Electricity & Magnetism II

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MWF 1-150

Grading
homework 250
midterm 100
final 150
500

Text Jackson
On reserve Landau & Lifshitz Classical theory of fields
Electrodynamics of continuous media
Born & Wolf "Optics"
Ryder QFT
Low Classical Field Theory
→ Thorne & Blandford "Modern Classical Physics"

Homework: Out Monday, due Weds 10 days later
Office hours: M 2-4:25
T 1-5

~~Final Monday May 5 1030~~

Other writing on web site
1) How light interacts with matter" - V Weisskopf, Sci Am
2) sophomore level special relativity notes
3) Link to translation of Einstein's 1905 SR article
4) Link to my QM notes
will add lecture notes for non-Jackson physics

Outline

radiation from

(~~3rd ed~~ ~~Ranges~~)

* I. $e^{i\omega t}$ sources without relativity 9.1-9.4 ~~9.P-9.4~~

* II. Scattering + diffraction 10.1-2 ~~10.3 10.3 10.4~~

10.5-10.8, 10.11

I mid term

III Special relativity Ch 11-12 - plus additional material

- done simply
- 4-vectors and all that
- covariant formalism of electrodynamics ($F_{\mu\nu}$)
- particle ~~motion in external fields (into radiation)~~
- useful "modern" classical field theory - Goldstone bosons, Higgs effect

IV Radiation and relativity - Ch 14-15 (selected topics)

- Larmor formula
- Synchrotron radiation (sparsely)
- Bremsstrahlung (~~sparsely~~ even more sparsely)

V Simple Quantum Field Theory

- quantizing the free EM field
- interaction of radiation + matter -
- radiative transitions in QM - recap I

* both uses MKS

I will use Gaussian units all semester.

Units

A real pain! Jackson continues MKS through Ch 10, then converts to CGS. I think I will make the switch now...

(+) simpler formulas, easy error checks (−) book is MKS
(+) connection w/ QM formulas (−) something new to learn
(~~[illegible]~~ ...)
(+) For most interesting ~~physical~~ ~~[illegible]~~ questions, it won't matter — Antenna pattern $\frac{1}{P}\frac{dP}{d\Omega}$

Cross sections $d\sigma/d\Omega$

Annoying part of CGS is for ~~di~~electrics, but we are really only going to be solving the wave eqn

Dictionary

	MKS	both	CGS
1)	$\nabla\cdot B = 0$		$\nabla\cdot B = 0$
2)	$\nabla\cdot E = \frac{\rho}{\epsilon_0}$	$\vec{E} = -\vec{\nabla}\phi$	$\nabla\cdot E = 4\pi\rho$

electrostatics $E = -\nabla\varphi \Rightarrow \nabla^2\varphi = -\frac{\rho}{\epsilon_0}$ or $\nabla^2\varphi = -4\pi\rho$

$\varphi(r) = \frac{q}{r}$ for pt charge

$\nabla^2 G(x,x') = -4\pi\delta^3(x-x')$

3) $\nabla\times B - \mu_0\epsilon_0\frac{\partial E}{\partial t} = \mu_0 J$

$\mu_0\epsilon_0 = \frac{1}{c^2}$

$\nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$ $\qquad$ $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$

note E & B have same units

Faraday $\nabla \times E + \frac{\partial B}{\partial t} = 0$ $\quad\quad$ $\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$

In both cases $\vec{B} = \vec{\nabla} \times \vec{A}$

$$E = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t} \qquad\qquad E = -\nabla \varphi - \frac{1}{c}\frac{\partial A}{\partial t}$$

again $\frac{\partial}{\partial x}, \frac{1}{c}\frac{\partial}{\partial t}$

note φ and A have same units.

(φ and $\vec{A}$ are components of a 4 vector, in rel. treatment, E & B components of rank 2 tensor)

~~Lorentz gauge~~

Gauge transformations

$$A \to A + \nabla \chi \qquad\qquad \vec{A} \to \vec{A} + \nabla \chi$$

$$\varphi \to \varphi - \frac{\partial \chi}{\partial t} \qquad\qquad \varphi \to \varphi - \frac{1}{c}\frac{\partial \chi}{\partial t}$$

Lorentz gauge becomes a bit more "natural"

$$\nabla \cdot A + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0 \qquad\qquad \nabla \cdot A + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0$$

In this semester we won't need the constitutive relations all that much (annoying 4π's) —

$$D = \epsilon E \qquad\qquad D = \epsilon E$$

$$D = \epsilon_0 E + P \qquad\qquad D = E + 4\pi P$$

$$B = \mu H \qquad\qquad B = \mu H$$

$$B = \mu_0 H + M \qquad\qquad B = H + 4\pi M$$

note in CGS all the fields have the same dimension — unlike MKS.

~~$\epsilon_0 = \mu_0 = 1$~~ $\quad$ $\epsilon = \mu = 1$ in free space

Gauge transformations

$A \to A + \nabla \chi$

$\varphi \to \varphi - \frac{\partial \chi}{\partial t}$

$A \to A + \nabla \chi$

$\varphi \to \varphi - \frac{1}{c}\frac{\partial \chi}{\partial t}$

Lorentz gauge is a bit more "natural"

$\nabla \cdot A + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$

$\nabla \cdot A + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0$

The constituent relations, other than

$D = \epsilon E$, $B = \mu H$

are annoying - but note for CGS, in free space $\epsilon = \mu = 1$.

Energy ~~dens~~ density - Poynting vector

$u = \frac{\epsilon_0}{2} E^2 + \frac{\mu_0}{2} H^2$

$\frac{1}{8\pi}(E^2 + B^2)$

$= \frac{1}{2}(E \cdot D + B \cdot H)$

$\frac{1}{8\pi}(E \cdot D + B \cdot H)$

$\vec{S} = \vec{E} \times \vec{H}$

$S = \frac{c}{4\pi}(\vec{E} \times \vec{H})$

For harmonic fields, usual $\frac{1}{2}$'s

$S \xrightarrow[\text{time ave}]{} \frac{c}{4\pi} \cdot \frac{1}{2}(\vec{E} \times \vec{H}^*)$

$u \to \frac{1}{16\pi}(E^2 + B^2)$

Lets find $\vec{E}$ + $\vec{B}$ for plane waves in free space in CGS

$$\vec{\nabla}\cdot\vec{E} = 0 \quad , \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t} \quad \left(\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \right)$$

$$\nabla\times \quad \left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0 \right)$$

$$\Downarrow$$

$$-\nabla^2 B - \frac{1}{c}\nabla\times\frac{\partial E}{\partial t} = 0$$

$$-\nabla^2\vec{B} + \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\begin{matrix}\vec{E}\\ \vec{B}\end{matrix}\right) = 0$$

