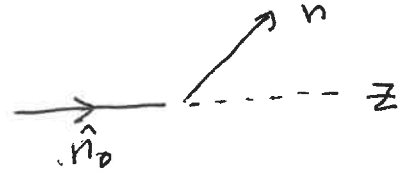
Scattering in E&M & QM - the big picture

Physics: beam (plane wave) interacts with target at origin, produces outgoing spherical wave

$$\psi = E_0 e^{ikz} + \frac{e^{ikr}}{r} F(\hat{n}, \hat{n}_0)$$



~~F is called the "scattering amplitude"~~

Typically $F \propto$ ~~intensity of~~ amplitude of initial beam so $F \equiv E_0 f$

$$\psi = E_0 e^{ikz} + E_0 \frac{e^{ikr}}{r} f(\hat{n}, \hat{n}_0)$$

f is called the "scattering amplitude"

Observe radiation away from z-axis

$$I \sim \frac{dP}{d\Omega} \sim r^2 \left| \frac{e^{ikr}}{r} F(\hat{n}, \hat{n}_0) \right|^2$$

Interesting quantity removes E_0

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{E_0^2} \frac{dP}{d\Omega} \longrightarrow \text{units typically area or "cross section"}$$

$$\sim |f(\hat{n}, \hat{n}_0)|^2$$

Techniques follow two paths

a) perturbative - Born approximation, Feynman graphs

b) "exact" - Direct solve of $(\nabla^2 + k^2)\psi = V(r)\psi(r)$

$(\nabla^2 + k^2)\vec{A} = \frac{4\pi}{c}\vec{J}$

exact results: "optical theorem" $\sigma_{TOT} = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{4\pi}{k} \operatorname{Im} f(\hat{n}_0, \hat{n}_0)$

Partial wave expansion

$$f(\theta) = \sum_\ell P_\ell(\cos\theta) f_\ell(k) = \sum_\ell P_\ell(\cos\theta)\left[\frac{e^{2i\delta_\ell(k)} - 1}{2ik}\right]$$

(more complicated in E+M due to $\vec{E}$, $\vec{B}$ vector nature though can always do this).

General simplifying limit: low energy, long wavelength

$kd \ll 1$

Typically
- in QM only one or 2 partial waves contribute
- "universal" behavior - only gross features of scatterer matter - one or 2 parameters

In QM, s-waves, scattering length + effective range

in E+M - dipole moment(s) all that matter, physical description of scattg follows.

Another simplifying limit - scatterer is weak

$\Rightarrow$ Born approximation

Scattering in the long-wavelength limit

physics idea: radiation due to oscillating dipole, but dipole moment induced by external $\vec{E}$ field E_0 - and proportional to it.

$$\frac{dP}{d\Omega} \sim \cancel{B^2} \frac{c}{8\pi} r^2 |E|^2 \propto E_0^2$$

~~Energy density~~ Flux in $\Phi_i = \frac{c}{8\pi} E_0^2$ $\left(\frac{\text{energy}}{\text{area-time}}\right)$

$$\frac{1}{\Phi_i} \frac{dP}{d\Omega} = \frac{\text{area} \times \text{time}}{\text{energy}} \times \frac{\text{energy}}{\text{time}} = \text{area}$$

$\equiv$ differential cross section $\frac{d\sigma}{d\Omega}$

Two variations on this idea
- direct use of induced (dipole) multipole moment
- Born approximation

First variation by example: Dielectric sphere, radius a, rel. permittivity ϵ $\equiv \epsilon/\epsilon_0$ in MKS, incident radiation has $\lambda \gg a$

$$\vec{E}_{in} = \hat{\epsilon}_0 E_0 e^{ik_0\hat{n}_0\cdot x}$$

is essentially uniform over sphere

$$\vec{B}_{in} = \hat{n}_0 \times \vec{E}_{in}$$

$\therefore$ induced dipole moment $\vec{p}(t) = \left[\frac{\epsilon - 1}{\epsilon + 2}\right] a^3 \vec{E}_{in}(t) \propto \hat{\epsilon}_0$

MKS $\frac{\epsilon}{\epsilon_0} \to \epsilon$ CGS

radiation field from dipole $\vec{B}_s = k^2 \frac{e^{ikr}}{r} \hat{n} \times \vec{p}$

$$\vec{E}_s = -\hat{n} \times \vec{B}_s = -k^2 \frac{e^{ikr}}{r}(\vec{p} - \hat{n}(\hat{n}\cdot\vec{p}))$$

First variation by example: Dielectric sphere, radius a, relative permittivity ϵ, incident radiation has $\lambda >> a$

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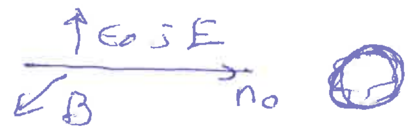


Induced dipole moment as in statics

$$\vec{p}(t) = \left[\frac{\epsilon - 1}{\epsilon + 2}\right] a^3 \vec{E}_{in}(t)$$

(CGS version of eq. 4-56; $\frac{\epsilon}{\epsilon_0} \to \epsilon$)

Note $\vec{p} \propto \hat{\epsilon}_0$!

