

## 7310 MIDTERM

Take your time and think before you write. Begin each problem on a separate piece of paper. Show all your work. When an explanation is requested, write complete sentences in grammatical English.

- 1) [40 points] (a) [30 points] Find the two-dimensional electrostatic Green's function for a slot ( $0 < y < a$ ,  $-\infty < x < \infty$ ) of the form

$$G(x, y; x', y') = \sum_m g_m(x, x') \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{m\pi y'}{a}\right) \quad (1)$$

and give an explicit formula for  $g_m(x, x')$ .

- (b) [10 points] Now a constant line charge  $\lambda$  oriented along the  $y$  axis is placed at  $x = 0$ . Both the sides at  $y = 0$  and  $y = a$  are grounded. Find the leading behavior of the electrostatic potential at large  $x$ .

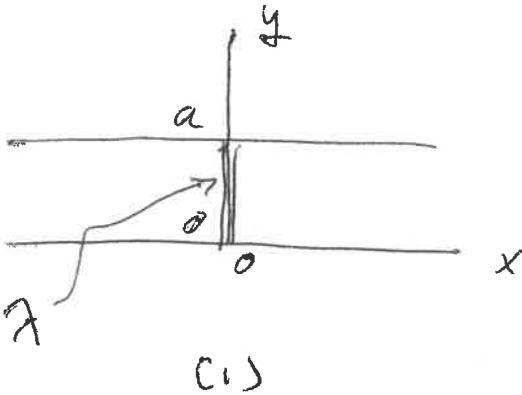
- 2) [30 points] An electron in one of the P shells of hydrogen has a charge density

$$\rho(r) = f(r) \cos^2 \theta \quad (2)$$

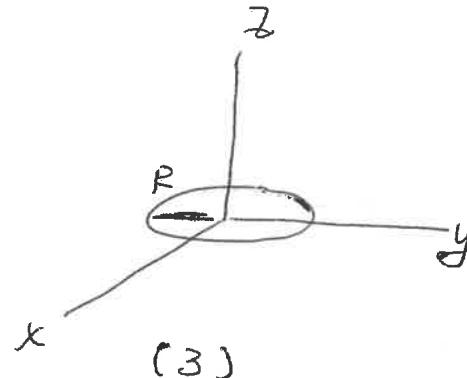
where  $f(r)$  is some function of the radius. Find all the nonzero multipoles of the charge distribution (either spherical multipoles or Cartesian ones, it's your choice). You would use these if you wanted to know the electrostatic potential of an atom. Express your answer in terms of the integrals

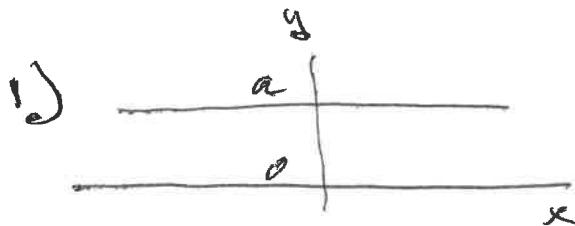
$$I_n = \int_0^\infty dr r^n f(r). \quad (3)$$

- 3) [30 points] A uniformly charged disc has radius  $R$  and total charge  $Q$  and is centered on the origin, in the  $x - y$  plane. Find an approximate formula for the potential at distances  $r \gg R$ , in the plane of the disc.



1





$$G(x, y; x', y') = \sum_m g_m(x, x') \sin \frac{m\pi y}{a} \sin \frac{m\pi y'}{a}$$

$$\text{and } S(y-y') = \frac{2}{a} \sum_m \sin \frac{m\pi y}{a} \sin \frac{m\pi y'}{a}$$

$$\nabla^2 G = -4\pi S^2(x-x') \Leftrightarrow -4\pi S(x-x') S(y-y')$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ so this is}$$

$$\begin{aligned} \nabla^2 G &= \sum_m \sin \frac{m\pi y}{a} \sin \frac{m\pi y'}{a} \left[ \frac{\partial^2 g}{\partial x^2} - \left( \frac{m\pi}{a} \right)^2 g \right] \\ &= \frac{2}{a} (-4\pi) S(x-x') \sum_m \sin \frac{m\pi y}{a} \sin \frac{m\pi y'}{a} \end{aligned}$$