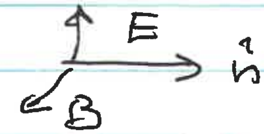
so if $\vec{E} = \hat{\epsilon}\, E_0\, e^{i(k\cdot x - \omega t)}$

- $k^2 - \frac{\omega^2}{c^2} = 0$
- $\vec{k}\cdot\hat{\epsilon} = 0$ from $\nabla\cdot E = 0$

$$B = \vec{B}_0\, e^{i(k\cdot x - \omega t)}$$

$$-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = \frac{i\omega}{c}\vec{B}_0 = \vec{\nabla}\times\vec{E} = i(\vec{k}\times\hat{\epsilon})\,E_0$$

or $\quad \vec{B} = \hat{n}\times\vec{E}\ \left(\times\frac{ck}{\omega}\right)$



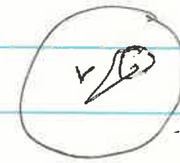
$\vec{H} = B = \hat{n}\times\vec{E}$ since $\omega = ck$

$$\vec{S} = \frac{1}{2}\frac{c}{4\pi}\vec{E}\times\vec{H}^* = \frac{c}{8\pi}|E_0|^2\,\hat{n}$$

$$u = \frac{1}{8\pi}|E_0|^2$$

The typical question we ask uses $\vec{S}$: what is the angular distribution of power radiated from ...

$$\frac{dP}{d\Omega} = r^2 \hat{n} \cdot \vec{S}$$



$$\frac{1}{P}\frac{dP}{d\Omega} \Rightarrow \text{"antenna pattern"}$$

Green's function for wave eqn - no boundaries

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -4\pi f(\vec{x}, t)$$

A Green's function would obey

$$\left(\nabla^2_\psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}\right) G(x,t;x',t') = -4\pi\, \delta^3(x-x')\,\delta(t-t')$$

and $\psi(x,t) = \int d^3x' d^3t'\, G(x,t;x',t')\, f(x',t')$

Many ~~paths~~ paths to an answer. Let's do one path by going into - and out of - ω, $\vec{k}$ space.

Define $R = x - x'$, $T = t - t'$, assume G depends on R & T.

$$G(R,T) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, e^{i\vec{k}\cdot\vec{R}} e^{-i\omega T}\, G(k,\omega)$$

$$\delta^3(R)\,\delta(T) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega\, e^{i\vec{k}\cdot\vec{R}} e^{-i\omega T}$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial T^2}\right] G(R,T) = \frac{1}{(2\pi)^4}\int d^3k\, d\omega \times\left[-k^2 + \frac{\omega^2}{c^2}\right] \times G(k,\omega) \times e^{+ik\cdot R} e^{-i\omega t}$$

$= \frac{1}{(2\pi)^4}$ (recess)

$$\left[-k^2 + \frac{\omega^2}{c^2}\right] G(k,\omega) = -4\pi$$

$$G(k,\omega) = \frac{4\pi}{k^2 - \frac{\omega^2}{c^2}} \quad \text{- almost!}$$

Now invert the FT in stages

$$G(R,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\left[\frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot\vec{R}}\,\frac{4\pi}{k^2-\frac{\omega^2}{c^2}}\right]$$

call $[\;] = G(R,\omega)$, 3-d Fourier transform

useful way to attack:

~~[illegible]~~. Pick z axis of coordinate system along R

$$d^3k = k^2\,dk\,d\cos\theta\,d\varphi$$

$$G(R,\omega) = \frac{4\pi}{(2\pi)^3}\int_0^{\infty}\frac{2\pi k^2\,dk}{k^2-\frac{\omega^2}{c^2}}\int_{-1}^{1} d\cos\theta\; e^{ikR\cos\theta}$$

$$= \frac{4\pi}{(2\pi)^2}\int_0^{\infty}\frac{k^2\,dk}{k^2-\frac{\omega^2}{c^2}}\left[\frac{e^{ikR}-e^{-ikR}}{ikR}\right]$$

$$= \frac{1}{i\pi R}\int_{-\infty}^{\infty}\frac{k\,dk\; e^{ikR}}{k^2-\frac{\omega^2}{c^2}} = \frac{1}{i\pi R}\int_{-\infty}^{\infty}\frac{k\,dk\; e^{ikR}}{(k-\frac{\omega}{c})(k+\frac{\omega}{c})}$$

Hmm - singularities! In order to define the integral, we move the poles off axis. (We can do this in several ways.). Cauchy then does the integral. We will then need a story - why is this sensible?

1) $\frac{\omega}{c} \to \frac{\omega}{c} + i\epsilon$; poles at $k = \frac{\omega}{c} + i\epsilon$, $k = -\frac{\omega}{c} - i\epsilon$

Look

To convert $\int_{-\infty}^{\infty}$ to $\oint$ with no contribution from the semicircle, notice $\exp iR(ik) \to 0$ in UHP. This encloses one pole at $k = \frac{\omega}{c} + i\epsilon$

$$G^{(+)}(R,\omega) = \frac{2\pi i}{i\pi R}\left(\frac{\omega + i\epsilon}{c}\right)\frac{e^{iR\left(\frac{\omega}{c} + i\epsilon\right)}}{2\frac{\omega}{c} + i\epsilon}$$

$$\xrightarrow[\epsilon \to 0^+]{} \frac{1}{R} \exp i\frac{\omega R}{c}$$

$$G^{(+)}(R,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega T}\,\frac{e^{i\frac{\omega R}{c}}}{R}$$

$$= \frac{\delta(T - R/c)}{R}$$

Signal at time $T = R/c$.

