7320 FINAL EXAM

Begin each problem on a separate piece of paper. Take your time and think before you write. Show all your work. When an explanation is required, write complete sentences in grammatical English.

1) [30 points] Compute the lifetime of classical matter: at time $t = 0$ an electron orbits a proton in a classical circular orbit of radius R. Compute how long it takes for the electron to spiral (non-relativistically) to $R = 0$, assuming that it always remains in a circular orbit as it radiates.

2) [30 points] The Greisen bound: there is thought to be an upper limit to the energy of galactic cosmic rays (protons) due to the inelastic scattering of protons off the 3 K cosmic microwave background, $p + \gamma = \Delta$. What is this bound? Use $m_p = 1$ GeV, $m_\Delta = 1.2$ GeV, and recall that room temperature (300 K) corresponds to a mean photon energy of 1/40 eV. As yet another hint, the bound is an energy much greater than the proton mass.

3) [30 points] Light of frequency ω scatters from a sphere of radius R made of paramagnetic material, which is parametrized by a local fluctuation in the magnetic susceptibility $\delta\mu$ which is a constant inside the sphere. (This is CGS language, in MKS, $\epsilon = \epsilon_0$ and it is μ/μ_0 which experiences a fluctuation away from unity.) Compute the (polarization summed and averaged) differential cross section in Born approximation for light scattering from the impurity.

4) [30 points] A particle moves relativistically along the x axis under a force due to a constant electric field $\vec{F} = \hat{x}eE_0$. It is at rest at $x = 0$ at $t = 0$. (a) [20 points] What is $x(t)$? Check your answer at very short and very long times. (b) [10 points] What is the total (instantaneous) electromagnetic power radiated at any time?

5) [30 points] A positron of initial velocity $v/c = \beta_0$ collides with a nucleus of charge Ze and scatters directly backward along its initial direction of motion. Find a formula for $d^2N/(d\omega d\Omega)$, the number of photons emitted per solid angle $d\Omega$ and frequency interval $d\omega$. Your answer should be valid when β_0 is close to unity and at low ω.

1) $P = -\frac{dE}{dt} = \frac{2}{3}\frac{e^2 a^2}{c^3}$

For circular motion, $E = \frac{1}{2}mv^2 - \frac{e^2}{r}$

$$\frac{mv^2}{r} = \frac{e^2}{r^2} \Rightarrow mv^2 = \frac{e^2}{r}$$

$$\rightarrow E = -\frac{1}{2}\frac{e^2}{r}$$

$a = \frac{v^2}{r} = \frac{e^2}{mr^2}$ so $P = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{e^2}{mr^2}\right)^2$

$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{-e^2}{2r}\right) = P$ so

$$\frac{e^2}{2r^2}\frac{dr}{dt} = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{e^2}{mr^2}\right)^2$$

$$\frac{dr}{dt} = \frac{4}{3}\frac{e^4}{m^2c^3}\frac{1}{r^2}$$

$$\int_0^R r^2 dr = \frac{4}{3}\frac{e^4}{m^2c^3}\int_0^t dT$$

$$\frac{1}{3}R^3 = \frac{4}{3}\frac{e^4}{m^2c^3}T$$

$$\therefore T = \frac{1}{4}\frac{m^2c^3}{e^4}R^3$$

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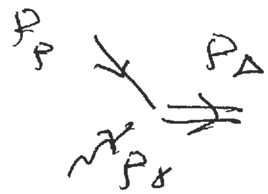
units: $\left(\frac{\hbar c}{e^2}\right)^2 \frac{1}{(\hbar c)^2} m^2c^4 \frac{R^3}{c} = \frac{\text{energy}^2}{(\text{energy} \times \text{length})^2} \frac{\text{length}^3}{\text{length}/\text{time}}$