$$\text{or } \frac{\partial^2 g}{\partial x^2} - k^2 g = -\frac{8\pi}{a} S(x-x') \text{ where } k^2 = \left( \frac{m\pi}{a} \right)^2$$

$$g=0 \text{ at } x \rightarrow \infty \text{ and } x \rightarrow -\infty \text{ so}$$

$$g = A e^{-kx_>} e^{+kx_<}$$

$x_>$  = greater of  $(x, x')$

$x_<$  = lesser of  $(x, x')$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left[ \frac{\partial^2 g}{\partial x^2} - k^2 g \right] = -\frac{8\pi}{a} \int_{x'-\epsilon}^{x'+\epsilon} dx S(x-x') = -\frac{8\pi}{a}$$

$$\left. \frac{dg}{dx} \right|_{x'+\epsilon} - \left. \frac{dg}{dx} \right|_{x'-\epsilon} = -\frac{8\pi}{a}$$

$$A \left[ -k e^{-kx'} e^{kx'} - (k e^{kx'} e^{-kx'}) \right] = -\frac{8\pi}{a} = -2kA$$

$$G(x, x') = \frac{4\pi}{a} \sum_m \sin \frac{m\pi y}{a} \sin \frac{m\pi y'}{a} \times \frac{\exp(-\frac{m\pi}{a}(x_s - x_{s'}))}{(m\pi/a)}$$

b)  $\Phi(x) = \frac{1}{4\pi\epsilon_0} \int d^2x' C(x') G(x, x')$

$$C(x') = \lambda S(x'), d^2x' = dx'dy'$$

$$\Phi(x, y) = \frac{1}{4\pi\epsilon_0} \int_0^a dy' \cdot \frac{4\pi}{a} \sum_m \sin \frac{m\pi y}{a} \sin \frac{m\pi y'}{a} \times \frac{\exp(-\frac{m\pi}{a}|x|)}{(m\pi/a)}$$

since  $x' = 0$  from the S-fn

~~P~~ The  $y'$  integral is

$$\int_0^a dy' \sin \frac{m\pi y'}{a} = -\frac{a}{m\pi} \cos \frac{m\pi y'}{a} \Big|_0^a$$

$$= -\frac{a}{m\pi} [\cos m\pi - 1] = \begin{cases} 0 & \text{if } m \text{ is even} \\ \frac{2a}{m\pi} & \text{if } m \text{ is odd} \end{cases}$$

but for  $x \gg a$  we only need  $m=1$

$$\Phi(x, y) \sim \frac{1}{4\pi\epsilon_0} \cdot \frac{2a}{\pi} \cdot \frac{4\pi}{a} \sin \frac{\pi y}{a} \frac{\exp(-\frac{\pi}{a}|x|)}{\pi/a}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \cdot \frac{8}{\pi a} \sin \frac{\pi y}{a} \exp\left(-\frac{\pi}{a}|x|\right)$$

$$2B) \quad C(x) = f(r) \cos^2 \theta$$

$$I_n = \int r^n f(r) dr$$

We need either spherical or Cartesian multipoles  
The key is to write  $C$  in terms of spherical harmonics:

$$\cos^2 \theta = \frac{2}{3} \left[ \frac{3 \cos^2 \theta - 1}{2} \right] + \frac{1}{3} = \frac{2}{3} P_2(\cos \theta) + \frac{1}{3} P_0(\cos \theta)$$

$$\text{and } Y_0^0 = \frac{1}{\sqrt{4\pi}}, \quad Y_2^0 = \sqrt{\frac{5}{4\pi}} P_2(\cos \theta),$$

$$\cos^2 \theta = \sqrt{4\pi} \left\{ \frac{1}{3} Y_0^0(\theta) + \frac{2}{3} \sqrt{\frac{1}{5}} Y_2^0(\theta) \right\}$$