Dipole radiates: $\vec{B}_s = k^2 \frac{e^{ikr}}{r} \hat{n} \times \vec{p}$

$$\vec{E}_s = -\hat{n} \times \vec{B}_s = -k^2 \frac{e^{ikr}}{r} (\vec{p} - \hat{n}(\hat{n} \cdot \vec{p})) \quad (*)$$

$$\frac{dP}{d\Omega} = \frac{c}{8\pi} r^2 \, \hat{n} \cdot \left(\vec{E}_{\cancel{out}\,s} \times \vec{B}^*_{\cancel{out}\,s} \right)$$

$$= \frac{c r^2}{8\pi} \left| \vec{E}_{\cancel{out}\,s} \right|^2$$

pause: New feature – polarization of incoming <u>and</u> outgoing radiation. Typically want to specify orientation of outgoing ~~φ~~ radiation in terms of ~~outg~~ external detector. – call this $\hat{\epsilon}$

$$\vec{E}_{\cancel{out}\,s} = \sum_\lambda \hat{\epsilon}_\lambda \, \underbrace{(\hat{\epsilon}_\lambda^* \cdot \vec{E}_{\cancel{out}\,s})}_{\text{projection on direction } \epsilon_\lambda}$$

$$\frac{dP}{d\Omega}(\hat{\epsilon}) = \frac{c r^2}{8\pi} \left| \hat{\epsilon}^*_{\cancel{out}} \cdot \vec{E}_{\cancel{out}\,s} \right|^2$$

$$\vec{E}_{out} \sim \vec{p} \sim \vec{E}_{in} \sim \hat{\epsilon}_0 \quad \text{initial pol}$$

$$\epsilon^* \cdot (\hat{n} \times (\hat{n} \times \vec{p})) = - \hat{\epsilon}^* \cdot \vec{p} \quad \text{since } \hat{\epsilon} \cdot \hat{n} = 0$$

see 8

and $\vec{p} = \hat{\epsilon}_0 p$

$$\frac{dP(\epsilon)}{d\Omega} = \frac{ck^4}{8\pi}\left|\epsilon^* \cdot \vec{p}\right|^2 = \frac{ck^4}{8\pi} p^2 \left|\hat{\epsilon}^* \cdot \hat{\epsilon}_0\right|$$

(using $\vec{p} = p\,\hat{\epsilon}_0$) $\quad p = \left(\frac{\epsilon-1}{\epsilon+2}\right) a^3$

$$\frac{dP_{scatt}}{d\Omega}\left(\vec{E}_{scatt} = \hat{\epsilon},\ \vec{E}_{in} = \hat{\epsilon}_0\right)$$

$$= \frac{ck^4 a^6}{8\pi}\left(\frac{\epsilon-1}{\epsilon+2}\right)^2 E_0^2 \left|\hat{\epsilon}\cdot\hat{\epsilon}_0\right|^2$$

Cross section defined by dividing out the flux of initial radiation

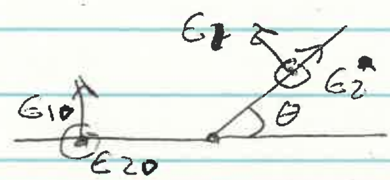
$$\Phi_i = \frac{c}{8\pi}\left|E_0\right|^2$$

$$\frac{1}{\Phi_i}\frac{dP}{d\Omega} = \frac{d\sigma}{d\Omega} = \frac{k^4\left|\epsilon^*\cdot p\right|^2}{E_0^2}$$

$$= \left|\frac{\epsilon-1}{\epsilon+2}\right|^2 k^4 a^6 \left|\hat{\epsilon}\cdot\hat{\epsilon}_0\right|^2$$

- $(ka)^4 a^2 =$ area (check units!)
- Rayleigh's law: $\sigma \sim \frac{1}{\lambda^4} \sim k^4 \sim \omega^4$

Summing/averaging polarizations: compute $\left|\hat{\epsilon}\cdot\hat{\epsilon}_0\right|^2$



ϵ \ ϵ_0	1 or ∥	2 or ⊥
1	$\cos^2\theta$	0
2	0	1

1 = in plane = ∥

2 = out of plane = ⊥

Assume initial beam is unpolarized - average over ϵ_{1o} & ϵ_{2o} → outgoing pol fixed in plane

$$\frac{d\sigma_{\parallel}}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma}{d\Omega}(1\to 1) + \frac{d\sigma}{d\Omega}(2\to 1)\right] = \frac{1}{2}\sigma_0 \cos^2\theta$$

out of plane pol

$$\frac{d\sigma_{\perp}}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma}{d\Omega}(1\to 2) + \frac{d\sigma}{d\Omega}(2\to 2)\right] = \frac{1}{2}\sigma_0$$

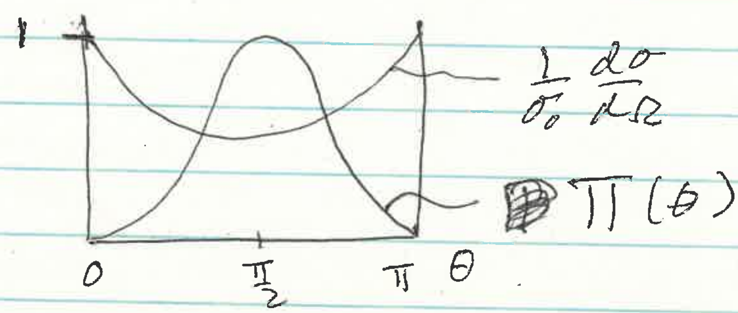
$$\frac{d\sigma}{d\Omega}(\underset{\text{ave}}{\text{unpol}} \to \underset{\text{pols}}{\text{all}}) = \frac{1+\cos^2\theta}{2}\sigma_0$$

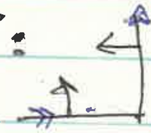
$$\sigma_0 = \left|\frac{\epsilon-1}{\epsilon+2}\right|^2 k^4 a^6$$

$$\sigma_{TOT} = \int \frac{d\sigma}{d\Omega}\, d\cos\theta\, d\varphi = \frac{2\pi}{2}\left[2 + \frac{2}{3}\right]\sigma_0 = \frac{8\pi}{3}\sigma_0$$

Polarization?

$$\Pi(\theta) = \frac{\frac{d\sigma_{\perp}}{d\Omega} - \frac{d\sigma_{\parallel}}{d\Omega}}{\frac{d\sigma_{\perp}}{d\Omega} + \frac{d\sigma_{\parallel}}{d\Omega}} = \frac{1-\cos^2\theta}{1+\cos^2\theta} = \frac{\sin^2\theta}{1+\cos^2\theta}$$