$$T = \frac{(mc^2)^2}{c}\left[\frac{R}{e^2}\right]^2 \cdot R \sim \frac{(5\times10^5\,\text{eV})^2}{3\times10^{18}\,\text{Å}/\text{sec}}\,\frac{1}{13.6\,\text{eV}} \times \text{Å}$$

$$T \sim 10^{-8}\ \text{sec.}$$

Greisen bound: $p + \gamma = \Delta$ means

$$p_\Delta = p_p + p_\gamma \quad \text{as 4-vector relation}$$



Square it: $p_\Delta^2 = m_\Delta^2 = (p_1 + p_\gamma)^2$

$$m_\Delta^2 = m_p^2 + 2 p_p \cdot p_\gamma \quad (p_\gamma^2 = 0)$$

or $$p_p \cdot p_\gamma = \frac{m_\Delta^2 - m_p^2}{2}$$

Now evaluate head-on collision in Earth's frame

$p_\gamma = [\omega, 0, 0, -\omega]$ massless photon

$p_p = [P, 0, 0, P]$: very high energy, treat as massless particle

$p_\gamma \cdot p_P = 2\omega P$ so

$$P = \frac{m_\Delta^2 - m_p^2}{4\omega}$$

Numbers: Room temp is 300 K, black body is 3 K, 300 K $= \frac{1}{40}$ eV

$$\omega = \frac{1}{40} \text{ eV} \times \frac{1}{100} = \frac{1}{4000} \text{ eV}$$

$m_\Delta = 1.2$ GeV, $m_p = 1$ GeV, GeV $= 10^9$ eV

$m_\Delta^2 - m_p^2 = 0.44 \text{ GeV}^2 = 0.44 \times 10^{18} \text{ eV}^2$

$$P = \frac{0.44 \times 10^{18} \text{ eV}^2}{\frac{1}{4} \times 10^{-3} \text{ eV}} \sim 10^{21} \text{ eV !}$$

Horrible un-necessary algebra

$$p_p \cdot p_\gamma = \omega \left[\sqrt{E^2 - m^2} + E \right]$$

p momentum

$$E + \sqrt{E^2 - m^2} = \frac{m_\Delta^2 - m_p^2}{2\omega}$$

$$\sqrt{E^2 - m^2} = \frac{m_\Delta^2 - m_p^2}{2\omega} - E$$

square: $$E^2 - m^2 = \left[\frac{m_\Delta^2 - m_p^2}{2\omega} \right]^2 - \frac{E(m_\Delta^2 - m_p^2)}{2\omega} + E^2$$

$$E = \frac{\left(\frac{m_\Delta^2 - m_p^2}{2\omega} \right)^2 - m_p^2}{\frac{m_\Delta^2 - m_p^2}{2\omega}}$$

The first term is huge because ω is tiny vs m_p. ~~This gives the~~ Neglect m_p, get

$$E \sim \frac{m_\Delta^2 - m_p^2}{2\omega} + \cdots$$

the answer on p. 2.1.

3) This is a Born approximation problem, nearly done in the book. Jackson 10.31 says

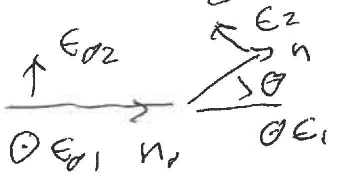
$$\frac{d\sigma}{d\Omega}(\vec{\epsilon}_0 \to \vec{\epsilon}) = |f(\theta)|^2$$

$$f(\theta) = \frac{k^2}{4\pi}(\hat{n} \times \hat{\epsilon}^*) \cdot (\hat{n}_0 \times \hat{\epsilon}_0)\, g(q)$$

$$g(q) = \int d^3x\, e^{i\vec{q}\cdot\vec{x}}\, \delta\mu(x)$$

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}|g(q)|^2 \left|(\hat{n} \times \hat{\epsilon}^*) \cdot (\hat{n} \times \hat{\epsilon}_0)\right|^2$$