- this is called a "retarded" Green's fn ($t - t' = \frac{|x - x'|}{c}$ so $t > t'$) wave
- it's an outgoing wave from (x', t') to (x, t)

(also note, $G \sim \frac{1}{R}$)

With $G^{(+)}$

$$\psi(x,t) = \int d^3x'\, dt\; f(x',t')\, G(x-x', t-t')$$

$$= \int d^3x' \frac{f\left(x', t' = t - \frac{|x-x'|}{c}\right)}{|\vec{x}-\vec{x}'|}$$

Emission out of the past, signal takes time $c(t-t') = |\vec{x}-\vec{x}'|$ to arrive

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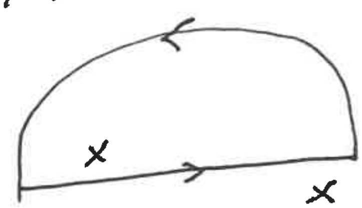
Another possibility: $\frac{\omega}{c} \to \frac{\omega}{c} - i\epsilon$



$$G^{(-)}(R,\omega) = \frac{2\pi i}{i\pi R} \frac{\left(-\frac{\omega}{c} + i\epsilon\right) \times \exp\left(-i\frac{\omega R}{c}\right)}{\left[-2\frac{\omega}{c}\right]}$$

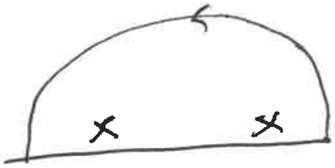
$$= \frac{1}{R} \exp\left(-i\frac{\omega}{c}R\right)$$

$$G^{(-)}(R,T) = \delta\left(T + \frac{R}{c}\right)\frac{1}{R}$$

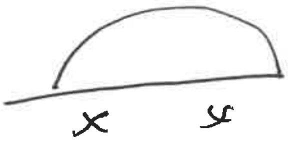
called an "advanced wave" -

$$t - t' + \frac{|x-x'|}{c} = 0$$

$t' > t$: source emits backwards in time



a mix of advanced + retarded



= 0

It's a 2nd order equation - from point of view of math, both solutions are present. See p.245 for (crazy) discussion - when might you use the advanced solution. (I've never encountered this)

Simple radiating systems : in Lorentz gauge

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\begin{pmatrix}\vec{A}\\ \Phi\end{pmatrix} = -\frac{4\pi}{c}\begin{bmatrix}\vec{J}\\ c\rho\end{bmatrix}$$

has $\vec{A}(x,t) = \frac{1}{c}\int d^3x'\,dt'\; G^{(+)}(x,t;x',t')\,\vec{J}(x',t')$

as a formal solution. Select $G^{(+)}$ for physical reason – causality.

$$\vec{A}(x,t) = \frac{1}{c}\int d^3x' \frac{1}{|x-x'|}\vec{J}\left(x', t' = t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

This can be difficult to integrate – so go immediately to special cases –

a very important one

$$\vec{J}(x,t) = \vec{J}(x)\,e^{-i\omega t}$$
$$\rho(x,t) = \rho(x)\,e^{-i\omega t}$$

Then $\vec{A}(x,t) = A(x)\,e^{-i\omega t}$ also

Follows from $J(x')\,e^{-i\omega\left[t - \frac{|x-x'|}{c}\right]}$ ~~$= J(x')e^{-i\omega t'}$~~

$\uparrow t'$

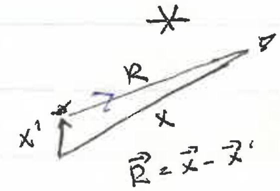
$J(x',t') = J(x')\,e^{-i\omega t'} =$

$$\vec{A}(x,t) = \frac{1}{c}\int d^3x' \, \frac{J(x')}{R} \, e^{-i\omega(t-R/c)}$$

$$= e^{-i\omega t} \, \frac{1}{c}\int d^3x' \, J(x') \, \frac{e^{ikR}}{R}$$

$\omega = ck$ - so $\vec{A}$ has the same time dependence as J,
$\vec{A}(x,t) = \vec{A}(x)\, e^{-i\omega t}$

$$\boxed{\vec{A}(x) = \frac{1}{c}\int d^3x' \, \vec{J}(x') \, \frac{e^{ikR}}{R}}$$



Most of Ch 9 begins with this formula.
There is a (identical) similar formula for Φ, but if we have

1) harmonic time dependence

2) away from sources, so $J = 0$

we dont need Φ.

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\vec{B} = \overbrace{\frac{4\pi}{c}\vec{J}}^{0} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{i\omega}{c}\vec{E} = -ik\vec{E}$$

$$A \to B \to E \to \vec{S}$$

Evaluating $\vec{A}(x)$ is still hard! so we go immediately to yet more special cases. What might interesting cases be?

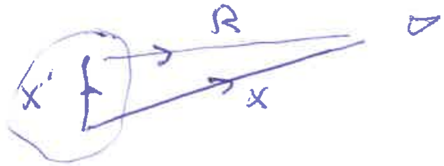
---

1) d = size of source, r = distance from source
Far from source? $r \gg d$

2) $d \ll \lambda$? (atom $d \sim Å$, $\lambda \sim 5000 Å$)
If $d \ll \lambda$ $\quad \frac{d}{c} \ll \frac{\lambda}{c} = T$ (period $\sim \frac{1}{\omega}$)
$d/T \ll c$, $\omega \sim \frac{1}{T}$ $\quad \omega d \ll c$ or $v \ll c$: non relativistic motion

3) $r \gg \lambda$ (many wavelengths from atom)

$$\vec{A}(x) = \frac{1}{c}\int d^3x'\, \vec{J}(x')\, \frac{e^{ikR}}{R}$$



$$\vec{R} = \vec{x} - \vec{x}'$$

There are 3 relevant distance scales

$r = |x| =$ distance to receiver

$\lambda = \frac{2\pi}{k} =$ wavelength of radiation

$d =$ size of source

Most often encounter

- Far-field or radiation zone or far zone

  $r \gg \lambda$ or $kr \gg 1$

  will find $E, B \sim 1/r$.

- $\lambda \gg d$ "long wavelength limit"

example - atoms - $d \sim$ Å, $\lambda =$ 1000's of Å

note - if $d \ll \lambda$

$$\frac{d}{c} \ll \frac{\lambda}{c} \equiv T = \text{period} \sim \frac{1}{\omega}$$

$$\frac{d}{c} \ll \frac{1}{\omega} \quad \text{or}$$

$$\omega d \ll c$$

$v \ll c$: nonrelativistic motion

nb 300 Mhz = 1 m, cell phones a few Ghz

Near zone: $\lambda \gg r \gg d$ or $kR = \frac{2\pi r}{\lambda} \ll 1$

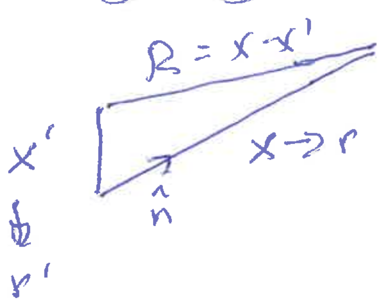
$$\vec{A}(x) \sim \frac{1}{c}\int d^3x' \, \frac{J(x')}{|x-x'|}$$

$A + \Phi$ given by static eqns, just with overall $e^{-i\omega t}$ dependence

ex: avalanche beacon: $\nu = 457$ kHz

$$\lambda = \frac{3\times 10^8 \text{ m/s}}{0.457\times 10^6/\text{s}} \sim 600 \text{ m.}$$

In the far zone the exponential dominates everything. If the size of the source is small



$$R = |\vec{x}-\vec{x}'| = \left[r^2 + r'^2 - 2\vec{x}\cdot\vec{x}'\right]^{1/2}$$

$$\simeq r - \frac{1}{r}\cdot\frac{1}{2}\, 2x\cdot x'$$

$$= r - \hat{n}\cdot\vec{x}'$$

$$A(\vec{x}) = \frac{e^{ikr}}{cr}\int d^3x' \, e^{-ik\hat{n}\cdot x'} J(x')$$

$\frac{e^{ikr}}{r}$ - generic formula for outgoing spherical wave.