Note 100% pol. $\perp$ at 90°:  $= 0$ in plane pol. vanishes

only $\odot_{\text{input}} \to \odot$ is non zero

Another example - perfectly conducting sphere

i) Recall $E_{in} = \frac{3}{\epsilon+2} E_0$ (4.55) $\to 0$, as

take $\epsilon \to \infty$, $\vec{p} = \frac{\epsilon-1}{\epsilon+2} a^3 E_0 \hat{\epsilon}_0 \to a^3 E_0 \hat{\epsilon}_0$

ii) Inside a perfect conductor $B = \mu H \to 0$ for AC field - see sec. 5-13 - so consider first, induced magnetic dipole moment for imperfect conductor, then take $\mu \to 0$. 5-112 says $B_{in} = \frac{3\mu}{\mu+2} B_0$,

$$\vec{m} = \frac{\mu-1}{\mu+2} a^3 \vec{B}_0 = \frac{\mu-1}{\mu+2} a^3 [\hat{n}_0 \times \hat{\epsilon}_0] E_0$$

$$\vec{m} \xrightarrow[\mu\to 0]{} -\frac{1}{2} a^3 (\hat{n}_0 \times \hat{\epsilon}_0) E_0 = -\frac{1}{2} \hat{n}_0 \times \vec{p}$$

$E_{scatt} = \vec{E}_{E_1} + \vec{E}_{m_1}$ - a superposition!

$$E_{E1} = -k^2 \frac{e^{ikr}}{r} \hat{n} \times (\hat{n} \times \vec{p}) = -k^2 a^3 E_0 \frac{e^{ikr}}{r} \hat{n} \times (\hat{n} \times \hat{\epsilon}_0)$$

$$E_{M1} = -k^2 \frac{e^{ikr}}{r} \hat{n} \times \vec{m} = +\frac{k^2 a^3}{2} E_0 \frac{e^{ikr}}{r} \hat{n} \times (\hat{n}_0 \times \hat{\epsilon}_0)$$

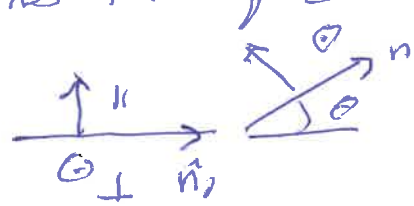
$$\frac{d\sigma}{d\Omega}(\hat{\epsilon}, \hat{n}; \hat{\epsilon}_0, \hat{n}_0) = k^4 a^6 \left| \epsilon^* \cdot \left(\hat{n} \times (\hat{n} \times \hat{\epsilon}_0) - \frac{1}{2} (\hat{n} \times (\hat{n}_0 \times \hat{\epsilon}_0) \right) \right|^2$$

$$= \frac{1}{\Phi_i} \frac{dP}{d\Omega}$$

$$= k^4 a^6 \left| \hat{\epsilon}^* \cdot \epsilon_0 - \frac{1}{2} (\hat{n} \times \epsilon)^* \cdot (\hat{n}_0 \times \hat{\epsilon}_0) \right|^2$$

The 2 terms interfere.

Recall



$$\begin{array}{c|cc} \text{in} \backslash \text{out} & \parallel & \perp \\ \hline \parallel & \cos\theta & 0 \\ \perp & 0 & 1 \end{array} \qquad \begin{array}{c|cc} \text{in} \backslash \text{out} & \parallel & \perp \\ \hline \parallel & 1 & 0 \\ \perp & 0 & \cos\theta \end{array}$$

$$\hat{\epsilon}\cdot\hat{\epsilon}_0 \qquad (n\times\epsilon)\cdot(n_0\times\epsilon_0)$$

$$\frac{d\sigma}{d\Omega}(\text{ave initial, final } \parallel) \equiv \frac{d\sigma_\parallel}{d\Omega} = \frac{k^4a^6}{2}\left|\cos\theta - \frac{1}{2}\right|^2$$

$$\frac{d\sigma_\perp}{d\Omega} = \frac{k^4a^6}{2}\left|1-\frac{1}{2}\cos\theta\right|^2 \quad \text{ave. initial final } \perp$$

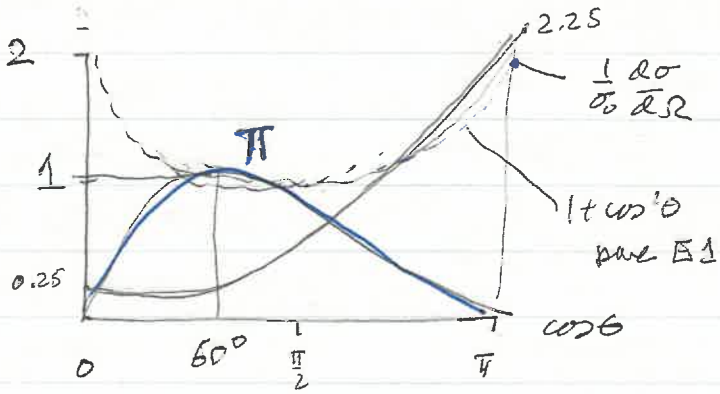
$$\left.\frac{d\sigma}{d\Omega}\right)_{\substack{\text{sum final}\\ \text{ave initial}}} = k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta\right]$$

$$\Pi(\theta) = \frac{3\sin^2\theta}{5(1+\cos^2\theta)-8\cos\theta} = \frac{\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega}}{\frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega}}$$

$$\sigma = k^4a^6 \cdot \frac{5}{8}\left[1+\frac{2}{3}\right] = \frac{25}{8}$$

$$\cos^2\theta - \cos\theta + \frac{1}{4} - \left(1 - \cos\theta + \frac{1}{4}\cos^2\theta\right)$$

$$\frac{3}{4}(\cos^2\theta - 1)$$



: note strong peaking backwards (reflection!)