We average initial polarizations, sum final ones



Note $\hat{n} \times \hat{\epsilon}_2 = \hat{\epsilon}_1$ for each
$\hat{n} \times \hat{\epsilon}_1 = -\hat{\epsilon}_2$ polarization

so the polarization part of the answer is like the $\hat{\epsilon}_0 \cdot \hat{\epsilon}^*$ of electric scattering

$\epsilon_0 \backslash \epsilon$	1	2
1	$\cos\theta$	0
2	0	1

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{1}{2}\sum_{\epsilon_0, \epsilon} \frac{d\sigma}{d\Omega}$$

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{k^4}{16\pi^2}\left[\frac{1+\cos^2\theta}{2}\right]|g(q)|^2.$$

$$g(q) = \int d^3x\, e^{iq\cdot x}\, \delta\mu(r)$$

$$= \delta\mu \int_0^R r^2 dr \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta\, e^{iqr\cos\theta}$$

(picking the z-axis to be along $\vec{q}$)

$$g = \delta\mu \cdot 2\pi \int_0^R r^2 dr \left[\frac{e^{iqr} - e^{-iqr}}{iqr} \right]$$

$$= 4\pi \frac{\delta\mu}{q} \int_0^R r\, dr\, \sin qr$$

$$= \frac{4\pi}{q^3} \int_0^{qR} x\, dx\, \sin x = \frac{4\pi}{q^3} \left[\sin qR - qR \cos qR \right]$$

$$\frac{d\bar{\sigma}}{d\Omega} = k^4 \left[\frac{1+\cos^2\theta}{2} \right] \left(\frac{\sin qR - qR\cos qR}{q^3} \right)^2$$

4) Particle begins at rest in constant E-field

$$\frac{d}{dt}\gamma m v = eE \Rightarrow \gamma v = \frac{e}{m}Et = \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$x(t) = \int v(t)\,dt \,..$$

$$\frac{v(t)/c}{\sqrt{1-\frac{v(t)^2}{c^2}}} = \frac{eEt}{mc} \Longrightarrow \frac{\beta^2}{1-\beta^2} = \left(\frac{eEt}{mc}\right)^2$$

$$\beta^2 = (1-\beta^2)\left[\frac{eEt}{mc}\right]^2$$

$$\left[1+\left(\frac{eEt}{mc}\right)^2\right]\beta^2 = \left(\frac{eEt}{mc}\right)^2$$

$$\Rightarrow \beta(t) = \frac{v(t)}{c} = \frac{eEt/mc}{\left[1+\left(\frac{eEt}{mc}\right)^2\right]^{1/2}}$$

$$x(t) = \frac{eE}{m}\int_0^t \frac{t'\,dt'}{\left[1+\left(\frac{eEt'}{mc}\right)^2\right]^{1/2}} \quad ; \quad u = 1+\left(\frac{eEt'}{mc}\right)^2$$

$$du = 2t\left(\frac{eE}{mc}\right)^2$$

$$x(t) = \frac{eE}{m}\left(\frac{mc}{eE}\right)^2\cdot\frac{1}{2}\int\frac{du}{\sqrt{u}} = \frac{mc^2}{eE}\left[1+\left(\frac{eEt'}{mc}\right)^2\right]^{1/2}\Bigg|_0^t$$

$$x(t) = \frac{mc^2}{eE}\left[\left(1+\left(\frac{eEt}{mc}\right)^2\right)^{1/2} - 1\right]$$

At small t, $x(t) = \frac{mc^2}{eE}\left[1+\frac{1}{2}\left(\frac{eEt}{mc}\right)^2 - 1\right]$

$$= \frac{eE}{m}\frac{1}{2}t^2 \;!$$

At big t $\quad x(t) \sim \frac{mc^2}{eE}\left(\frac{eEt}{mc}\right) \sim ct\,!$

Power radiated?

