Units etc

Systems of units are human constructs. They are designed to accomplish at least two goals:

- Produce order unity numbers for calculations
- Build in physics constraints which are known to be true

This sounds very peculiar! In an undergraduate class, one might begin with electricity and magnetism as separate interactions with their own unit conventions, and eventually discover that $\mu_0\epsilon_0 = 1/c^2$. But you are not undergraduates, and this unification was done long ago. Why not move on?

You will encounter (at least) three unit systems in electrodynamics. They are MKS/MKSA/SI (kilograms, etc), CGS, and Lorentz-Heaviside (which is CGS with re-arranged 4π's). Lorentz-Heaviside is the standard in quantum field theory. They differ in

- Relation of charge and current to force
- Relative normalization of E and B

All these unit systems maintain the "known physics" of special relativity, in particular, that light moves at velocity c. Charge is conserved in electromagnetism and the unit systems maintain that, too.

In addition, you will encounter "natural units," where $c = 1$ and $\hbar = 1$. More on this below.

Everyone starts simply with the continuity equation for the current density $\vec{J}$ and the charge density ρ:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0. \tag{1}$$

No magnetic monopoles:

$$\nabla \cdot \vec{B} = 0 \tag{2}$$

and this introduces the vector potential, $\vec{B} = \vec{\nabla} \times \vec{A}$. It's the same convention for all unit systems.

Faraday's law is

$$\vec{\nabla} \times \vec{E} + k_3 \frac{\partial \vec{B}}{\partial t} = 0 \tag{3}$$

and the constant k_3 sets the relative dimensionality of E and B. Introducing brackets to label the dimensionality of a quantity,

$$\frac{[E]}{[B]} = k_3 \frac{[L]}{[T]} \tag{4}$$

(In CGS, $k_3 = 1/c$ and E and B have the same units.)

Gauss' law relates E and ρ:

$$\vec{\nabla} \cdot \vec{E} = 4\pi k_1 \rho. \tag{5}$$

Always writing $\vec{E} = -\vec{\nabla}\Phi + \ldots$, the field and potential for a point charge are

$$\vec{E} = k_1 \frac{q}{r^2}\hat{r}; \qquad \Phi = k_1 \frac{q}{r}. \tag{6}$$

We can define units for q and use k_1 to convert electric forces to "ordinary forces" (whatever that means – think about it!). This is what MKS does. Alternatively, we can define k_1 and the units of charge come from units of mechanical force plus k_1. This is what CGS does: in CGS $k_1 = 1$ and $[q^2] = [energy \times length]$ since energy is $q\Phi$. Since $\hbar c$ has the same units of $[energy \times length]$, the fine structure constant $e^2/(\hbar c) = 1/137$ is a pure dimensionless number in CGS.

Continuity plus Gauss' law tells us that

$$\vec{\nabla} \cdot [\vec{J} + \frac{1}{4\pi k_1}\frac{\partial \vec{E}}{\partial t}] = 0. \tag{7}$$

We use the term in square brackets, the sum of ordinary current and displacement current, to source the magnetic field. We know the curl of B is proportional to the current density, so we write

$$\vec{\nabla} \times \vec{B} = 4\pi A k_2 [\vec{J} + \frac{1}{4\pi k_1}\frac{\partial \vec{E}}{\partial t}] \tag{8}$$

where A is another constant. The J term by itself can be used to give B for a long straight wire and then from the magnetic part of the Lorentz force law, the force per unit length between two parallel wires a distance d apart is

$$\frac{dF}{dl} = \frac{BI'}{A} = 2k_2 \frac{II'}{d}. \tag{9}$$

I is a current, so regardless of the units of q, $[I] = [q/t]$. k_2 is like k_1, in that it connects a mechanical force to a magnetic quantity. Forces are also forces, so we can compare the units of electrostatic forces with magnetic ones, and we discover

$$[\frac{k_1}{k_2}] = [\frac{l^2}{t^2}] \tag{10}$$

The experimental ratio of magnetostatic to electrostatic forces fixes this ratio to be c^2. In the convention table, this constraint is hardwired. (Maybe long ago you would have let it be a free parameter.)

Also

$$[E] = [F/q] = [\frac{BId}{Aq}] = \frac{[B]}{A}\frac{d}{t} \tag{11}$$

or

$$\frac{E}{B} = \frac{1}{A}\frac{d}{t} \tag{12}$$

System	k_1	k_2	A	k_3	α	$V(r)$
Gaussian (CGS)	1	$1/c^2$	c	$\frac{1}{c}$	$\frac{e^2}{\hbar c}$	$\frac{e^2}{r}$
Lorentz-Heaviside	$1/(4\pi)$	$1/(4\pi c^2)$	c	$\frac{1}{c}$	$\frac{1}{4\pi}\frac{e^2}{\hbar c}$	$\frac{e^2}{4\pi r}$
MKSA	$\frac{1}{4\pi\epsilon_0} = 10^{-7}c^2$	$\frac{\mu_0}{4\pi} = 10^{-7}$	1	1	$\frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c}$	$\frac{e^2}{4\pi\epsilon_0 r}$

Table 1: Constants in various conventions (modified from Jackson). The fine structure constant uses the physical electron charge ($1.6 \times 10^{-19} C$ in MKS – you almost never need to know that!) It is the dimensionless number $1/137.035\ldots$. The last column is the formula for the potential energy between two charges a distance r apart.

again tells you the ratio of dimensions of E and B. You can also get this from the Lorentz force law,

$$\vec{F} = q[\vec{E} + \frac{\vec{v}}{A} \times \vec{B}]. \tag{13}$$

Note also, k_3 has units of $1/A$ from Faraday's law.