Spherical multipoles are

$$B_{em} = \int d^3x C(x) Y_e^m(\theta, \phi) r^\ell$$

and from orthogonality the only nonvanish ones

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$$\text{are } B_{00} = \frac{\sqrt{4\pi}}{3} I_2 \quad B_{20} = \sqrt{4\pi} \frac{2}{3} \sqrt{\frac{1}{5}} I_4.$$

The Cartesian ones take a bit longer, but we can still use orthogonality

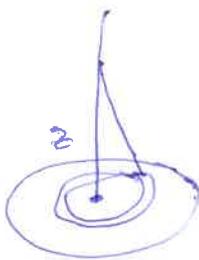
$$Q = \int d^3r C(r) = \frac{1}{3} \cdot 4\pi \cdot I_2, \text{ since } P_0(\cos \theta) \geq 1$$

$$Q_{zz} = \int d^3r C(r) \cdot r^2 \cdot (3z^2 - r^2) = \int d^3r C(r) r^4 \cdot \underbrace{\frac{2}{2\ell+1}}_{\ell=2} \times 2 \cdot \left[ \frac{3 \cos^2 \theta - 1}{2} \right]$$

$$= 2 \cdot I_4 \cdot \frac{2}{3} \cdot \frac{2}{5} \cdot 2\pi = \frac{16\pi}{15} I_4$$

$$Q_{xx} + Q_{yy} + Q_{zz} = 0, \quad Q_{xx} = Q_{yy} = -\frac{1}{2}, \quad Q_{zz} = -\frac{8\pi}{15} I_4$$

3)



$$\Phi(z) = \frac{\sigma}{4\pi\epsilon_0} \int_0^R \frac{2\pi c dc}{\sqrt{c^2 + z^2}}$$

$$= \frac{2\pi\sigma}{4\pi\epsilon_0} \left[ \sqrt{c^2 + z^2} \right]_0^R$$

$$= \frac{2\pi\sigma}{4\pi\epsilon_0} \left( \sqrt{z^2 + R^2} - |z| \right)$$

$$y^2 = c^2 + z^2$$

$$dy = \frac{c dc}{\sqrt{c^2 + z^2}}$$

$$(z^2 + R^2)^{1/2} = |z| \left( 1 + \frac{1}{2} \frac{R^2}{|z|^2} + \frac{1}{8} \left( \frac{R^2}{|z|^2} \right)^2 \right)$$

$$\sigma = \frac{\sigma}{\pi R^2}$$

$$\Phi = \underbrace{\frac{1}{2\epsilon_0} \frac{\sigma}{\pi R^2}}_{\text{constant}} \left[ \frac{1}{2} \frac{R^2}{|z|} - \frac{1}{8} \frac{R^4}{|z|^3} + \dots \right]$$

$$= \frac{\sigma}{4\pi\epsilon_0} \left[ \frac{1}{2} \frac{R^2}{|z|} - \frac{1}{4} \frac{R^2}{|z|^3} + \dots \right]$$

$$\Phi(r, \theta) = \frac{\sigma}{4\pi\epsilon_0} \left[ \frac{1}{r} - \frac{R^2}{4r^3} P_2(\cos\theta) + \frac{P_4(\cos\theta)}{r^5} \right]$$

~~In the planes~~  $\theta = \frac{\pi}{2}$   $\frac{3\cos^2\theta - 1}{2} = -\frac{1}{2}$

$$\Phi = \frac{\sigma}{4\pi\epsilon_0} \left[ \frac{1}{r} + \frac{1}{8} \frac{R^2}{r^3} + \dots \right]$$

To use a Greens function you need  $\rho(x)$  in spherical coords - it is

$$\rho(\vec{r}) = \sigma \Theta(R - r \sin\theta) \delta(z)$$

$$= \sigma \Theta(R - r \sin\theta) \delta(r \cos\theta)$$

$$= \frac{\sigma}{r} \delta(\cos\theta) \Theta(R - r)$$

check:

$$\int_0^{2\pi} d\phi \int_{-1}^1 d\cos\theta \int_0^R r^2 dr \frac{\sigma}{r} \delta(\cos\theta)$$

$$= 2\pi \int_0^R r dr \cdot \sigma = \pi R^2 \sigma$$

Note the  $1/r$ !

7310 Midterm

Fall 2023

ave = 67 σ = 23