Further, if $kd \ll 1$, expand the exponential.

$$\vec{A}(x) = \frac{e^{ikr}}{cr} \sum_{l} \frac{1}{l!} \int d^3x' \, (-ik\hat{n}\cdot\vec{x}')^l \, \vec{J}(x') \qquad *$$

This is related to a multipole expansion: $\hat{n}\cdot x' = r'\cos\theta$

lth term $\sim (kd)^l = \left(2\pi \frac{d}{\lambda}\right)^l \ll 1$

i.e. d/λ controls convergence – $\frac{1}{1000}$ for atoms.

Let's look at the multipole expansion, one term at a time.

0) Electric monopole moments don't radiate
(back to basics)

$$\Phi(x,t) = \int d^3x' \, \rho(x', t-\tfrac{R}{c}) \frac{1}{R}$$

[I cheated, but $A \leftrightarrow J$, $\Phi \leftrightarrow \rho$ = same eqn]

$$\frac{1}{R} = \frac{1}{r}\sum_l \left(\frac{r'}{r}\right)^l P_l(\cos\theta) \rightarrow \frac{1}{r} \text{ for } l=0$$

$$\Phi_0(x,t) = \frac{1}{r}\int d^3x' \, \rho(x', t-R/c) = \frac{Q(t-R/c)}{r}$$

$= \frac{Q}{r}$ since the total charge is independent of t.

$\rightarrow$ Charge monopole is static

or

Fields behaving as $e^{-i\omega t}$ have no monopole term

No radiation, $E \sim \frac{1}{r^2}$

1) Electric dipole radiation

Consider $l=0$ term in expansion for A

$$\vec{A}(x) \to \vec{A}_{dipole}(x) = \frac{e^{ikr}}{cr} \int d^3x' \, \vec{J}(x')$$

$\vec{J}$ often poorly known - trade this for ρ

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = i\omega\rho$$ ☆

$$\int d^3x \, J_i(x) = \int d^3x \left[\vec{\nabla} x_i\right] \cdot \vec{J}$$ trick

$$= \int d^3x \, \vec{\nabla} \cdot (x_i \vec{J}) - \int d^3x \, x_i \vec{\nabla} \cdot \vec{J}$$

First term is surface integral. If no boundaries, go far away, $J \to 0 \Rightarrow 0$. 2nd term is ☆

$$\int d^3x \, \vec{J}(x) = -i\omega \int d^3x \, \rho(x) \vec{x}$$

$$= -i\omega \vec{p}$$ usual electric dipole moment!

$$\boxed{\vec{A}(x) = -ik \vec{p} \frac{e^{ikr}}{r}}$$

$$\vec{A}(x) = -ik\,\vec{p}\,\frac{e^{ikr}}{r}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -ik\left(\vec{\nabla}\,\frac{e^{ikr}}{r}\right)\times\vec{p}$$

$r = \sqrt{x^2+y^2+z^2}$

Note $\vec{\nabla} r = \sum_j \hat{j}\,\frac{x_j}{r}$ so $\vec{\nabla} r\cdot\vec{\nabla} r = \sum_j \frac{x_j^2}{r^2} = 1$

this means $\vec{\nabla} r = \hat{n}$, unit vector along r.

$$\nabla\frac{e^{ikr}}{r} = \hat{n}\left[ik - \frac{1}{r}\right]\frac{e^{ikr}}{r} = \hat{n}\,ik\left[1 + \frac{i}{kr}\right]\frac{e^{ikr}}{r}$$

$kr \gg 1$ in far field - drop the last term.

(replace $\vec{\nabla}$ by $ik\hat{n}$!)

$$\vec{B} = k^2\,\frac{e^{ikr}}{r}\,\hat{n}\times\vec{p}$$ note $\frac{1}{r}$

$$\vec{E} = \frac{i}{k}\,\vec{\nabla}\times\vec{B} = -\hat{n}\times\vec{B} = \vec{B}\times\hat{n}$$

$$= -k^2\,\frac{e^{ikr}}{r}\left[\hat{n}\times(\hat{n}\times\vec{p})\right]$$ note $\frac{1}{r}$

$$(\hat{n}\times(\hat{n}\times\vec{p}))_i = \epsilon_{ijk}\, n_j\,(n\times p)_k = \epsilon_{ijk}\epsilon_{k\ell m}\, n_j n_\ell p_m$$

$$= \epsilon_{ijk}\epsilon_{\ell mk}\, n_j n_\ell p_m$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\, n_j n_\ell p_m$$

$$= n_i\,\hat{n}\cdot\vec{p} - \vec{p}_i\, n\cdot n$$

$$\vec{E} = -k^2\,\frac{e^{ikr}}{r}\left[\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}\right]$$

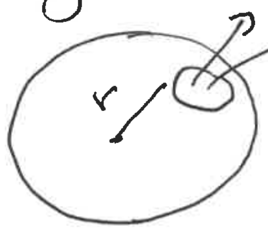
can check $\vec{B} = \hat{n}\times\vec{E}$ — its only the 2nd term)

note $\hat{n}\cdot\vec{E} = \hat{n}\cdot\vec{B} = 0$

Poynting vector

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \mathrm{Re}\, \vec{E} \times \vec{B}^* = \text{time averaged flux.}$$