Finally, if $\vec{J} = 0$ the two curl equations give you

$$\nabla^2 \vec{B} - [k_3 A \frac{k_2}{k_1}] \frac{\partial^2 \vec{B}}{\partial t^2} = 0. \tag{14}$$

The object in the brackets must equal $1/c^2$ to get radiation moving at the velocity of light. From the static force ratio, $k_2/k_1 = 1/c^2$ and $k_3 A = 1$.

We can summarize useful results in Table 1.

The 10^{-7} is an exact number. Notice that if you are working in MKS, you never need to know ϵ_0. If you are working in CGS you never need to know e. If you are only doing electrostatics, or only doing magnetostatics, you can forget that there is any connection between electricity and magnetism. That is what we will do this semester. In that case, MKS is nice: physical batteries deliver voltages in volts, an ampere is an interesting current. If you are doing atomic physics, MKS is not nice, and you will run into a lot of big exponents which make it easy to make mistakes, and hard to keep track of the relative sizes of various effects.

Essentially all quantum field theory is done using Lorentz - Heaviside units. The reason is that most perturbative calculations are done in momentum space (not coordinate space). The quantities $1/q^2$ and $1/(4\pi r)$ are Fourier transforms of each other. In Lorentz - Heaviside units the annoying 4π's are pushed into places where one does not need to go (usually) – typically into coordinate space formulas.

All of this discussion was done at the level of Maxwell's equations, and the Lorentz force law makes a fifth equation. Is everything consistent? This is very annoying to check. The right way to tell this story is to start with a Lagrange density for electrodynamics, and to think of Maxwell's equations as equations of motion which arise from varying the Lagrangian. The Lorentz force law comes from thinking about momentum conservation in terms of field variables. We will do that next semester. (For now, it is overkill.) But, knowing that this is coming, I can say a few more things about units.

First of all, E and B are not independent: they are elements of a rank -two tensor, called the field strength tensor, $F^{\mu\nu}$. The vector and scalar potentials are also not independent; they are components of a rank - one tensor (a vector). Under a Lorentz transformation, the potentials mix (as do the components of E and B). That being the case, it seems absurd to me to think of E and B, or Φ and A, as having different units.

In fact, when you are working at this level, it is absurd to give quantities like Φ and A different names. Treat them as components of one object, with an index! Then it is easy to build in the symmetries which you want your dynamics to encode. Again, the purpose of notation is to make your life easy, not to complicate it.

Second, the distinction between "charge units"' and "mechanical units" seems very capricious. Where do mechanical forces come from, after all? The language of modern quantum field theory does not make this distinction. The Lagrangians we use are built out of physical degrees of freedom (for electrodynamics, these are the potentials Φ and A) and a set of coupling constants, quantities which multiply terms in the Lagrangian. For a free nonrelativistic particle,

$$S = \int dt L(q, dq/dt) = \int dt \frac{1}{2} m (\frac{dq}{dt})^2. \tag{15}$$

The coordinate q is the analog of the field variable and m is the analog of a coupling constant. Some coupling constants (like m) carry an engineering dimension and some (like the charge) are dimensionless. (I am now thinking about $e^2/(\hbar c)$, of course.) But never mind, they are all coupling constants.

All the constants (K's, A, the 4π's of CGS vs Lorentz - Heaviside in the Maxwell equations will take care of themselves when you write a Lagrange density and vary it: CGS versus Lorentz-Heaviside is just

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{16}$$

versus the choice

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}. \tag{17}$$

Doing relativistic electrodynamics in MKS is just a mess and I don't know anyone who does serious calculations in those units.

Finally, natural units, $c = \hbar = 1$. You probably won't see these in a first year graduate course, but when you get to the real world, this will be all that there is.

With $c = 1$ you are treating time as a component of $x = (x_0, \vec{x}) = (ct, \vec{x})$ so you can measure time or distance in cm or sec, and multiply or divide by $c = 3 \times 10^8 m/s$ to get back to high school units. We are using

$$[L] = [T]. \tag{18}$$

Next, $\hbar = 1$. This means that $[E] = 1/[T] = 1/[L]$. An energy is an inverse time. Use $\hbar c = 2000$ eV-Angstroms or 200 MeV-fm to convert energies into inverse lengths or inverse times.

In fact, if you are doing atomic physics, even if you are keeping $\hbar$ and c around, multiply and divide by $\hbar c$, convert e^2 into $e^2/(\hbar c)$, convert the electron mass into $m_e c^2 = 0.511$ KeV, remember the formula for the Bohr radius and the Rydberg (1/2 Angstrom and 13.6 eV), work in terms of Angstroms and eV's, and never never never think about MKS values of anything.