$$\frac{5}{8}(1+\cos^2\theta) - \cos\theta \quad \nearrow \cos\theta = 1 \quad \frac{5}{4} - 1 = \frac{1}{4}$$

$$\searrow \cos\theta = -1 \quad \frac{5}{4} + 1 = \frac{9}{4}$$

Perturbation theory for scattering

motivation: fluctuation in $\epsilon(x)$, $\mu(x)$ induces scattering

Need to go macro for a bit: in CGS

$$\underline{\vec{D} = \epsilon \vec{E}} = \vec{E} + 4\pi \vec{P} \; ; \; \underline{\vec{B} = \mu \vec{H}} = \vec{H} + 4\pi \vec{M}$$

$\vec{D} = \vec{E}$ and $\vec{B} = \vec{H}$ in free space

If there are no macroscopic charges or currents

$$\vec{\nabla}\cdot\vec{D} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

1) $$\nabla\times[\nabla\times(\vec{D}-\vec{E})] = \underbrace{\nabla(\vec{\nabla}\cdot\vec{D})}_{0} \underbrace{- \nabla^2\vec{D}}_{D \text{ terms}} + \underbrace{\nabla\times\left(\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right)}_{E \text{ term}}$$

2) $$\frac{\partial}{\partial t}\left(\frac{\partial \vec{D}}{\partial t}\right) = c\frac{\partial}{\partial t}(\nabla\times\vec{H})$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{D} = -\nabla\times\left[\nabla\times(\vec{D}-\vec{E}) + \frac{1}{c}\frac{\partial}{\partial t}\nabla\times(\vec{B}-\vec{H})\right]$$

idea - assume RHS is known, solve for LHS

$$e^{-i\omega t} \rightarrow (\nabla^2 + k^2)\vec{D} = \text{source}(x)$$

$$\vec{D}(x) = \underbrace{\vec{D}_0(x)}_{\text{no source}} + \int d^3x' \, G_k(\vec{x},\vec{x}') \cdot \text{source}(x')$$

$$G_k(\vec{x},\vec{x}') = -\frac{1}{4\pi}\frac{e^{ikR}}{R} \; ; \; R = |x-x'|, \; k = \frac{\omega}{nc}$$

$$\simeq -\frac{1}{4\pi}\frac{e^{ikr}}{r} e^{-ik\hat{n}\cdot x'} \quad \text{in radiation zone}$$

so $\vec{D}(x) = \vec{D}_0(x) + \vec{D}_{scatt}(x)$.

Temporarily set $\mu = 1$ everywhere so no $B-H$

$$\text{source} = -\vec{\nabla}\times(\vec{\nabla}\times(\vec{D}-E))$$

$$\vec{D}_{sc} = \frac{e^{ikr}}{4\pi r}\int d^3x'\, e^{-ik\hat{n}\cdot\vec{x}'}\, \nabla'\times\left(\nabla'\times\left(\vec{D}(x') - \vec{E}(x')\right)\right)$$

Integrate by parts <u>twice</u> - ∇' hits $e^{-ik\hat{n}\cdot x'}$

$$\vec{D}_{sc} = \frac{e^{ikr}}{r}\,\hat{n}\times\left(\hat{n}\times\vec{f}_s(\hat{n})\right)$$

$$\vec{f}_s(\hat{n}) = -\frac{k^2}{4\pi}\int d^3x'\, e^{-ik\hat{n}\cdot x'}\left[\vec{D}(x') - \vec{E}(x')\right]$$

This is ~~called~~ the scattering amplitude

In the radiation zone, assume $\epsilon \to 1$ (so $\vec{D} = \vec{E}$)

$$\frac{dP}{d\Omega} = \frac{c}{8\pi}\, r^2\, \hat{n}\cdot\left[\vec{D}_{sc}\times\vec{B}_{sc}\right]$$

$$\vec{B}_{sc} = \hat{n}\times\vec{D}_{sc}$$

$$\frac{dP}{d\Omega} = \frac{c r^2}{8\pi}\,|\vec{D}_{sc}|^2 \quad \text{or} \quad |\hat{\epsilon}^*\cdot\vec{D}_{sc}|^2 \text{ for particular detected pol. } \vec{\epsilon}$$

$$\epsilon^*\cdot\left(\hat{n}\times(\hat{n}\times\vec{F})\right) = -\epsilon^*\cdot F \quad \text{since } \epsilon\cdot n = 0$$

$$\frac{d\sigma}{d\Omega} = \frac{|\epsilon^*\cdot\vec{F}_s(\hat{n})|^2}{|\vec{D}_0|^2}$$

Pause to reflect. $\vec{F}$ depends on $\vec{E} = \vec{E}_0 + \vec{E}_{sc}$, ~~B~~ $\vec{D} = \vec{D}_0 + \vec{D}_{sc}$. But if we actually knew the scattered fields, we wouldn't have to do this.

~~In QM, $T = \langle \phi_f | V | \psi_i \rangle$, $\frac{d\sigma}{d\Omega} = |T|^2$~~

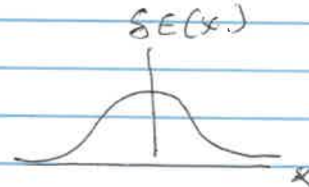
~~ϕ_f = free particle final state, ψ_i = full initial state~~

However, just as in QM, we can introduce the Born (Rayleigh 1881) approximation, valid as $\epsilon - 1 \to 0$

$$\vec{D} \to \vec{D}_0 \quad \text{in the integral}$$
$$\vec{E} \to \vec{E}_0$$

To do this, write

$$\vec{E} = E_0 \hat{\epsilon}_0 \, e^{i \vec{k}_0 \cdot \vec{x}}$$

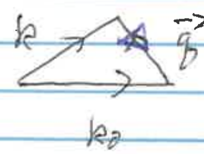


$$\delta\epsilon(x) = \epsilon(x) - 1$$

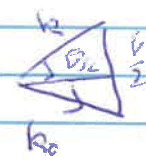
$$\vec{F}_{born} = -\frac{k^2 E_0 \hat{\epsilon}_0}{4\pi} \int d^3x' \, e^{i(\vec{k}_0 - \vec{k}) \cdot x'} \, \delta\epsilon(x')$$

$$= -\frac{k^2 E_0 \hat{\epsilon}_0}{4\pi} \, \delta\epsilon(\vec{q})$$ Fourier transform of $\delta\epsilon(x)$!