Integrate over a big sphere



$$\frac{dP}{d\Omega} = r^2\, \hat{n} \cdot \langle \vec{S} \rangle$$

$$= r^2 \frac{c}{8\pi}\, \hat{n} \cdot (\vec{E} \times \vec{B}^*)$$

$$\hat{n} \cdot (\vec{E} \times \vec{B}^*) = n_i\, \epsilon_{ijk} E_j (n \times E^*)_k$$

$$= n_i\, \epsilon_{ijk} \epsilon_{klm} E_j n_l E^*_m$$

$$= n_i\, \epsilon_{ijk} \epsilon_{lmk}\, n_l E_j E^*_m$$

$$= n_i (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, n_l E_j E^*_m$$

$$= n_i \left[n_i |E|^2 - E^*_i (n \cdot E) \right] = |E|^2$$

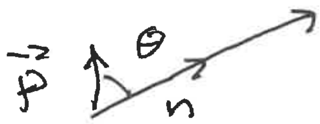
$$\frac{dP}{d\Omega} = r^2 \frac{c}{8\pi} |E|^2$$

$$r^2 |E|^2 = k^4 \left| \hat{n}(\hat{n} \cdot \vec{p}) - \vec{p} \right|^2 =$$

$$= k^4 \left[|\hat{n} \cdot \vec{p}|^2 - 2 |n \cdot p|^2 + |\vec{p}|^2 \right]$$

$$\frac{dP}{d\Omega} = \frac{c k^4}{8\pi} \left[|\vec{p}|^2 - |\hat{n} \cdot \vec{p}|^2 \right]$$

Special case: $\vec{p}$ fixed in space

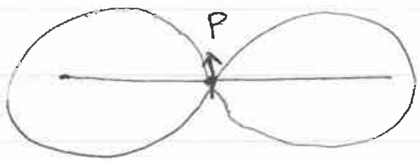


$$\vec{p}\cdot\hat{n} = p \cdot \cos\theta$$

$$\frac{dP}{d\Omega} = \frac{c k^4}{8\pi} |p|^2 \sin^2\theta$$

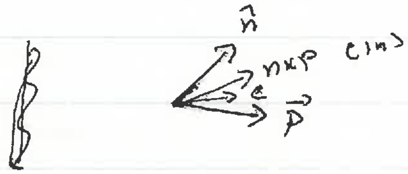
and $\int_{-1}^{1} d\cos\theta\,[1-\cos^2\theta] = 2 - \frac{2}{3} = \frac{4}{3}$

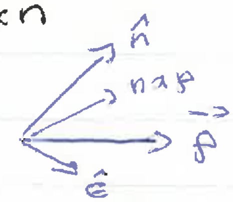
$$P = \frac{8\pi}{3}\,\frac{c}{8\pi}\,|p|^2 k^4$$ total power radiated



antenna pattern

Polarization of radiation? Thats along the direction of $\vec E$: $\quad \vec\epsilon \propto \vec E \sim (n\times p)\times n = \hat n(\hat n\cdot\hat p) - \vec p$





We could also measure the intensity of radiation in a particular direction, with a particular polarization

$$\vec E = \sum_\lambda \hat\epsilon_\lambda(\hat\epsilon^*_\lambda\cdot\vec E) = \sum \hat\epsilon_\lambda E_\lambda \quad\Big|\quad |n\rangle = \sum_\lambda |\lambda\rangle\langle\lambda|n\rangle$$

$E(\hat\epsilon) = \hat\epsilon^*\cdot\vec E$ projects on $\hat\epsilon$ polarization

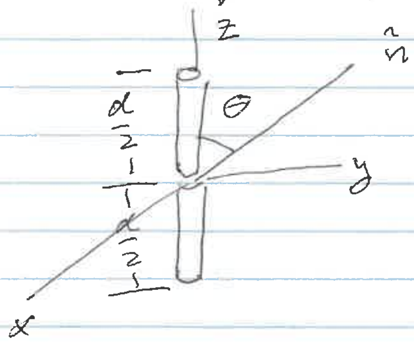
$$\frac{dP}{d\Omega} = r^2\hat n\cdot(\vec E\times\vec B^*) = n\cdot(E\times(n^*\times E^*))$$

$$\sim |n\times E|^2 \sim |E|^2 \quad \text{since } n\cdot E = 0$$

$$\frac{dP(\epsilon)}{d\Omega} = \frac{c r^2}{8\pi}|\epsilon^*\cdot\vec E|^2$$ for polarization $\vec\epsilon$.

$$\frac{dP}{d\Omega} = \frac{c r^2}{8\pi}|E|^2$$

Simple example - short antenna done badly

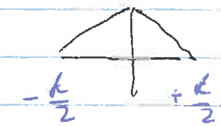


Make ansatz (you have to do this - otherwise it's a boundary value problem)

$$I = I(z)\, e^{-i\omega t} = \int J\, d^2x_\perp$$

$$I(z = \pm \tfrac{d}{2}) = 0$$

so $$I(z) = I_0 \left[1 - \frac{2|z|}{d} \right]$$



To find dipole moment

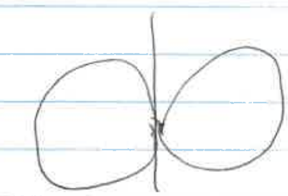
$$i\omega\rho = \frac{dI}{dz} =$$

$$\rho = \pm \frac{2 I_0}{\omega d}\, i \qquad + \text{top}, - \text{bottom}$$

$$p_z = \int_{-d/2}^{d/2} z\,\rho(z)\,dz = \frac{2 I_0}{\omega d}\, i \times 2 \times \frac{1}{2}\frac{d^2}{4} = i\,\frac{I_0 d}{2\omega}$$

$\omega = ck$ so

$$\frac{dP}{d\Omega} = \frac{c}{8\pi}\left(\frac{I_0 d}{2ck}\right)^2 k^4 \sin^2\theta$$



$$= \frac{I_0^2}{32\pi c}(kd)^2 \sin^2\theta$$

$P = \frac{1}{2} I_0^2 R$. R = "radiation resistance"

$$= \frac{(kd)^2}{6c}$$

R in ohms $= 30c \cdot R_{cgs} = 5(kd)^2\,\Omega$

half-wave antenna : $kd = \frac{2\pi}{\lambda} d = \pi$ (dangerous!)

$$R = 50\,\Omega$$

Magnetic dipole and Electric Quadrupole Radiation

Both come from the $\ell=1$ term in the multipole expansion

$$\vec{A}(x) = ik\frac{e^{ikr}}{cr}\int d^3x'\,(\hat{n}\cdot\vec{x}')\,\vec{J}(x')$$

$\nabla\cdot J + \frac{\partial\rho}{\partial t} = 0$

Note: 1) we'd like to trade in J for ρ. Recall though that $\vec{J} = \vec{J}_t + \vec{J}_L$ (transverse/longitudinal) and $\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot\vec{J}_L$. We have to treat J_t explicitly

$\nabla\cdot J_t = 0$

$\nabla\times J_L = 0$

2) Procedure is so messy we can't go further.