$$P = -\frac{2}{3}\frac{e^2}{m^2c^3}\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}$$

$$= \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\left[\left(\frac{d\vec{p}}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]$$

$\frac{d\vec{p}}{dt} = e\mathcal{E}$ for electric field $\mathcal{E}$

energy $E^2 = p^2c^2 + m^2c^4$ so

$$\frac{E}{c}\frac{dE}{dt} = \frac{c^2 p}{c}\frac{dp}{dt}$$

$$\frac{1}{c}\frac{dE}{dt} = c\,\frac{p}{E}\frac{dp}{dt} \quad ; \quad \frac{cp}{E} = \beta = \frac{v}{c} \text{ so}$$

$$\frac{1}{c}\frac{dE}{dt} = \frac{v}{c}\frac{dp}{dt}$$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2 \cdot \gamma^2\left(1 - \frac{v^2}{c^2}\right)$$

$$= \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2 \quad \text{(this is 14.27!)}$$

$$= \frac{2}{3}\frac{e^2}{m^2c^3}(e\mathcal{E})^2 .$$

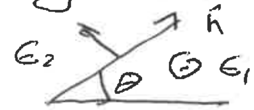
5) Positron collides with nucleus. In the extreme relativistic limit and at low frequencies and small angles,

$$\frac{d^2I}{d\Omega\, d\omega} = \frac{e^2}{4\pi^2 c}\left| \hat{\epsilon}^* \cdot \left(\frac{\vec{\beta}_f}{1-\vec{\beta}_f\cdot\hat{n}} - \frac{\beta_i}{1-\vec{\beta}_i\cdot\hat{n}} \right)\right|^2$$

$\vec{\beta}_i = \vec{\beta}_0$, $\vec{\beta}_f = -\vec{\beta}_0$: let them lie on the z axis

$$(\ \ ) = -\left(\frac{1}{1+\beta_0\cos\theta} + \frac{1}{1-\beta_0\cos\theta}\right)\vec{\beta}_0 = -\frac{2\vec{\beta}_0}{1-\beta_0^2\cos^2\theta}$$

To make life easy while summing polarizations, pick them as



so $\hat{\epsilon}_1\cdot\vec{\beta}_0 = 0$, $\hat{\epsilon}_2\cdot\vec{\beta}_0 = \beta_0\sin\theta$

$$\frac{d^2I}{d\Omega\, d\omega} = \frac{e^2}{4\pi^2 c}\cdot\frac{4\beta_0^2\sin^2\theta}{(1-\beta_0^2\cos^2\theta)^2}$$

$I(\omega) = \hbar\omega N(\omega)$ so

$$\frac{d^2N}{d\Omega\, d\omega} = \frac{e^2}{\hbar c}\frac{1}{4\pi^2}\frac{1}{\omega}\cdot\frac{4\beta_0^2\sin^2\theta}{(1-\beta_0^2\cos^2\theta)^2}$$

Grooming: $\beta_0^2 = 1-\frac{1}{\gamma^2}$, $\sin\theta \sim \theta$, $\cos^2\theta = 1-\theta^2$