$\vec{q} = \vec{k}_0 - \vec{k}$ = wave # transfer in scattering



and



$|q| = 2k \sin\frac{\theta}{2}$

$$\frac{d\sigma}{d\Omega}(\hat{\epsilon}, \hat{\epsilon}_0) = \frac{k^4}{16\pi^2} \left| \delta\epsilon(q) \, \hat{\epsilon}^* \cdot \hat{\epsilon}_0 \right|^2$$

For magnetic interaction, re-insert $\delta\mu$

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} \left| \cdots + \int d^3x' \, e^{iq \cdot x'} \, \delta\mu(x') \left[(\hat{n} \times \hat{\epsilon}) \cdot (\hat{n}_0 \times \hat{\epsilon}_0) \right] \right|^2$$

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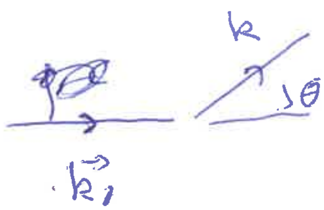
$$\frac{d\sigma}{d\Omega}(\vec{k},\hat{\epsilon};\vec{k}_0,\hat{\epsilon}_0) = \frac{k^4}{16\pi^2}\left|\int d^3x\; e^{i\vec{q}\cdot\vec{x}}\right.$$

$$\left[\delta\epsilon(x)\,\hat{\epsilon}^*\cdot\epsilon_0 + \delta\mu(x)\,(\hat{n}\times\hat{\epsilon}^*)\cdot(\vec{n}_0\times\hat{\epsilon}_0)\right]$$

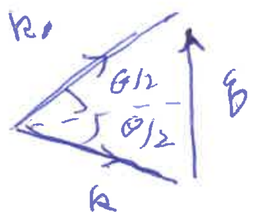
$$\delta\epsilon(x) = \epsilon(x) - 1 \qquad \left(\text{or } \frac{\epsilon(x)-\epsilon_0}{\epsilon_0}\right.$$

$$\delta\mu(x) = \mu(x) - 1 \qquad \left.\frac{\mu(x)-\mu_0}{\mu_0}\right)$$



$$\vec{q} = \vec{k}_0 - \vec{k}$$

$$|k_0| = |k| = \frac{\omega}{c}$$



$$\frac{q}{2} = k\sin\frac{\theta}{2}$$

or $q = 2k\sin\frac{\theta}{2}$ in magnitude

Physical picture:

- In some limited region of space $\delta\epsilon, \delta\mu \neq 0$
- Incident waves scatter from this region
- $\delta\epsilon, \delta\mu$ small so inside the scatterer

  E + B are approximately plane waves

Output: result valid for all k, not just in long wavelength limit but assumes $\delta\epsilon/\epsilon$, $\delta\mu/\mu$ small (answer is first order in $\delta\epsilon/\epsilon$)

~~Examples~~ What happens in the long wavelength limit?

Assume size of $\delta\epsilon \sim a$. $e^{iqx} \sim e^{iqa} \sim 1$

($|q| = 2k\sin\frac{\theta}{2}$, small $ka \to$ small qa

$$\left.\frac{d\sigma}{d\Omega}\right)_{Born} \simeq \frac{k^4}{16\pi^2}\left| \hat{\epsilon}^* \cdot \hat{\epsilon}_0 \int d^3x\, \delta\epsilon(x) \right|^2$$

Rayleigh again - universal behavior!

For a uniform sphere of radius a the integral is

$\frac{4\pi}{3} a^3 [\epsilon - 1]$. ~~Exact result from before~~

$$\left.\frac{d\sigma}{d\Omega}\right)_{Born} = \left[\frac{\epsilon-1}{3}\right]^2 k^4 a^6 \left|\hat{\epsilon}^* \cdot \hat{\epsilon}_0\right|^2$$

Compare to dielectric sphere $\frac{\epsilon-1}{3} \leftrightarrow \frac{\epsilon-1}{\epsilon+2}$;

note if $\epsilon = 1 + \delta\epsilon$, ~~$\epsilon+2 = 3 + $ small~~

$$\frac{\epsilon-1}{\epsilon+2} = \frac{\delta\epsilon}{3+\delta\epsilon} = \frac{\delta\epsilon}{3}\left[1 - \frac{1}{3}\delta\epsilon + \cdots\right]$$

Many scatterers

We have been assuming that the target is located at the ~~orig~~ origin of the coordinate system. What if it's not? Let's assume the scatterer is located at $\vec{x}_0$.

Incident wave at scatterer: $\vec{E}_{in} = \vec{\epsilon}_0 E_0 e^{i(k\hat{n}_0\cdot\vec{x}_0)}$ ($\vec{n}_0$ = beam direction). It induces a current or dipole moment with the same phase factor - that is, in

$$\vec{A}(x) = \frac{1}{c}\int \vec{J}(x')\,\frac{e^{ik|x-x'|}}{|x-x'|}\,d^3x'$$

$$J(x') \propto e^{ik\hat{n}_0\cdot\vec{x}_0}$$

Now let's write $\vec{x}' = \vec{x}_0 + \vec{x}''$ (thinking of the center of the scatterer as $\vec{x}_0$): $J(x') = e^{ik\hat{n}_0\cdot x_0}\vec{J}(x'')$

$$k|\vec{x}-\vec{x}'| = k|\vec{x}-\vec{x}_0-\vec{x}''| = kr - k\hat{n}\cdot\vec{x}_0 - k\hat{n}\cdot\vec{x}''$$

$$A(x) = \frac{e^{ikr}}{r}\int \underbrace{e^{+ik\hat{n}_0\cdot\vec{x}_0}}_{\text{from } J}\;\underbrace{e^{-ik\hat{n}\cdot\vec{x}_0}}_{\text{from } k|x-x'|}\;e^{-ik\hat{n}\cdot x''}\,\vec{J}(x'')\,d^3x''$$