To proceed, observe random fact

$$\hat{n}\times(\hat{n}'\times\vec{J}) = \hat{n}'(\hat{n}\cdot\vec{J}) - \underbrace{(\hat{n}\cdot\hat{n}')}\vec{J}$$

$*$ term in the $\int x' \Leftrightarrow n'(x')$

shuffle x, $\frac{1}{2}x\frac{1}{2}J$

$$(\hat{n}\cdot\hat{n}')\vec{J} = \underbrace{\frac{1}{2}\left((\hat{n}\cdot\hat{n}')\vec{J} + (\hat{n}\cdot\vec{J})\hat{n}'\right)} - \frac{1}{2}\hat{n}\times(\hat{n}'\times J)$$

happens to be electric quadrupole and mag dipole

~~Magnetization~~ Magnetic moment is

$$\vec{m} = \int d^3x\,\vec{\mathcal{M}} \quad \text{w/}\quad \vec{\mathcal{M}}(x) = \frac{1}{2c}(\vec{x}'\times\vec{J})$$

c in CGS defn. so

$$\vec{A}(x) = ik(\hat{n}\times\vec{m})\,e^{ikr}/r$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = ik\,\hat{n}\times\vec{A} = -k^2\frac{e^{ikr}}{r}\,\hat{n}\times(\hat{n}\times\vec{m})$$

$$\vec{E} = \frac{i}{k}\vec{\nabla}\times\vec{B} = -\hat{n}\times\vec{B} = -k^2\frac{e^{ikr}}{r}\,\hat{n}\times\vec{m}$$

Compare angular forms

	E1	M1
$\vec{B}$	$\hat{n} \times \vec{p}$	$-\hat{n} \times (\hat{n} \times \vec{m})$
$\vec{E}$	$-\hat{n} \times (\hat{n} \times \vec{p})$	$-\hat{n} \times \vec{m}$

so it's easy to write down

$\frac{dP}{d\Omega}$	$\frac{ck^4}{8\pi} \lvert \hat{n} \times (\hat{n} \times \vec{p}) \rvert^2$	$\frac{ck^4}{8\pi} \lvert \hat{n} \times (\hat{n} \times \vec{m}) \rvert^2$
P (linear $\vec{p}, \vec{m}$)	$\frac{ck^4 p^2}{3}$	$\frac{ck^4 m^2}{3}$

Electric Quadrupole Radiation

$$\vec{A}(x) = \frac{ik\,e^{ikr}}{2cr}\int d^3x' \left[\underbrace{(\hat{n}\cdot\vec{x}')\vec{J}}_{\text{first term}} + \vec{x}'(\hat{n}\cdot\vec{J}) \right] \qquad (1)$$

$$\text{First term} = \int d^3x' (\hat{n}\cdot\vec{x}')\vec{J}_i = \int d^3x' (\hat{n}\cdot\vec{x}')\left[(\vec{\nabla}x_i')\cdot\vec{J}\right]$$

rearrange derivatives

$$= \int d^3x' \, \vec{\nabla}'\cdot\left[(\hat{n}\cdot\vec{x}')\, x_i' \vec{J}\right] - \int d^3x' \, x_i' \nabla'\cdot\left[(\hat{n}\cdot x')\vec{J}\right]$$

First piece is a surface integral, gives zero.

$$\nabla'\cdot\left[(\hat{n}\cdot\vec{x}')\vec{J}\right] = \{\vec{\nabla}'(\hat{n}\cdot\vec{x}')\}\cdot\vec{J} + (\hat{n}\cdot x')\,\vec{\nabla}'\cdot\vec{J}$$

$$= \hat{n}\cdot\vec{J} + (n\cdot x')\,\vec{\nabla}'\cdot\vec{J}$$

multiply by $-x_i'$, return to (1)

$$\vec{A} = \frac{ik\,e^{ikr}}{2cr}\int d^3x' \left[-\vec{x}'(\hat{n}\cdot\vec{J}) - \vec{x}'(\hat{n}\cdot\vec{x}')\,\nabla'\cdot J + \vec{x}'(\hat{n}\cdot\vec{J}) \right]$$

!

$\nabla'\cdot J = i\omega\rho$ so this integral is

$$-i\omega \int d^3x' \, \vec{x}'(\hat{n}\cdot\vec{x}')\rho(x) \qquad \omega = ck$$

$$\vec{A}_{E2} = \underbrace{-\frac{k^2}{2}\frac{e^{ikr}}{r}\int d^3x' \, \vec{x}'(\hat{n}\cdot\vec{x}')\rho(x)}$$

next page

It's easiest (and conventional) to work with

$$\vec{B}_{E2} = ik\,\hat{n}\times\vec{A}$$

and it also helps to know the answer in advance!

Recall Quadrupole tensor

$$Q_{ij} = \int d^3x' \, \rho(x') \left[3x_i' x_j' - \delta_{ij} x'^2 \right] \quad ; \text{ note } \sum_i Q_{ii} = 0$$

Define "quadrupole vector" $q_i(\vec{n}) = Q_{ij} n_j$

$$q_i(\hat{n}) = \int \left(3x_i' x_j' n_j - (x')^2 n_i \right) \rho \, d^3x'$$

$$= \int \left[3x_i' (x' \cdot n) - n_i x'^2 \right] \rho \, d^3x' \quad \text{or } \vec{q} = \int \left[3\vec{x}'(\vec{x}' \cdot \hat{n}) - x'^2 \hat{n} \right] \rho$$

Notice

$$\hat{n} \times \vec{q}(n) = \int \left[3 (\hat{n} \times \vec{x}')(\hat{n} \cdot \vec{x}') - x'^2 \underbrace{\hat{n} \times \hat{n}}_{\text{zero}} \right] \rho \, d^3x'$$

$(6\,\hat{n} \times \vec{A})$

i.e.

$$B_{E2} = \frac{-ik^3}{6} \frac{e^{ikr}}{r} \hat{n} \times \vec{q}(n)$$

$$E_{E2} = -\hat{n} \times B_{E2} = \frac{ik^3}{6} \hat{n} \times (\hat{n} \times \vec{q}(n)) \frac{e^{ikr}}{r}$$

This is a miracle! Look at the dipole formulas, they are the same, except $\vec{p} \Leftrightarrow \vec{q}(\hat{n})$