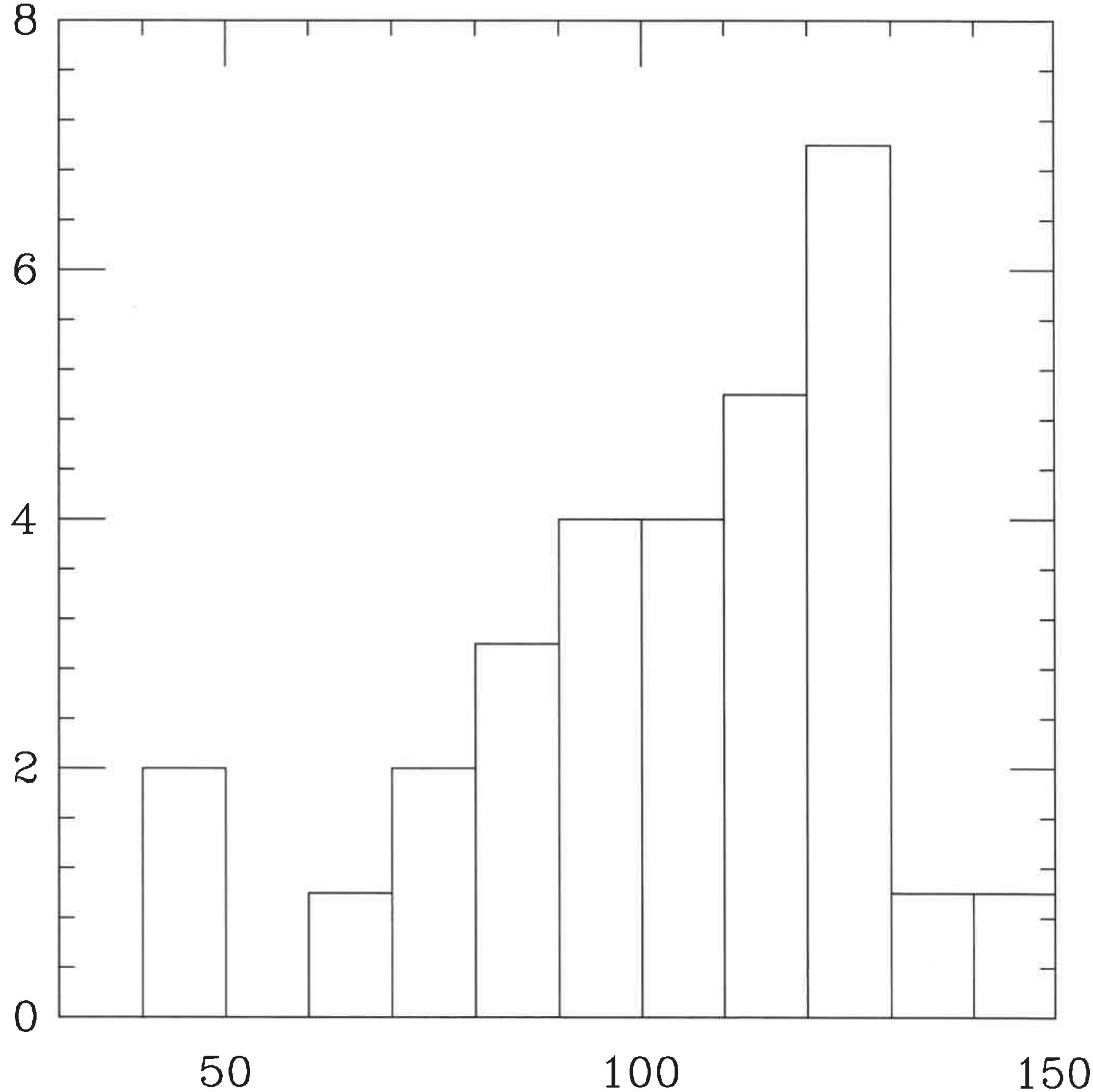
$$1-\beta_0^2\cos^2\theta = 1-\left(1-\frac{1}{\gamma^2}\right)(1-\theta^2) = \frac{1}{\gamma^2}+\theta^2 = \frac{1}{\gamma^2}(1+\gamma^2\theta^2)$$

$$\frac{d^2N}{d\Omega\, d\hbar\omega} = \frac{e^2}{\hbar c}\frac{1}{\pi^2}\frac{1}{\hbar\omega}\cdot\gamma^2\left[\frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^2} + \begin{matrix}(\theta\to\pi-\theta \\ \text{for} \\ \text{backwards} \\ \text{term})\end{matrix}\right]$$

7320 Spring 2024

Final ave = 101

σ = 25



7320 overall
Spring 2024

ave = 382/500
σ = 37