Thus the scattered wave has an overall phase

$$\exp i(k\hat{n}_0 - k\hat{n})\cdot\vec{x}_0 \equiv \exp i\vec{q}\cdot\vec{x}_0$$

$\vec{q}$ = wave number transfer. Usually we ignore this factor because we square A and it drops out. But suppose we have many scatterers - the scattered waves will add coherently

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{E_0^2}\left|\sum_j \vec{\epsilon}^*\cdot\vec{p}_j \exp i\vec{q}\cdot\vec{x}_j\right|^2$$

, x_j = location of jth scatterer

Of course, this argument is quite general, and amounts to the statement that the scattering amplitude from a source at location $\vec{x}_j$ is related to the scattering amplitude for a source at the origin by

$$\vec{F}(\hat{n}, \vec{x}_j) = e^{i\vec{q}\cdot\vec{x}_j}\, \vec{F}(\hat{n}, 0)$$

Let's suppose for simplicity that the scatterers are all identical, so that we can factorize $\frac{d\sigma}{d\Omega}$

$$\frac{d\sigma}{d\Omega} = \frac{|\hat{e}\cdot\vec{F}(\hat{n},0)|^2}{E_0^2} \left|\sum_j e^{i\vec{q}\cdot\vec{x}_j}\right|^2$$

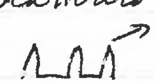
$$\equiv \frac{d\sigma_0}{d\Omega}(\epsilon)\, F(\vec{q})$$

$$F(\vec{q}) = \left|\sum_j e^{i\vec{q}\cdot\vec{x}_j}\right|^2 \equiv \text{"Structure factor"}$$

or "Form factor"

Many kinds of physics are driven by the structure factor - lets look at some examples

1) regular array of scatterers → Bragg peaks (homework set 1)



2) Random array of scatterers

Write $\quad F(q) = \sum_{ij} e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)}$ ~~sum over pairs~~.

For a random orientation of scattering centers

$$\sum_{i\neq j} e^{i q\cdot(x_i - x_j)} = 0 \quad \text{(random phases)}$$

$$F(q) = \sum_{i=1}^{N} 1 = N \quad \text{only diagonal terms contribute}$$

$$\frac{d\sigma}{d\Omega} = N \frac{d\sigma_0}{d\Omega} \equiv \text{Incoherent scattering}$$

but in the forward direction $\vec{q} = \vec{k} - \vec{k}_0 \Rightarrow 0$

$$e^{i q \cdot x} = 1 \quad \text{so} \quad F(q) = \left| \sum_{i=1}^{N} 1 \right|^2 = N^2$$

$$\frac{d\sigma}{d\Omega} \sim N^2 \frac{d\sigma_0}{d\Omega} \equiv \text{Coherent scattering}$$

If d = size of scattering region, to get this requires

$q d \sim k \theta d \ll 1$

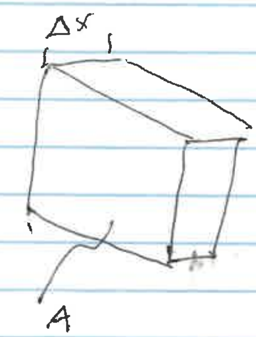
$$\theta \ll \frac{1}{kd} = \frac{\lambda}{2\pi} \frac{1}{d}$$

Light scattering on 1 m cube of air - coherency wants

$$\theta \ll \frac{500 \times 10^{-9}\,\text{m} \; (= \lambda)}{1\,\text{m}} \frac{1}{2\pi} = 10^{-7}\,\text{rad} = 10^{-5}\,\text{degrees}.$$

Also attenuation of beam

Power loss traversing a volume = $\left[\text{incident flux} = \frac{\text{power-in}}{\text{area}} \right]$



$\times \left[\frac{\text{power loss per mol.}}{\text{incident unit flux}} = \sigma \right]$

$\times \left[N = \#\text{ of molecules} = (\text{density } n) \times A \Delta x \right]$

$$\Delta P = -\frac{P}{A} \cdot \sigma \cdot n A \Delta x$$

$$\frac{\Delta P}{P} = -n\sigma \Delta x \Rightarrow P(\Delta x) = P_0 e^{-n\sigma \Delta x} = P_0 e^{-\frac{\Delta x}{\lambda}}$$

$$\lambda \equiv \text{attenuation length} = \frac{1}{n\sigma}$$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_0}{d\Omega} F(q)$$

Next, suppose we have a continuous distribution of scatterers

$$F(q) = \left| \sum_j e^{i q x_j} \right|^2 \to \left| \int d^3x \, \delta n(x) \, e^{i q \cdot x} \right|^2$$

$\delta n(x)$ = fluctuation in local matter density
(no scattering if n = constant!)

$$\delta n(x) = n(x) - \bar{n}, \quad \bar{n} = \frac{1}{V} \int d^3x \, n(x)$$

basically Born formula – $F(q)$ (F.T. of $\delta n(x)$)^2

~~Finally~~, suppose we scatter off an ensemble of scatterers (example – thermal ensemble) – must average scattering over ensemble

$$F(q) = \left\langle \left| \int d^3x \, \delta n(x) \, e^{i q \cdot x} \right|^2 \right\rangle$$

Expand – write twice

$$F(q) = \left\langle \int d^3x \, \delta n(x) \, e^{i q \cdot x} \int d^3y \, \delta n(y) \, e^{-i q \cdot y} \right\rangle$$

keep →
$$= \int d^3x \, d^3y \, e^{i q \cdot (x-y)} \langle \delta n(x) \delta n(y) \rangle$$

If system is translationally invariant

$$\langle \delta n(x) \delta n(y) \rangle = f(x-y) = \langle \delta n(x-x') \delta n(0) \rangle$$

$$\int d^3x \, d^3y = \int d^3y \, d^3(x-y) = V \int d^3(x-y)$$

$$F(q) = V \int d^3r \, e^{i q \cdot r} \langle \delta n(r) \delta n(0) \rangle$$

Extremely important
– scattering is off fluctuations in medium

– structure factor is Fourier transform of a correlation function, ensemble average of correlation of density fluctuations at 2 points separated by r

⇒ Light scattering reveals structure of matter!