So

$$\frac{dP}{d\Omega} = \frac{ck^6}{288\pi} |\hat{n} \times (\hat{n} \times \vec{q})|^2 = \frac{ck^6}{288\pi} \left[|\vec{q}|^2 - |\hat{n} \cdot \vec{q}|^2 \right]$$

An actual antenna pattern can be very complicated. The total power radiated is not so bad.

$$|q|^2 = \sum_{\alpha\beta\gamma} Q^*_{\alpha\beta} Q_{\alpha\gamma} n_\beta n_\gamma$$

$$|n \cdot q|^2 = \sum_{\alpha\beta\gamma\delta} Q^*_{\alpha\beta} Q_{\gamma\delta} n_\alpha n_\beta n_\gamma n_\delta$$

$\delta_{\beta\alpha}$ is ~~[illegible]~~ ~~[illegible]~~ parity

$$\int n_\beta n_\gamma \, d\Omega = \delta_{\beta\gamma} \cdot \frac{4\pi}{3} \qquad \left(\int \sum_\gamma n_\gamma^2 \, d\Omega = 4\pi \right)$$

It happens, $\int n_\alpha n_\beta n_\gamma n_\delta \, d\Omega = \frac{4\pi}{15}\left[\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} \right]$

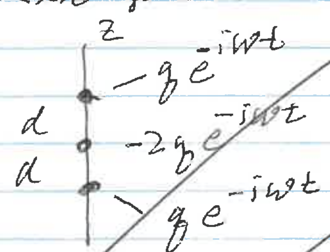
so
$$\int d\Omega \left[|q|^2 - |n\cdot q|^2 \right] = 4\pi \Big[\frac{1}{3} \sum_{\alpha\beta} |Q_{\alpha\beta}|^2 - \frac{1}{15}\Big(\underbrace{\sum_\alpha Q^*_{\alpha\alpha} \sum_\beta Q_{\beta\beta}}_{\text{zero}} + 2 \sum_{\alpha\beta} |Q_{\alpha\beta}|^2 \Big)$$

$$= \frac{4\pi}{15}\left[5-2\right] \sum_{\alpha\beta} |Q_{\alpha\beta}|^2 \qquad 4\pi \cdot \frac{3}{15}$$

$$P = \left(\frac{1}{288\pi} \; \frac{4\pi}{3} \right) c k^6 \sum_{\alpha\beta} |Q_{\alpha\beta}|^2$$

$$\frac{1}{360}$$

Example :



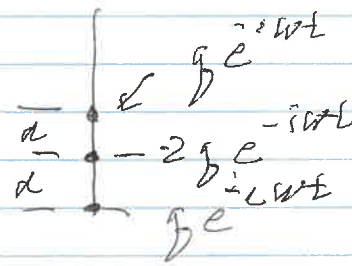
$$Q_{zz} = q\left[2\cdot 3d^2 - 2d^2 \right] = 4qd^2 = Q_0$$

$$Q_{xx} = q\left[0 - 2d^2 \right] = -2qd^2 = -\frac{1}{2} Q_0$$
$$= Q_{yy}$$

$$\frac{dP}{d\Omega} = \frac{ck^6}{288\pi}\left[|Q|^2 - |Q\cdot n|^2\right]$$

$$P = \frac{1}{360} ck^6 \sum_{\alpha\beta} |Q_{\alpha\beta}|^2$$

Example:



$$Q_{ij} = q\left[3x_i x_j - \delta_{ij} x^2\right]$$

$$Q_{zz} = q\left[2\cdot 3d^2 - 2d^2\right] = 4qd^2 \equiv Q_0$$

$$Q_{xx} = q\left[0 - 2d^2\right] = -2qd^2 = Q_{yy} = -\tfrac{1}{2} Q_0$$

$Q(n)_i = \sum_j \hat{Q}_{ij} \hat{n}_j$. Write $\hat{Q}_{ij} = \underbrace{\hat{e}_i \hat{e}_j}_{\text{unit vectors}} \underbrace{Q_{ij}}_{\text{numbers}}$

$$Q(n)_i = \sum_j \hat{e}_i (\hat{n}\cdot\hat{e}_j) Q_{ij}$$

Here Q is diagonal, $Q(n)_i = \hat{e}_i (\hat{n}\cdot\hat{e}_i) Q_{ii}$

$$|Q|^2 = \sum_i (\hat{n}\cdot\hat{e}_i)^2 Q_{ii}^2$$

$$n\cdot Q = \sum_i (\hat{n}\cdot\hat{e}_i)^2 Q_{ii}$$

$$|n\cdot Q|^2 = \left|\sum_i (n\cdot e_i)^2 Q_{ii}\right|^2$$

$$\begin{array}{l} |\vec{g}|^2 \\ - \\ |\hat{n}\cdot\vec{g}|^2 \end{array} = Q_0^2 \left\{ \overset{zz}{\cos^2\theta} + \frac{1}{4}\sin^2\theta\,(\overset{xx}{\cos^2\varphi} + \overset{yy}{\sin^2\varphi}) - \left[\cos^2\theta - \frac{1}{2}\sin^2\theta(\cos^2\varphi + \sin^2\varphi)\right]^2 \right\}$$

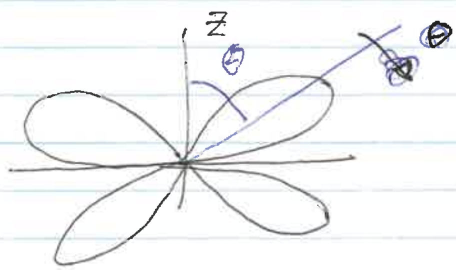
$$= Q_0^2 \left\{ \cos^2\theta + \frac{1}{4}\underbrace{\sin^2\theta}_{\frac{1}{4}(1-\cos^2\theta)} - \left[\cos^2\theta - \frac{1}{2}\sin^2\theta\right]^2 \right\}$$

$$= Q_0^2 \left[\frac{1}{4} + \frac{3}{4}\cos^2\theta - \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)^2 \right]$$

$$= \dots \frac{9}{4}\cos^2\theta\,(1-\cos^2\theta) = \frac{9}{4}\cos^2\theta\sin^2\theta$$

$$= \frac{9}{16}\sin^2 2\theta$$

$$\frac{dP}{d\Omega} = \frac{ck^6}{288\pi} Q_0^2 \cdot \frac{9}{6}\sin^2 2\theta = \frac{ck^6 Q_0^2}{512\pi}\sin^2 2\theta$$



