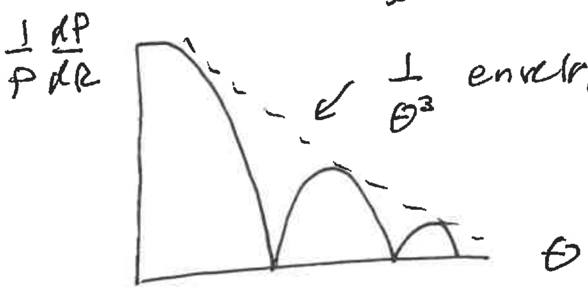
power $\propto |E|^2$, incident flux $P_i = \frac{c}{8\pi} E_0^2 \times \pi a^2$

$$\frac{1}{P_i}\frac{dP}{d\Omega} \sim \frac{(ka)^2}{4\pi}\left[\frac{J_1(\xi a)}{\xi a}\right]^2 \times \text{polarization factors}$$

Diffraction minima at zeros of J_1. First one is at $\xi a = 3.832 \simeq ka\theta_1$

$\Rightarrow$ measure θ_1, know k, get a

At large x $\quad \frac{J_1(x)}{x} \sim \sqrt{\frac{2}{\pi x^3}} \cos\left(x - \frac{3\pi}{4}\right)$



Other ~~[illegible]~~ points discussed in Jackson

- Transmission coefficient $T = \int_{\text{Forward}} d\Omega \, \frac{1}{P}\frac{dP}{d\Omega}$

- $ka \ll 1$ or $\frac{a}{\lambda} \ll 1$ limit

- Scalar theory – basically the same result at small θ

$$\frac{1}{P}\frac{dP}{d\Omega} = (\text{our answer}) \times \begin{cases} 1 & \text{scalar} \\ 1 - \sin^2\theta\cos^2\varphi & \hat{\epsilon}_0 = \hat{y} \\ 1 - \sin^2\theta\sin^2\varphi & \hat{\epsilon}_0 = \hat{x} \end{cases}$$

Resolving power of optical instrument

Circular aperture, Fraunhofer limit

$$\vec{E}(x) = \frac{ik}{2\pi} E_0 \frac{e^{ikr}}{r} \left[\hat{n} \times (\hat{z} \times \hat{\epsilon}_0) \right] \cancel{\oint_{aper} dA \, e^{i}} \; I$$

$$I = \int_{\text{aperture in } x\text{-}y \text{ plane}} dA' \, e^{i(\vec{k}_0 - \vec{k}) \cdot x'}$$

$$\int dA' = \int_0^a \rho' d\rho' \int_0^{2\pi} d\varphi'$$

~~Normal incidence: $\vec{x}' = (x', y', 0)$~~

Circular aperture: $\vec{x}' = (\rho' \cos\varphi', \rho' \sin\varphi', 0)$

normal incidence $\vec{k}_0 = (0, 0, k_0) \Rightarrow \vec{k}_0 \cdot x' = 0$

$$\vec{k} = k \left[\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta \right]$$

~~Normal incidence?~~

Exercise in special functions (homework problem)

$$\frac{1}{P} \frac{dP}{d\Omega} \sim \frac{(ka)^2}{4\pi} \left[\frac{J_1(qa)}{qa} \right]^2 \times \text{polarization factors}$$

$$qa = ka \sin\theta \sim ka\theta$$

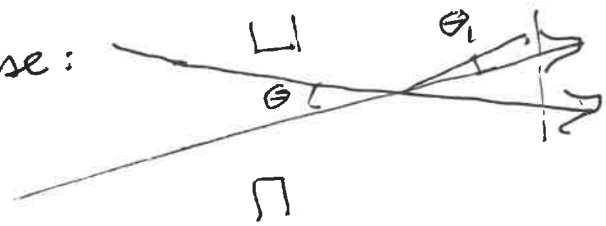
(R-2

minima of $\frac{J_1(x)}{x}$ are $x = 1.22\pi, 2.33\pi \ldots$

$$x = ka\theta = \frac{2\pi}{\lambda} a\theta$$

first minimum $\theta_1 = 0.61 \frac{\lambda}{a}$

use:



2 offset rays can be distinguished if $\theta > \theta_1$

→ Rayleigh criterion for resolving power of optical instrument, $\Delta\theta \sim 0.61 \frac{\lambda}{a}$ or $1.22 \frac{\lambda}{D}$

example: the eye, $\lambda \sim 5.6 \times 10^{-5}$ cm

diameter of pupil 1.5 - 6 mm

$5 \cdot 10^{-4} > \Delta\theta > 1 \times 10^{-4}$ radian

1′34″ to 0′24″

(Born + Wolfe)

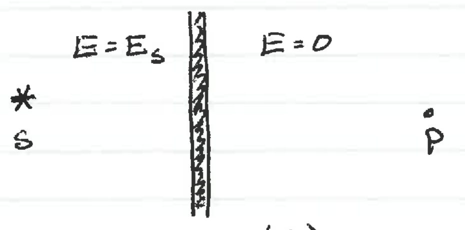
Babinet's Principle a la Feynman (V I : 31-10)

Light comes through the holes in an opaque screen.

Similar problem

Replace holes by sources uniformly distributed over hole - so the diffracted wave is the same as though the hole were a new source. Strange: the hole is where there are no sources - no accelerating charges.

What is an opaque screen? "Opaque" - no field at P.



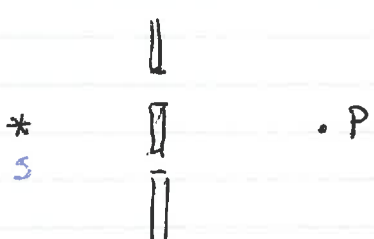
(a)

What is field at P? It is due to field of source E_s, plus fields from all the other charges around.

Charges in screen are set in motion by E_s, they must generate a field which exactly cancels E_s on back side (side toward observer) of screen. It's a miracle, this exact cancellation!

For if there weren't exact cancellation, the residual field would accelerate more charges in the screen, they would radiate - you would see something at P.

Now what about a wall with holes? Field at P = field due to S + field due to charges in walls.



(b)

$$E_P = E_S + E_{wall}$$

~~[illegible]~~

Now plug up the holes with the same kind of material as the walls. (primed situation)

primed situation – case c

(c)

c) $E_p' = 0 = \underset{E_p'' + E_{wall}}{E_S} + \left(E'_{wall} + E'_{plug}\right)$

$E_p = \left(E_{wall} - E'_{wall}\right) - E'_{plug}$

Now if the holes are not too small we'd expect $E'_{wall} = E_{wall}$ (correction is from the edge of the holes)
and in that case

$$E_p = -E'_{plug}$$

Field at P when there are holes in screen (case b)

= – (Field produced by the part of an opaque screen located ~~by the~~ where the holes are)

It's an amazing forward backward argument –

~~[illegible]~~ arises because the source is cancelled by the motion of charges everywhere in the screen.