Note as $q \to 0$ $F(q) = V \int d^3r \langle \delta n(r)\, \delta n(0) \rangle$

$$= \int d^3r \int d^3r' \langle (n(r) - \bar{n})(n(r') - \bar{n}) \rangle$$

$$= \langle N^2 \rangle - \langle N \rangle^2$$

$\equiv (\Delta N)^2$: number fluctuation (in the sense of Grand Canonical Ensemble)

where $\langle N^2 \rangle = \langle (\int d^3r\, n(r))^2 \rangle$

For ideal gas, $\langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle$.

Magnetic scattering sees the local spin density

$$F_{spin}(q) = \int d^3r \langle (\sigma(r) - \bar{\sigma})(\sigma(0) - \bar{\sigma}) \rangle e^{iq\cdot r}$$

$\lim_{q \to 0} F(q) = \langle \sigma^2 \rangle - \langle \sigma \rangle^2 =$ magnetic susceptibility χ

$$\chi = \left.\frac{\partial M}{\partial h}\right)_{h=0}$$ M = magnetization, h = applied field

$\Rightarrow \lim_{q \to 0} F(q) \longleftrightarrow$ susceptibility (in sense of stat mech)

Application
~~Critical opalescence~~ II

~~approach~~ – in a gas – appeal to stat mech (grand canonical ensemble)

$$\langle N^2\rangle - \langle N\rangle^2 = N kT \rho\, K_T$$

ρ = density, K_T = "isothermal compressibility"

$$K_T = \frac{1}{V}\,\frac{1}{\left[-\frac{\partial p}{\partial V}\right)_T\Big]}$$

Ideal gas – $p = \frac{NkT}{V}$, $-\frac{\partial p}{\partial V} = \frac{NkT}{V^2}$

$$\langle N^2\rangle - \langle N\rangle^2 = NkT\rho\left[\frac{1}{V}\,\frac{1}{\frac{NkT}{V^2}}\right] =$$

$$= \rho V$$
$$= N$$

$(\langle N^2\rangle - \langle N\rangle^2 = N$ for ideal gas –$)$

return to non-ideal case in a moment, –

First – Blue sky

Blue sky - scattering ~~of~~ on density fluctuations of air molecules.

Assume dipole moment $\vec{p}_\perp = \gamma E_0 \hat{\epsilon}_0$

γ = molecular polarizability (units a^3 - recall $\frac{\epsilon-1}{\epsilon+2} a^3$)

CGS dielectric constant $\epsilon = 1 + 4\pi \gamma_{mol} \times \rho$

ρ = density = $\frac{\text{\# of molecules}}{\text{unit volume}}$; $\gamma_{mol} \cdot \rho = a^3 \cdot \frac{1}{a^3}$

$$\frac{d\sigma}{d\Omega} = k^4 \gamma_{mol}^2 \, |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 \, F(q)$$

$|\epsilon^* \cdot \epsilon_0|^2 \to \frac{8\pi}{3}$, incoherent scattering - or

scattering on fluctuations - $F(q) = N$

$$\sigma = \frac{8\pi}{3} k^4 \gamma_{mol}^2 \cdot N \equiv \sigma_0 \cdot N$$

or (mass on spring problem) $\sigma = \frac{8\pi}{3} r_e^2 \left(\frac{\omega}{\omega_0}\right)^4 \cdot N$

blue sky : $\sigma_{blue} \gg \sigma_{red}$

red sunset : $\lambda = \frac{1}{\rho \sigma_0} \sim \frac{1}{k^4}$; $\lambda_{red} \gg \lambda_{blue}$

attenuation length

Avogadro's number

1900 issue - what is it?

(equivalently what is $\rho = \frac{\text{atoms}}{\text{volume}}$ ~~[illegible]~~)

~~express~~ what is γ_{mol}?

eliminate micro physics quantities in terms of macroscopic ones.

$$\cancel{\gamma} = \cancel{\quad} \gamma_{mol} = \frac{\epsilon - 1}{4\pi\rho}$$

index of refraction $n = \sqrt{\epsilon}$

$$\cancel{n^2} \; \epsilon = n^2$$

$$\cancel{n^2} \; \epsilon - 1 = n^2 - 1 = (n-1)(n+1)$$
$$\simeq 2(n-1)$$

$$\gamma_{mol} = \frac{n-1}{2\pi\rho}, \quad \sigma_0 = \frac{8\pi}{3} k^4 \frac{(n-1)^2}{4\pi^2\rho^2}$$

$$\sigma_0 = \frac{2}{3\pi}\left(\frac{n-1}{\rho}\right)^2 k^4$$

$$\lambda = \frac{1}{\rho\sigma_0} \; ; \text{ given known } n, \lambda$$

$\Rightarrow \rho$

$\Rightarrow$ Avogadro's #.