- Direct integration $P = \int \frac{dP}{d\Omega} d\Omega$ reproduces

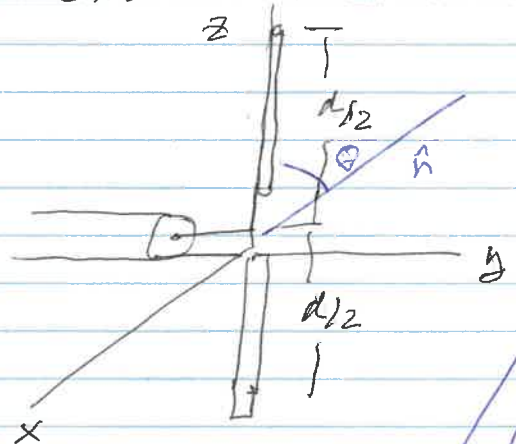
$$P = \frac{ck^6}{360}\sum_{ij}|Q_{ij}|^2 = \frac{ck^6}{360} Q_0^2 \left(1 + \frac{1}{4} + \frac{1}{4}\right) = \frac{ck^6 Q_0^2}{240}$$

$$\left.\begin{array}{l} \text{Note } E_1 \sim k^4 \\ E_2, M_1 \sim k^4 (kd)^2 \end{array}\right\} \text{E1 dominates if its there}$$

Thin-wire antennas can often be solved outside multipole approx. directly from

$$\vec{A}(x) = \frac{1}{c}\frac{e^{ikr}}{r}\int d^3x'\, e^{-ik\hat{n}\cdot\vec{x}'}\vec{J}(x')$$

Jackson looks at center-fed antenna



$$\vec{J}(\vec{x}') = \hat{z}\, I(z')\,\delta(x')\,\delta(y')\,\theta\left(\frac{d}{2}-|z|\right)$$

Need a model for $I(z')$ -

J. suggests

$$I(z) = I_0 \sin\left(k\left(\frac{d}{2}-|z|\right)\right)$$

$$A(\vec{x}) = \hat{z}\,\frac{I_0}{c}\frac{e^{ikr}}{r}A_z$$

$$A_z = \int_{-d/2}^{d/2} dz' \sin\left[k\left(\frac{d}{2}-|z'|\right)\right]e^{-ikz'\cos\theta}$$

Integrals are of form $\int e^{\pm ik\left(\frac{d}{2}-z'\right)}e^{\pm ikz'\cos\theta}\,dz'$

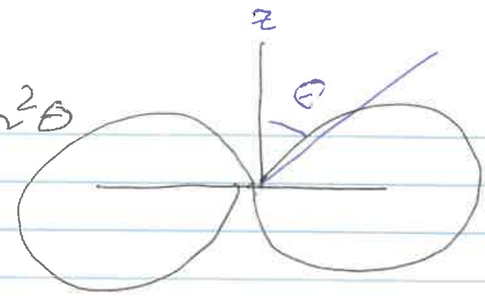
$$\vec{B} = ik\,\hat{n}\times\vec{A}\,,\quad \vec{E} = -\hat{n}\times\vec{B} \qquad \vec{A} = \hat{z}A_z$$

$$\frac{dP}{d\Omega} = \frac{c}{8\pi}r^2(E\times B^*)\cdot\hat{n} = \frac{ck^2}{8\pi}|A_z|^2$$

$$\times\,\hat{n}\cdot\left[\left(-\hat{n}\times(\hat{n}\times\hat{z})\right)\cdot(\hat{n}\times\hat{z})^*\right]$$

Vector products same as for linear electric dipole along $\hat{z}$

it is $[\hat{z}^2 - (\hat{n}\cdot\hat{z})^2] = \sin^2\theta$



but A_z modifies it.

$$A_z = \frac{2I_0}{c}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]$$

after long messy calculation. This is exact, includes all multipoles. Useful test bed

1) In long wavelength limit, $kd = \frac{2\pi d}{\lambda} \to 0$,

expand $[\;] = \dfrac{1 - \frac{1}{2}\left[\frac{kd}{2}\cos\theta\right]^2 - \left(1 - \frac{1}{2}\left(\frac{kd}{2}\right)^2\right)}{\sin^2\theta}$

$$= \frac{\frac{1}{2}\left(\frac{kd}{2}\right)^2\left[-\cos^2\theta + 1\right]}{\sin^2\theta} = \text{constant}$$

recover full dipole formula (have to work to get constant prefactors but they are perfect, too)

2) Can see how well dipole formula works, when it shouldn't!

Consider half-wavelength dipole

$kd = \pi$ — kills $\frac{kd}{2}$ term in A_z

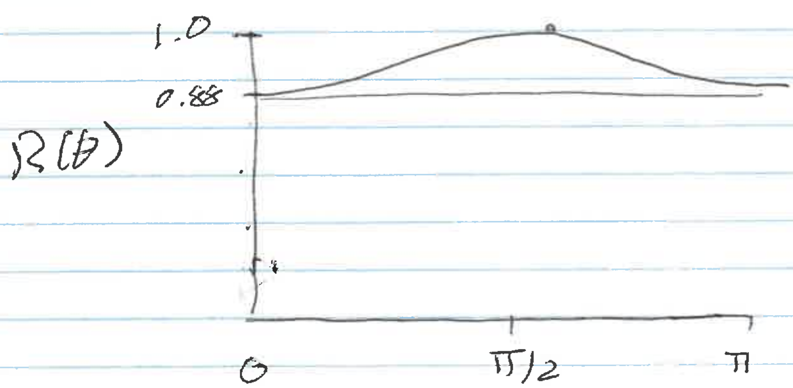
$$\frac{dP}{d\Omega} = \frac{I_0^2}{2\pi c} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

Dipole formula is $\frac{I_0^2}{2\pi c} \frac{\pi^4}{64} \sin^2\theta$

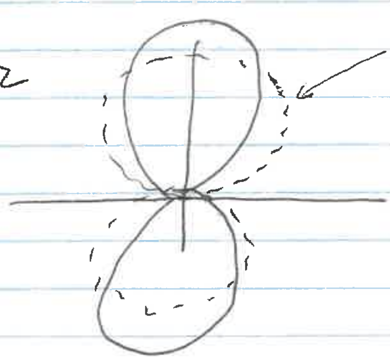
from $I = \sin k\left(\frac{\ell}{2} - |z|\right)$

$$\frac{\frac{dP}{d\Omega}}{\left.\frac{dP}{d\Omega}\right)_{dipole}} = \frac{64}{\pi^4} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^4\theta} = 0.66\, R(\theta)$$

$\frac{64}{\pi^4} \sim 0.66$



or



dipole is a little fatter