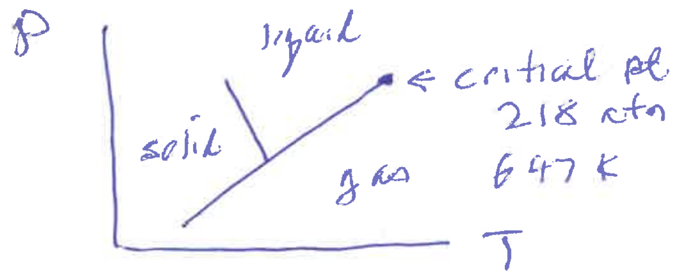
Critical Opalescence

$F(q) = \langle N^2\rangle - \langle N\rangle^2 = NkT\rho k_T$

ρ = density k_T = isothermal compressibility

$$k_T = \frac{1}{V}\,\frac{1}{\left[-\frac{\partial P}{\partial V}\right)_T\Big]}$$

real ~~gas of~~ material (water)



Water – first order lines
2nd order critical pt

at $*$, $\left.\frac{\partial P}{\partial V}\right)_T = 0$, k_T diverges, $F(q)$ diverges as $q \to 0$

At a special P, T, $\frac{d\sigma}{d\Omega}$ becomes very large

This is called "critical opalescence" – it is associated with existance of 2nd order critical point.

Aside in Grand Canonical Ensemble

$$\langle N^2\rangle - \langle N\rangle^2 = kT \cdot V \cdot \rho \cdot \frac{\partial \rho}{\partial P}\Big|_T$$

ρ = density

$\rho V = N$

$$\langle N^2\rangle - \langle N\rangle^2 = NkT\,\rho \cdot \left[\frac{1}{\rho}\frac{\partial \rho}{\partial P}\Big|_T\right]$$

$$\equiv NkT\,\rho\,\kappa_T$$

$$\kappa_T = \frac{1}{\rho}\frac{\partial \rho}{\partial P}\Big)_T \quad \alpha \quad -\frac{1}{V}\frac{\partial V}{\partial P}\Big)_T = -\frac{1}{V\left(\frac{\partial P}{\partial V}\right)_T}$$

$$= -\frac{N}{V}\frac{\partial\left(\frac{V}{N}\right)}{\partial P} = -\rho\frac{\partial\left(\frac{1}{\rho}\right)}{\partial P} = \frac{\rho}{\rho^2}\frac{\partial \rho}{\partial P}$$

Ideal gas $P = \frac{NkT}{V}$

$$-\frac{\partial P}{\partial V} = \frac{NkT}{V^2}$$

$$\langle N^2\rangle - \langle N\rangle^2 = NkT\rho\left[\frac{1}{V}\frac{1}{\left(\frac{NkT}{V^2}\right)}\right]$$

$$= \rho V = N\,!$$

Last remarks about scattering

The homework problem:

1) "classical electron radius": $\frac{e^2}{r} = mc^2$, $r = \frac{e^2}{mc^2} = \frac{e^2}{\hbar c}\frac{\hbar c}{mc^2}$

$r_e \sim 10^{-13}$ cm

2)
$$\frac{d\sigma}{d\Omega} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 |\epsilon^*\cdot\epsilon_0|^2 \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \frac{\omega^2\Gamma^2}{m^2}}$$

$\omega \ll \omega_0$ $\quad \sigma = \frac{8\pi}{3} r_e^2 \left(\frac{\omega}{\omega_0}\right)^4$

$\omega \gg \omega_0$ $\quad \sigma = \frac{8\pi}{3} r_e^2$

either way $\ll 10^{-26}\ \mathrm{cm}^2$

3) In QM, similar formula: ω_0 replaced (roughly) by energy differences of atomic levels

$$T_{fi} = \frac{m}{\hbar}\sum_n \frac{\langle f|\epsilon\cdot x|n\rangle\langle n|\epsilon_0\cdot x|i\rangle}{\omega - \omega_{ni}} - \frac{\langle f|\epsilon_0\cdot x|n\rangle\langle n|\epsilon\cdot x|i\rangle}{\omega + \omega_{ni}}$$

$$\frac{d\sigma(i\to f)}{d\Omega} \sim r_e^2\,\omega^4\,|T_{fi}|^2$$

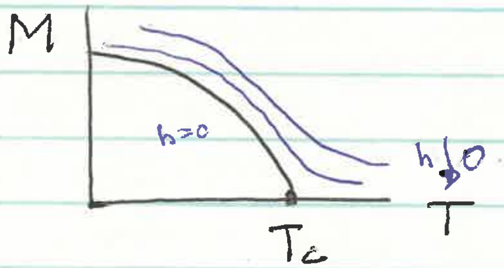
Now critical opalescence.

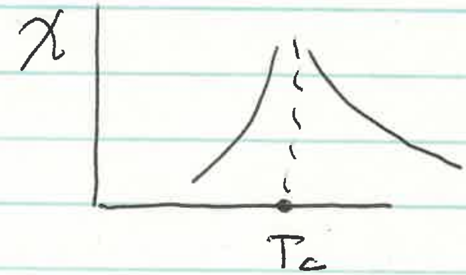
$$\frac{\hbar^2}{m s^2} = \frac{\hbar^2}{m c^2}$$

$F_{spin}(q) = \int d^3r\, e^{i q\cdot r} \langle(\sigma(r)-\bar\sigma)(\sigma(0)-\bar\sigma)\rangle$ CS-10

$\lim_{q\to 0} F_{spin}(q) = \langle\sigma^2\rangle - \langle\sigma\rangle^2 = \chi = \partial M/\partial h|_{h=0}$, $M = \langle\sigma\rangle$

Similar behavior in magnetic system





$\chi \sim |T-T_c|^{-\gamma}$ so F_{spin} becomes very large as $T \to T_c$.

Finally, useful to know:

$$F(q) \sim \int \langle \delta n(x)\delta n(0)\rangle e^{iq\cdot x} d^3x$$

diverges at critical point – $\langle\delta n(x)\,\delta n(0)\rangle$ very broad

Ornstein-Zernike parameterization

$$\langle \delta n(r)\delta n(0)\rangle \sim \frac{e^{-r/\xi}}{r} \qquad \xi \equiv \text{"correlation length"}$$

$$F(q) = \frac{1}{q^2 + \frac{1}{\xi^2}} \qquad \text{do the math!}$$

$\xi \to \infty$ at critical point, $F(q=0)$ diverges as $q \to 0$

(long range correlation at critical pt.)